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Solution of boundary value problems for the heat equation with a piecewise constant coefficient using the Fourier method

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Abstract. This paper considers some initial boundary value problems for the heat equation in a bounded segment with a piecewise constant coefficient. Using the method of separation of variables, the problem under consideration is reduced to a spectral problem and eigenvalues and eigenfunctions of the resulting spectral problem are found. It is shown that the system of eigenfunctions forms a Riesz basis. Next, we prove a theorem on the existence and uniqueness of solutions to the initial-boundary value problems under consideration.

Keywords. Heat equation, discontinuous coefficients, eigenvalues, eigenfunctions, method of separation of variables.

Introduction

Heat conduction problems with discontinuous coefficients have been well studied for a long time. It should be noted the works [1–5], which are closest in theme to our work. In the work of A.A. Samarsky [1] using the methods of Green's function and heat potentials, the correctness of the first initial-boundary value problem for the heat equation with a discontinuous coefficient was proved. And in the work of Kazakhstan mathematicians E.I. Kim and B.B. Baimukhanov [2] by the method of potentials, by reducing to an integral equation, the correctness of the first initial-boundary value problem for a two-dimensional heat equation with a discontinuous heat conductivity coefficient in a half-space was proved. In [3–5], using

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heat potentials, the existence of classical solutions to various boundary value problems for parabolic equations was proved.

In the case without discontinuity, the spectral theory of these problems is almost completely constructed. Here we can mention the works [6–16].

In this work, the solution to some initial boundary value problems for the heat equation with a piecewise constant coefficient is justified by the method of separation of variables and the existence and uniqueness theorem of the solution is proved.

1 Formulation of the problem

We consider an initial boundary value problem for the following heat equation with a piecewise constant coefficient

$$Lu \equiv \left\{ \begin{array}{ll} u_t - k_1^2 u_{xx}, & 0 < x < x_0 \\ u_t - k_2^2 u_{xx}, & x_0 < x < l \end{array} \right\} = f(x,t), \tag{1}$$

in the domain $\Omega = \{(x,t) : 0 < x < l,\ 0 < t < T\}$, with the initial condition

$$u(x,0) = \varphi(x), \quad 0 \leq x \leq l, \tag{2}$$

with conjugation conditions

$$u(x_0 - 0, t) = u(x_0 + 0, t), \qquad 0 \leq t \leq T, \tag{3}$$

$$k_1 u_x(x_0 - 0, t) = k_2 u_x(x_0 + 0, t), \quad 0 \leq t \leq T, \tag{4}$$

and with one of the following boundary conditions of the form

$$u(0,t) = u(l,t) = 0, \qquad 0 \leq t \leq T, \tag{5}$$

$$u(0,t) = u_x(l,t) = 0, \qquad 0 \leq t \leq T, \tag{6}$$

$$u_x(0,t) = u(l,t) = 0, \qquad 0 \leq t \leq T, \tag{7}$$

$$u_x(0,t) = u_x(l,t) = 0, \qquad 0 \leq t \leq T. \tag{8}$$

The point x_0 is a strictly internal point of interval $0 < x_0 < l$. The coefficients $k_i > 0$, where $i = 1,\ 2$.

Problem (1)–(8) models the process of temperature field propagation in a thin rod of length l, consisting of two sections: $0 < x < x_0$ and $x_0 < x < l$ with different thermophysical characteristics.

The function $f(x,t)$ is continuous, $\varphi(x)$ is a twice continuously differentiable function that satisfies Boundary conditions (5)–(8) and Conjugation conditions (3)–(4).

2 Solution method

First, we consider Problem (1)–(5). The solution to Problem (1)–(5) is sought in the form $u(x,t) = Y(x) \cdot T(t) \neq 0$. Substituting into Equation (1) (for $f(x,t) = 0$) and Conditions (3)–(5), and separating the variables, we obtain the following spectral problem

$$LY(x) = \left\{ \begin{array}{ll} -k_1^2 Y''(x), & 0 < x < x_0 \\ -k_2^2 Y''(x), & x_0 < x < l \end{array} \right\} = \lambda Y(x), \tag{9}$$

$$Y(0) = Y(l) = 0 \tag{10}$$

$$Y(x_0 - 0) = Y(x_0 + 0), \quad k_1 Y'(x_0 - 0) = k_2 Y'(x_0 + 0). \tag{11}$$

The function $T(t)$ is a solution to the equation $T'(t) + \lambda T(t) = 0$.

The general solution to Equation (9) has the form

$$\left\{ \begin{array}{ll} Y(x) = c_1 \cos \mu_1 x + c_2 \sin \mu_1 x, & 0 < x < x_0, \\ Y(x) = d_1 \cos \mu_2 x + d_2 \sin \mu_2 x, & x_0 < x < l, \end{array} \right. \tag{12}$$

where $\mu_i = \frac{\sqrt{\lambda}}{k_i}$, for $i = 1, 2$.

Substituting the general solution (12) into Boundary conditions (10) and Conjugation conditions (11), and taking into account that $\mu_1 k_1 = \mu_2 k_2 = \sqrt{\lambda}$, we obtain

$$\left\{ \begin{array}{l} c_1 = 0, \\ d_1 \cos(\mu_2 l) + d_2 \sin(\mu_2 l) = 0, \\ c_1 \cos(\mu_1 x_0) + c_2 \sin(\mu_1 x_0) - d_1 \cos(\mu_2 x_0) - d_2 \sin(\mu_2 x_0) = 0, \\ -c_1 \sin(\mu_1 x_0) + c_2 \cos(\mu_1 x_0) + d_1 \sin(\mu_2 x_0) - d_2 \cos(\mu_2 x_0) = 0. \end{array} \right.$$

After cumbersome but simple calculations, we find the characteristic determinant of the system:

$$\Delta(\lambda) = \sin\left(\frac{\sqrt{\lambda}}{r}\right) = 0, \tag{13}$$

where

$$r = \frac{k_1 k_2}{k_2 x_0 + k_1 (l - x_0)}. \tag{14}$$

From Equation (13) one can find eigenvalues and eigenfunctions.

So, $\lambda_n = (\pi n r)^2$, where $n = 1, 2, \ldots$

$$Y_n(x) = C \left\{ \begin{array}{ll} \sin\left(\frac{\pi n r x}{k_1}\right), & 0 < x < x_0, \\ (-1)^{n+1} \sin\left(\frac{\pi n r (l - x)}{k_2}\right), & x_0 < x < l, \end{array} \right. \tag{15}$$

where r is determined by Formula (14).

Lemma 1. Spectral problem (9)–(11) is non-self-adjoint.

Proof. We find an adjoint problem to Problem (9)–(11). Given the following formula

$$Y''(x)Z(x) = \big(Y'(x)Z(x) - Y(x)Z'(x)\big)' + Z''(x)Y(x),$$

we obtain

$$\int_0^l Z(x)LY(x)dx = -\int_0^{x_0} Z(x)k_1^2Y''(x)dx - \int_{x_0}^l Z(x)k_2^2Y''(x)dx =$$
$$= -k_1^2Z(x_0-0)Y'(x_0-0) + k_1^2Z(0)Y'(0) + k_1^2Z'(x_0-0)Y(x_0-0)+$$
$$+ k_1^2Z'(0)Y(0) - k_2^2Z(l)Y'(l) + k_2^2Z(x_0+0)Y'(x_0+0) + k_2^2Z'(l)Y(l)-$$
$$- k_2^2Z'(x_0+0)Y(x_0+0) + \int_0^l Y(x)LZ(x)dx.$$

Using Boundary conditions (10) and Conjugation conditions (11), we obtain an adjoint problem. So, the adjoint problem has the following form:

$$LZ(x) = \left\{ \begin{array}{ll} -k_1^2Z''(x), & 0 < x < x_0 \\ -k_2^2Z''(x), & x_0 < x < l \end{array} \right\} = \lambda Z(x),$$

$$\left\{ \begin{array}{l} Z(0) = 0, \\ Z(l) = 0, \end{array} \right.$$

$$k_1Z(x_0-0) = k_2Z(x_0+0), \quad k_1^2Z'(x_0-0) = k_2^2Z'(x_0+0).$$

Thus, we have shown that problem (9)–(11) is not self-adjoint.

Lemma 2. The next spectral problem is self-adjoint.

$$Lv(x) = \left\{ \begin{array}{ll} -k_1^2v''(x), & 0 < x < x_0 \\ -k_2^2v''(x), & x_0 < x < l \end{array} \right\} = \lambda v(x), \tag{16}$$

$$v(0) = v(l) = 0, \tag{17}$$

$$k_1^{\frac{1}{2}}v(x_0-0) = k_2^{\frac{1}{2}}v(x_0+0), \quad k_1^{\frac{3}{2}}v'(x_0-0) = k_2^{\frac{3}{2}}v'(x_0+0). \tag{18}$$

Proof. We find an adjoint problem

$$\int_0^l w(x)Lv(x)dx = -\int_0^{x_0} w(x)k_1^2v''(x)dx - \int_{x_0}^l w(x)k_2^2v''(x)dx =$$
$$= -k_1^2w(x_0-0)v'(x_0-0) + k_1^2w(0)v'(0) + k_1^2w'(x_0-0)v(x_0-0)-$$
$$- k_1^2w'(0)v(0) - k_2^2w(l)v'(l) + k_2^2w(x_0+0)v'(x_0+0) + k_2^2w'(l)v(l)-$$
$$- k_2^2w'(x_0+0)v(x_0+0) + \int_0^l v(x)Lw(x)dx.$$

Using Boundary conditions (17) and Conjugation conditions (18), we obtain

$$\int_0^l w(x)Lv(x)dx = -k_2^{\frac{3}{2}}v'(x_0+0)\left(k_1^{\frac{1}{2}}w(x_0-0) - k_2^{\frac{1}{2}}w(x_0+0)\right) + k_1^2 v'(0)w(0) -$$
$$-k_2^2 v'(l)w(l) + k_1^{\frac{1}{2}}v(x_0-0)\left(k_1^{\frac{3}{2}}w'(x_0-0) - k_2^{\frac{3}{2}}w'(x_0+0)\right) + \int_0^l v(x)Lw(x)dx.$$

So, an adjoint problem has the form:

$$Lv(x) = \left\{ \begin{array}{ll} -k_1^2 w''(x), & 0 < x < x_0 \\ -k_2^2 w''(x), & x_0 < x < l \end{array} \right\} = \lambda w(x),$$

$$w(0) = w(l) = 0,$$

$$k_1^{\frac{1}{2}}w(x_0-0) = k_2^{\frac{1}{2}}w(x_0+0), \quad k_1^{\frac{3}{2}}w'(x_0-0) = k_2^{\frac{3}{2}}w'(x_0+0).$$

Thus, we have shown that Problem (16)–(18) is self-adjoint.

Now we find eigenvalues and eigenfunctions of Problem (16)–(18).

It can be shown that the eigenvalues of Problems (16)–(18) and (9)–(11) coincide, and the eigenfunctions differ by a piecewise constant factor. That is, $\lambda_n = (\pi n r)^2$, where $n = 1, 2, \ldots$

$$v_n(x) = C \left\{ \begin{array}{ll} \frac{1}{\sqrt{k_1}} \sin\left(\frac{\pi n r x}{k_1}\right), & 0 < x < x_0, \\ \frac{(-1)^{n+1}}{\sqrt{k_2}} \sin\left(\frac{\pi n r (l-x)}{k_2}\right), & x_0 < x < l. \end{array} \right.$$

This shows that the eigenfunctions of problem (9)–(11) and (16)–(18) are related by the following equality:

$$v_n(x) = \alpha(x)Y_n(x), \quad \text{where } \alpha(x) = \left\{ \begin{array}{ll} \frac{1}{\sqrt{k_1}}, & 0 < x < x_0, \\ \frac{1}{\sqrt{k_2}}, & x_0 < x < l. \end{array} \right.$$

Main result

Since $v_n(x)$ are the eigenfunctions of a self-adjoint problem, the system $\{v_n(x)\}$ of eigenfunctions forms a Riesz basis in $L_2(0,l)$ (orthogonal basis, if normalized, then orthonormal basis). If we choose C equal to $\sqrt{2r}$, then the system of eigenfunctions $\{v_n(x)\}$ forms an orthonormal basis.

$$v_n(x) = \sqrt{2r} \left\{ \begin{array}{ll} \frac{1}{\sqrt{k_1}} \sin\left(\frac{\pi n r x}{k_1}\right), & 0 < x < x_0, \\ \frac{(-1)^{n+1}}{\sqrt{k_2}} \sin\left(\frac{\pi n r (l-x)}{k_2}\right), & x_0 < x < l. \end{array} \right.$$

So, $v_n(x) = AY_n(x)$, where $AY(x) = \alpha(x)Y(x)$, $A : L_2(0,l) \to L_2(0,l)$ is a bounded operator and there exists A^{-1}, which is also bounded. This implies that the system of eigenfunctions $\{Y_n(x)\}$ also forms a Riesz basis.

Further, if we consider Problem (1)–(4) with Boundary conditions (6), (7) and (8), then, respectively, the eigenvalues and eigenfunctions have the form:

$\lambda_n = \left(\frac{\pi(2n+1)r}{2}\right)^2$, where $n = 0,\ 1,\ 2,\ \ldots$

$$Y_n(x) = C\begin{cases} \sin\left(\frac{\pi(2n+1)rx}{2k_1}\right), & 0 < x < x_0, \\ (-1)^n \cos\left(\frac{\pi(2n+1)r(l-x)}{2k_2}\right), & x_0 < x < l, \end{cases}$$

$\lambda_n = \left(\frac{\pi(2n+1)r}{2}\right)^2$, where $n = 0,\ 1,\ 2,\ \ldots$

$$Y_n(x) = C\begin{cases} \cos\left(\frac{\pi(2n+1)rx}{2k_1}\right), & 0 < x < x_0, \\ (-1)^n \sin\left(\frac{\pi(2n+1)r(l-x)}{2k_2}\right), & x_0 < x < l, \end{cases}$$

$\lambda_n = (\pi nr)^2$, where $n = 0,\ 1,\ 2,\ \ldots$

$$Y_n(x) = C\begin{cases} \cos\left(\frac{\pi nrx}{k_1}\right), & 0 < x < x_0, \\ (-1)^n \cos\left(\frac{\pi nr(l-x)}{k_2}\right), & x_0 < x < l. \end{cases}$$

We use the following notation for individual parts of the domain Ω:

$$\Omega_0 = \{(x,t):\ 0 < x < x_0,\ \ 0 < t < T\},\quad \Omega_l = \{(x,t): x_0 < x < l,\ \ 0 < t < T\}.$$

We denote by W the linear manifold of functions from the class

$$u\,(x,t) \in C\left(\overline{\Omega}\right) \cap C^{2,1}\left(\overline{\Omega_0}\right) \cap C^{2,1}\left(\overline{\Omega_l}\right)$$

that satisfy all Conditions (1)–(5).

A function $u(x,t)$ from the class $u\,(x,t) \in W$ is called a *classical solution* to Problem (1)–(5), if it satisfies Equation (1) and all Conditions (2)–(5) in the usual, continuous sense.

The following theorem holds.

Theorem 1. For any function $\varphi\,(x) \in C[0,l] \cap C^2[0,x_0] \cap C^2[x_0,l]$ and for any function $f\,(x,t) \in C\left(\overline{\Omega}\right) \cap C^{2,1}\left(\overline{\Omega_0}\right) \cap C^{2,1}\left(\overline{\Omega_l}\right)$, satisfying Boundary conditions (5) and Conjugation conditions (3)–(4), there is a unique classical solution $u\,(x,t) \in C\left(\overline{\Omega}\right) \cap C^{2,1}\left(\overline{\Omega_0}\right) \cap C^{2,1}\left(\overline{\Omega_l}\right)$ to Problem (1)–(5).

Proof. First, we construct a formal solution to Problem (1)–(5) using the method of separation of variables, in the form of a series of eigenfunctions.

Application of the method of separation of variables to the solution of the heat equation with a piecewise constant coefficient leads to a spectral problem for an ordinary differential equation with the discontinuous coefficient (9)–(11).

We have proven that the system of eigenfunctions of Problem (6)–(8) forms the Riesz basis in $L_2(0,l)$. Then, we write the solution to Problem (1)–(5) in the form:

$$u(x,t)=\sum_{n=1}^{\infty} Y_n(x)T_n(t), \tag{19}$$

where $Y_n(x)$ is determined by Formula (15). The right sides of Equation (1) and Initial condition (2) are also expanded into series:

$$f(x,t)=\sum_{n=1}^{\infty} Y_n(x)f_n(t), \tag{20}$$

$$\varphi(x)=\sum_{n=1}^{\infty} \varphi_n Y_n(x), \tag{21}$$

where $\varphi_n=\int_0^l \varphi(x)Y_n(x)dx$.

Suppose that $\varphi_n=\frac{\tilde{\varphi}_n}{n}$, and the number series $\sum_{n=1}^{\infty} |\tilde{\varphi}_n|^2$ converges. We estimate the function

$$f_n(t)=\sup_{n\in N}\sup_{0\leq t\leq T}|f_n(t)|=\sup_{n\in N}\sup_{0\leq t\leq T}\left|\int_0^l f(x,t)Y_n(x)dx\right|\leq$$
$$\leq \sup_{0\leq t\leq T} C\int_0^l |f(x,t)|\,dx\leq Cl\sup_{(x,t)\in\bar{\Omega}}|f(x,t)|\leq C_1<\infty.$$

Substituting Series (19), (20) and (22) into Problem (1)–(5), we obtain

$$\sum_{n=1}^{\infty} Y_n(x)T_n'(t)+\sum_{n=1}^{\infty}\lambda_n Y_n(x)T_n(t)=\sum_{n=1}^{\infty} Y_n(x)f_n(t), \quad 0<x<l,$$

$$\sum_{n=1}^{\infty} Y_n(x)T_n(0)=\sum_{n=1}^{\infty} Y_n(x)\varphi_n, \quad 0\leq x\leq l,$$

(we will discuss the possibility of term-by-term differentiation of a series later).

We require that the equalities written above be fulfilled term by term:

$$T_n'(t)+\lambda_n T_n(t)=f_n(t),$$

$$T_n(0)=\varphi_n.$$

The solution to the Cauchy problem for a first-order inhomogeneous differential equation has the form:

$$T_n(t) = \varphi_n e^{-\lambda_n t} + \int_0^t f_n(\tau) e^{-\lambda_n (t-\tau)} d\tau.$$

Thus, we have formally constructed a series

$$u(x,t) = \sum_{n=1}^{\infty} Y_n(x) T_n(t) = \sum_{n=1}^{\infty} \left(\varphi_n e^{-\lambda_n t} + \int_0^t f_n(\tau) e^{-\lambda_n (t-\tau)} d\tau \right) Y_n(x). \tag{22}$$

We prove the existence of a solution to Problem (1)–(5). We show that Series (22) converges uniformly, for this we estimate the n-th term as follows:

$$\sup_{(x,t)\in\bar{\Omega}} |Y_n(x) T_n(t)| \le C \sup_{0\le t\le T} |T_n(t)| =$$
$$= C \sup_{0\le t\le T} \left| \int_0^l f_n(\tau) e^{-\lambda_n (t-\tau)} d\tau \right| + C \left| \varphi_n e^{-\lambda_n t} \right| \le$$
$$\le C \sup_{0\le t\le T} |f_n(t)| \sup_{0\le t\le T} \int_0^l e^{-\lambda_n (t-\tau)} d\tau + C |\varphi_n| \le$$
$$\le C \cdot C_1 \sup_{0\le t\le T} \frac{1}{\lambda_n} \left(1 - e^{-\lambda_n t} \right) + C \frac{|\varphi_n|}{n} \le \frac{C_2}{\lambda_n} + C_3 |\tilde{\varphi}_n|^2 + \frac{C_3}{n^2}.$$

Since the majorizing series $\sum_{n=1}^{\infty} \left(|\tilde{\varphi}_n|^2 + \frac{C}{n^2} \right)$ converges, Series (22) converges uniformly on $\bar{\Omega}$ to some function $u(x,t)$.

Now we show the possibility of term-by-term differentiation of Series (22). To do this, it is sufficient to show the uniform convergence of the series

$$\sum_{n=1}^{\infty} Y_n(x) T_n'(t), \quad \sum_{n=1}^{\infty} Y_n'(x) T_n(t), \quad \sum_{n=1}^{\infty} Y_n''(x) T_n(t).$$

This provides the possibility of termwise differentiation, and since $Y_n(x)T_n(t) \in C_{x,t}^{2,1}(\Omega)$ for any $n \in N$, we obtain $u(x,t) \in C_{x,t}^{2,1}(\Omega)$. We find

$$f_n'(t) = \int_0^l f_t'(x,t) Y_n(x) dx,$$

and

$$\sup_{n\in N} \sup_{0\le t\le T} \left| f_n'(t) \right| \le \sup_{0\le t\le T} C \int_0^l \left| f_t'(x,t) \right| dx \le Cl \sup_{(x,t)\in\bar{\Omega}} \left| f_t'(x,t) \right| \le C_4 < \infty.$$

We choose and fix an arbitrary small $t_0 > 0$. We estimate the n-th term of the series

$$\sum_{n=1}^{\infty} Y_n(x)T_n^{'}(t) \sup_{0\leq x\leq l} t_0 \leq t \leq T \left|Y_n(x)T_n'(t)\right| \leq C \sup_{t_0\leq t\leq T} \left|T_n'(t)\right| =$$

$$= C \sup_{t_0\leq t\leq T} \left|-\lambda_n\varphi_n e^{-\lambda_n t} + f_n(0)e^{-\lambda_n t} + \int_0^l f_n'(\tau)e^{-\lambda_n(t-\tau)}d\tau\right| \leq$$

$$\leq C\left|\varphi_n\right|\lambda_n e^{-\lambda_n t_0} + C_1 e^{-\lambda_n t_0} + C_2 \sup_{t_0\leq t\leq T}\int_0^l e^{-\lambda_n(t-\tau)}d\tau \leq$$

$$\leq C\left|\varphi_n\right|\lambda_n e^{-\lambda_n t_0} + C_1 e^{-\lambda_n t_o} + \frac{C_2}{\lambda_n}.$$

At the same time, we have taken into account that

$$T_n'(t) = -\lambda_n\varphi_n e^{-\lambda_n t} + f_n(t) - \lambda_n\int_0^t f_n(\tau)e^{-\lambda_n(t-\tau)}d\tau =$$

$$= -\lambda_n\varphi_n e^{-\lambda_n t} + f_n(0)e^{-\lambda_n t} + \int_0^t f_n'(\tau)e^{-\lambda_n(t-\tau)}d\tau.$$

Thus, the series $\sum_{n=1}^{\infty} Y_n(x)T_n'(t)$ is majorized by a numerical convergent series, therefore the series $\sum_{n=1}^{\infty} Y_n(x)T_n'(t)$ converges uniformly on the set $(0 \leq x \leq l,\ t_0 \leq t \leq T)$, $t_0 > 0$.

Due to its arbitrariness of t_0, this means the existence of

$$u_t(x,t) = \sum_{n=1}^{\infty} Y_n(x)T_n'(t) \in C(\bar{\Omega}).$$

Now we consider the following series in order to estimate its n-th term:

$$\sum_{n=1}^{\infty} Y_n''(x)T_n(t).$$

Using the equality $Y_n''(x) = \lambda_n Y_n(x)$, we obtain $|Y_n''(x)| = |\lambda_n Y_n(x)| \leq C\lambda_n$. Then

$$\sup_{0\leq x\leq l}\left|Y_n''(x)T_n(t)\right| \leq C\lambda_n\left|T_n(t)\right| = C\left|\lambda_n\varphi_n e^{-\lambda_n t} + \lambda_n\int_0^l f_n(\tau)e^{-\lambda_n(t-\tau)}d\tau\right| \leq$$

$$\leq C\left|\varphi_n\right|\lambda_n e^{-\lambda_n t} + C\left|\lambda_n\int_0^t f_n(\tau)e^{-\lambda_n(t-\tau)}d\tau\right| \leq$$

$$\leq C\left|\varphi_n\right|\lambda_n e^{-\lambda_n t} + C\left|f_n(t) - f_n(0)e^{-\lambda_n t} - \int_0^t f_n'(\tau)e^{-\lambda_n(t-\tau)}d\tau\right| \leq$$

$$\leq C\left|\varphi_n\right|\lambda_n e^{-\lambda_n t} + C\left|f_n(t)\right| + C\cdot C_1 e^{-\lambda_n t} + \frac{C_2}{\lambda_n}.$$

From the smoothness of the function $f(x,t)$, required when constructing a formal solution, it follows that $|f_n(t)| = \frac{|\tilde{f}_n(t)|}{n}$, and moreover, the series $\sum_{n=1}^{\infty} \left|\tilde{f}_n(t)\right|^2$ converges for any fixed $t \in [0,T]$. Therefore, the following inequality is true:

$$\sup_{0\le x\le l} \left|Y_n''(x)T_n(t)\right| \le C\lambda_n |T_n(t)| \le C|\varphi_n|\lambda_n e^{-\lambda_n t} + C\left|\tilde{f}_n(t)\right|^2 + \frac{C_1}{n^2} + C_3 e^{-\lambda_n t} + \frac{C_2}{\lambda_n}.$$

So, the series $\sum_{n=1}^{\infty} Y_n''(x)T_n(t)$ converges uniformly, hence there exists

$$u_{xx}(x,t) = \sum_{n=1}^{\infty} Y_n''(x)T_n(t).$$

Since $u_{xx}(x,t) = \frac{1}{k_i^2}(u_t(x,t) - f(x,t))$, then from the conditions $u_t(x,t) \in C(\bar{\Omega})$ and $f(x,t) \in C(\bar{\Omega})$ it follows that $u_{xx}(x,t) \in C(\bar{\Omega})$. Thus, we have proven the existence of the solution from the class

$$u(x,t) \in C(\bar{\Omega}) \cap C_{x,t}^{2,1}(\bar{\Omega}_0) \cap C_{x,t}^{2,1}(\bar{\Omega}_l).$$

Now we prove the uniqueness. Let $\tilde{u}(x,t)$ and $\hat{u}(x,t)$ be solutions to Problem (1)–(5). Then their difference $u(x,t) = \tilde{u}(x,t) - \hat{u}(x,t)$ satisfies the homogeneous heat conduction equation with homogeneous initial and boundary conditions and conjugation conditions. Then from the representation of solution (22) it follows that $u(x,t) = 0$, since $\varphi(x) = 0$ and $f(x,t) = 0$. Thus, Theorem 1 is proven.

Similar theorems can be proven for Initial boundary value problem (1)–(4) with Boundary conditions (6), (7) and (8).

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Қойлышов Ү.Қ., Садыбеков М.А. КОЭФФИЦИЕНТІ ҮЗІЛІСТІ ЖЫЛУӨТКІЗГІШТІК ТЕҢДЕУ ҮШІН ШЕКАРАЛЫҚ ЕСЕПТЕРДІ ФУРЬЕ ӘДІСІМЕН ШЕШУ

Бұл мақалада шенелген аралықта коэффициенті бөлікті-тұрақты жылуөткізгіштік теңдеу үшін кейбір бастапқы шекаралық есептер қарастырылады. Айнымалыларды ажырату әдісін қолдана отырып, қойылған мәселе спектрлік есепке келтіріледі және алынған спектрлік есептің меншікті мәндері мен меншікті функциялары табылады. Меншікті функциялар жүйесінің Рисс базисін құрайтыны көрсетілді. Әрі қарай, қойылған бастапқы-шекаралық есептердің шешімдерінің бар және жалғыз екендігі туралы теореманы дәлелдейміз.

Кілттік сөздер. Жылуөткізгіштік теңдеу, үзілісті коэффициенттер, меншікті мәндер, меншікті функциялар, айнымалыларды ажырату әдісі.

Койлышов У.К., Садыбеков М.А. РЕШЕНИЕ КРАЕВЫХ ЗАДАЧ ДЛЯ УРАВНЕНИЯ ТЕПЛОПРОВОДНОСТИ С КУСОЧНО-ПОСТОЯННЫМ КОЭФФИЦИЕНТОМ МЕТОДОМ ФУРЬЕ

В данной статье рассматриваются некоторые начально-краевые задачи для уравнения теплопроводности в ограниченном отрезке с кусочно-постоянным коэффициентом. Методом разделения переменных, поставленная задача сведена к спектральной задаче и найдены собственные значения и собственные функции полученной спектральной задачи. Показана, что система собственных функции образует базис Рисса. Далее доказана теорема существование и единственности решения поставленных начально-краевых задач.

Ключевые слова. Уравнение теплопроводности, разрывные коэффициенты, собственные значения, собственные функции, метод разделения переменных.