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Criteria for the uniqueness of a solution for some differential-operator equation

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Abstract. In this article, the symmetric operator L, corresponding to the boundary value problem is represented as the difference of two commuting operators A and B. The uniqueness of the solution is guaranteed if the spectra of the operators A and B do not intersect and the domain of the operator B is given by non-degenerate boundary conditions. In contrast to the existing papers, the criterion for the uniqueness of the boundary value problem formulated in this paper is satisfied even when the system of root functions of the operator B does not form a basis in the corresponding space. At the same time, only the closedness of the linear operator A is required.

Keywords. symmetric operator part, Sturm–Liouville equation, non-degenerate boundary conditions, uniqueness of solution, operator eigenvalues, complete orthogonal systems, operator spectrum.

1 Introduction

In this paper, we consider a differential-operator equation of the form

$$-\frac{\partial^2 u}{\partial t^2} + q(t)u = Au + f(t), \quad 0 < t < T < \infty \tag{1}$$

with non-degenerate boundary conditions in time

$$\Gamma_i(u) = a_{i1}u(0) + a_{i2}u'(0) + a_{i3}u'(T) + a_{i4}u'(T) = 0, \quad i = 1, 2. \tag{2}$$

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It is also assumed that the operator A does not depend on t and is a closed linear operator in a separable Hilbert space H. In this paper, no other restrictions on the operator A are assumed. Recall that boundary conditions (2) are called non-degenerate if one of the following three requirements is satisfied:

1) $\begin{vmatrix} a_{14} & a_{12} \\ a_{24} & a_{22} \end{vmatrix} \neq 0,$

2) $\begin{vmatrix} a_{14} & a_{12} \\ a_{24} & a_{22} \end{vmatrix} = 0, \quad \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} + \begin{vmatrix} a_{13} & a_{12} \\ a_{23} & a_{22} \end{vmatrix} \neq 0,$

3) $\begin{vmatrix} a_{14} & a_{12} \\ a_{24} & a_{22} \end{vmatrix} = 0, \quad \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} + \begin{vmatrix} a_{13} & a_{12} \\ a_{23} & a_{22} \end{vmatrix} = 0, \quad \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \neq 0.$

Otherwise, the coefficients a_{ij} of the boundary conditions are arbitrary and can be complex numbers. The coefficient $q(t)$ of the differential expression on the left side of (1) is assumed to be an integrable complex-valued function on $[0, T]$.

The main goal of this article is to establish a criterion for the uniqueness of the solution of problem (1)–(2). There are various ways to prove uniqueness. Usually, the maximum principle [1] and its various generalizations like the Hopf [2] and Zaremba-Giraud [3] principles are effective means of proving uniqueness. For task (1)–(2), these principles may not be fulfilled. Therefore, we need a different toolkit, different from the extreme principle.

We note the work of I.V. Tikhonov [4], devoted to uniqueness theorems in linear non-local problems for abstract differential equations. I.V. Tikhonov's method uniqueness proof is based on the "quotient method" for entire functions of exponential type. [5] studied the question of the uniqueness of the solution of the heat equation with a non-local condition expressed as an integral over time on a fixed interval. They succeeded in giving a complete description of the uniqueness classes in terms of the behavior of solutions as $|x| \to \infty$. In this paper, I.V. Tikhonov's method is adapted for operators whose differential part is a second-order operator with general boundary conditions.

In this paper, the operator symmetric L corresponding to the boundary value problem (1)–(2) is represented as the difference of two commuting operators A and B. The uniqueness of the solution is guaranteed if the spectra of the operators A and B do not intersect and the domain of the operator B is given by non-degenerate boundary conditions. In contrast to the existing papers [6–8], the criterion for the uniqueness of the boundary value problem (1)–(2) formulated in this paper is satisfied even when the system of root functions of the operator B does not form a basis in the corresponding space. At the same time, only the closedness of the linear operator A is required. For example, in our case, the unbounded operator A may be non-semibounded or have an empty spectrum. Note that in the papers [6–8] the operator A was required to be semibounded, while the operator B must have a system of root functions forming a basis.

The method of proving the uniqueness of the solution of the boundary value problem (1)–(2) is based on the method of guiding functionals by M.G. Krein [9] with their subsequent estimation when the spectral parameter infinitely increases in the complex region.

2 On the spectral properties of the Sturm–Liouville operator on a segment

In this section, we consider the boundary value problem generated on the interval $(0, T)$ by the Sturm–Liouville equation

$$-w''(t) + q(t)w(t) = \mu w(t), \quad 0 < t < T < \infty \tag{3}$$

and the following two boundary conditions

$$\Gamma_i(w) = a_{i1}w(0) + a_{i2}w'(0) + a_{i3}w(T) + a_{i4}w'(T) = 0, \quad i = 1, 2, \tag{4}$$

where $q(t)$ is an integrable complex-valued function, a_{ik} are arbitrary complex numbers.

Further, the fundamental system of solutions to Equation (3) determined by the initial data $c(\mu, 0) = s'(\mu, 0) = 1$, $c'(\mu, 0) = s(\mu, 0) = 0$, will be denoted by $c(\mu, t), s(\mu, t)$.

We introduce the characteristic function by the formula

$$\chi(\mu) = J_{12} + J_{34} + J_{13}s(\mu, T) + J_{14}s'(\mu, T) + J_{32}c(\mu, T) + J_{42}c'(\mu, T), \tag{5}$$

where $J_{ij} = a_{1i}a_{2j} - a_{2i}a_{1j}$ is the determinant composed of the i-th and j-th columns of the coefficient matrix of the boundary conditions

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}.$$

We denote by $w_1(t, \mu), w_2(t, \mu)$ the solutions of the homogeneous equation (3) and the boundary conditions

$$\Gamma_1(w_1) = \Gamma_2(w_2) = 0,$$

$$\Gamma_1(w_2) = \Gamma_2(w_1) = \chi(\mu).$$

The eigenvalue μ_n of the boundary value problem (3)–(4) is called an eigenvalue of multiplicity p if μ_n is a multiplicity p root of the function $\chi(\mu)$.

Because

$$\frac{\partial^k}{\partial \mu^k}\Gamma_i(w_j) = \Gamma_i\left(\frac{\partial^k}{\partial \mu^k} w_j\right),$$

then the functions

$$w_{i,k}(t) = \frac{1}{k!}\frac{\partial^k}{\partial \mu^k} w_j(t, \mu) \tag{6}$$

for $\mu = \mu_n$ satisfy both boundary conditions (4) if $0 \le k \le p-1$. The functions $w_{i,0}(t), \dots, w_{i,p-1}(t)\,(i=1,2)$ form a chain in which the first nonzero function $w_{i,l_i}(t)$ is its eigenfunctions, and the following are its associated functions. Differentiating equation (3) k times with respect to μ, we conclude that the eigenfunction and associated functions of the chain satisfy the equations

$$-w''_{i,k}(t) + q(t)w_{i,k}(t) = \mu_n w_{i,k}(t) + w_{i,k-1}(t), \quad 0 < t < T$$

and the boundary conditions (4). To avoid misunderstandings, we emphasize that both chains, $w_{1,0}(t), \dots, w_{1,p-1}(t)$ and $w_{2,0}(t), \dots, w_{2,p-1}(t)$, may consist of the same functions. For us, it is only essential that, in addition to eigenfunctions and associated functions, the chains can include only functions that are identically equal to zero.

Denote by B the operator given by the differential expression $Bw(t) = -w''(t) + q(t)w(t)$ and the domain given by the boundary conditions (4).

Let $\sigma(B)$ be the spectrum, that is, the set of all eigenvalues μ_n of the boundary value problem (3)–(4), p_n– their multiplicity. According to the previous function

$$\frac{1}{k!}\frac{\partial^k}{\partial \mu^k} w_j(t,\mu)\Big|_{\mu=\mu_n}, \quad 0 \le k \le p_n - 1, \quad \mu_n \in \sigma(B), \quad i = 1,2$$

are either identically equal to zero or are eigenfunctions or associated functions of this boundary value problem.

The operator has a dense domain in the space $L_2(0,T)$. Therefore, there is a unique adjoint operator B^*. The action of the adjoint operator B^* is given by the formula

$$B^*\tau(t) = -\tau''(t) + \overline{q}(t)\tau(t), \tag{7}$$

where $\overline{z}$ means the conjugate of the complex number z.

Let the domain of the operator be given by the boundary forms $V_1(\cdot)$ and $V_2(\cdot)$, i.e.

$$D(B^*) = \left\{\tau \in W_2^2[0,T]: \ V_1(\tau) = 0, \ V_2(\tau) = 0\right\}. \tag{8}$$

Here, the boundary forms of the adjoint problem have the following form

$$V_i(\tau) = a^*_{i1}\tau(0) + a^*_{i2}\tau'(0) + a^*_{i3}\tau(T) + a^*_{i4}\tau'(T) = 0, \quad i = 1,2. \tag{9}$$

We introduce a fundamental system of solutions $\{R_1(t,\overline{\mu}), R_2(t,\overline{\mu})\}$ of a homogeneous adjoint equation

$$-R''_s(t,\overline{\mu}) + \overline{q}(t)R_s(t,\overline{\mu}) = \overline{\mu}\,R_s(t,\overline{\mu}), \quad 0 < t < T \tag{10}$$

satisfying the Cauchy condition at zero

$$R_1(0,\overline{\mu}) = 1, \quad R_2(0,\overline{\mu}) = 0, \quad R'_1(0,\overline{\mu}) = 0, \quad R'_2(0,\overline{\mu}) = 1. \tag{11}$$

Note that all solutions $\{R_i(t,\overline{\mu}),\, i=1,2\}$ are entire functions of $\overline{\mu}$. Denote by $\chi^*(\overline{\mu})$ the characteristic determinant given by the formula

$$\chi^*(\overline{\mu}) = \det(V_\nu(R_j)).$$

The zeros, taking into account their multiplicities of the characteristic determinant $\chi^*(\overline{\mu})$, represent the eigenvalues of the adjoint operator B^*.

We also introduce $\tau_i(t,\overline{\mu})$ for $i=1,2$ solutions of the homogeneous adjoint equation (10) with heterogeneous conditions

$$V_j(\tau_s) = \delta_{j,s}\cdot\chi^*(\overline{\mu}), \quad j=1,2, \tag{12}$$

where $\delta_{j,s}$ is the Kronecker symbol.

Let μ_0 be the zero of the characteristic determinant $\chi(\mu)$ and its multiplicity equals m_0. Then for any $s=1,2$ in the ordered row

$$\left[\tau_s(t,\overline{\mu_0}), \frac{1}{1!}\frac{\partial}{\partial\overline{\mu}}\tau_s(t,\overline{\mu_0}),\ldots,\frac{1}{(m_o-1)!}\frac{\partial^{m_0-1}}{\partial\overline{\mu}^{m_0-1}}\tau_s(t,\overline{\mu_0})\right] \tag{13}$$

the first non-zero function represents the eigenfunction of the operator B^*, and the subsequent members of the row give the chain of associated functions generated by it.

In what follows, the eigenvalues of the operator B^* will be denoted by $\overline{\mu}_\nu$, $\nu\geq 1$, and the corresponding eigenvalues and associated functions by $\tau_\nu(t)$, $\nu \geq 1$.

In work [10], the following assertion was proved.

Theorem 1. [10] *Let the domain of the operator B be given by non-degenerate boundary conditions. Then the domain of definition of the adjoint operator B^* is also given by non-degenerate boundary conditions.*

We also need the following assertion [10].

Theorem 2. [10] *Let the operator B be generated by non-degenerate boundary conditions. Then the system of eigenfunctions and associated functions of the operator B is a complete system in the space $L_2(0,T)$.*

Applying Theorems 2.1 and 2.2 to the adjoint operator B^*, we can formulate the assertion.

Theorem 3. *Let one of the requirements 1),2),3) be satisfied. Then the system of eigenfunctions and associated functions of the operator B^* is complete in the space $L_2(0,T)$.*

For further purposes, it is convenient for us to reformulate Lemmas 1.3.1 and 1.3.2, as well as Corollaries 1 and 2 from the monograph [10] in the following form.

Lemma 1. [10] *For all functions $f(t) \in L_1(0,T)$ the following equalities hold:*

$$\lim_{|\rho|\to\infty} e^{-|Im\rho T|}\int_0^T f(t)\cos\rho t dt = \lim_{|\rho|\to\infty} e^{-|Im\rho T|}\int_0^T f(t)\sin\rho t dt = 0. \tag{14}$$

We denote by $R_{1T}(t,\lambda), R_{2T}(t,\lambda)$, $(\lambda = \overline{\mu})$ the solution of Equation (11) with the initial data

$$R_{1T}(T,\lambda) = R'_{2T}(T,\lambda) = 1,$$
$$R'_{1T}(T,\lambda) = R_{2T}(T,\lambda) = 0.$$

Corollary 1. [10] *For all functions $f(x) \in L_1(0,T)$ the following equalities hold:*

$$\lim_{|\rho|\to\infty} e^{-|Im\rho T|}\int_0^T f(t)R_1(t,\rho^2)dt = \lim_{|\rho|\to\infty} e^{-|Im\rho T|}\int_0^T f(t)R_{1T}(t,\rho^2)dt = 0.$$

$$\lim_{|\rho|\to\infty} e^{-|Im\rho T|}\int_0^T f(t)\rho R_2(t,\rho^2)dt = \lim_{|\rho|\to\infty} e^{-|Im\rho T|}\int_0^T f(t)\rho R_{2T}(t,\rho^2)dt = 0. \tag{15}$$

Corollary 2. [10] *For all functions $f(t) \in L_1(0,T)$ the following equalities hold:*

$$\lim_{|\rho|\to\infty} e^{-|Im\,\rho T|}\int_0^T f(t)\tau_i(t,\rho^2)dt = 0, \quad i = 1,2. \tag{16}$$

Lemma 2. [10] *If the boundary conditions in the boundary value problem (3)–(4) are non-degenerate, then there exists a constant $C > 0$ and a sequence of infinitely expanding contours K_n on the ρ–planes on which the following inequalities hold:*

$$|\chi^*(\overline{\rho}^2)| > |\rho|^{-1}\, C\, e^{|Im\,\overline{\rho}T|}, \quad \overline{\rho} \in K_n. \tag{17}$$

3 Main result and its proof

In this section, we formulate and prove a criterion for the uniqueness of the solution of the boundary value problem (1)–(2). In accordance with the notation of Section 1, the boundary value problem (1)–(2) can be written in the operator form

$$Bu(t) = Au(t) + f(t), \quad t \in (0,T). \tag{18}$$

Here, the operator B acts on the variable t and its spectral properties are given in Section 1. The operator A is a closed linear operator in a separable Hilbert space H and does not depend on t.

Theorem 4. *Let the matrix of boundary coefficients*

$$J = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$$

rank 2 is subject to the requirement: at least one of the numbers J_{42}, $J_{14} + J_{32}$ and J_{13} is different from zero, where $J_{kj} = a_{1k}a_{2j} - a_{2k}a_{1j}$ and assume that the operator A is a closed linear operator in a separable Hilbert space H and does not depend on t. Then the homogeneous operator equation

$$Bu = Au$$

has only the trivial solution $u \in D(B) \cap D(A)$ if and only if

$$\sigma(B) \cap \sigma(A) = \varnothing,$$

where $\sigma(B)$ and $\sigma(A)$ are spectra of the operators B and A, respectively.

Proof. Proof of necessity. Let μ_n be some eigenvalue of the operator B (with own function $w_n(t)$) is also an eigenvalue of the operator A. That is, there is an eigenvalue λ_s of the operator A, which is the same as the eigenvalue μ_n of the operator B. Suppose that the eigenvalue λ_s of the operator A corresponds to the eigenvalue v_s. Then the function $u(t) = w_n(t) \cdot v_s$ will be a non-trivial solution of the homogeneous equation (17). The necessity of the requirements of Theorem 1 is proved.

Proof of sufficiency. Let none of $\{\mu_n,\, n \geq 1\}$ eigenvalues of the operator B is not an eigenvalue of the operator A. In other words, if λ_s is an arbitrary eigenvalue of the operator A, then $\chi(\lambda_s) \neq 0$.

We show that the solution $u(t)$ of the homogeneous operator equation (18) is identically equal to zero in the space $L_2((0,T);H)$.

To do this, we introduce functions with values in the Hilbert space H for $j = 1, 2$

$$F_j(\overline{\mu}) = \int_0^T \overline{\tau_j(t,\overline{\mu})}\, u(t)dt. \tag{19}$$

Functions of the type $F_1(\overline{\mu})$ and $F_2(\overline{\mu})$ were introduced by M.G. Krein in [9] and are called guiding functionals.

According to the Lagrange formula [11] the functions $AF_j(\overline{\mu})$, for $j = 1, 2$, can be rewritten as:

$$AF_j(\overline{\mu}) = \int_0^T \overline{\tau_j(t,\overline{\mu})} \cdot Au(t)dt = \int_0^T \overline{\tau_j(t,\overline{\mu})} \cdot Bu(t)dt =$$

$$\int_0^T u(t)\ \left(\overline{-\tau_j''(t,\overline{\mu}) + \overline{q(t)}\tau_j(t,\overline{\mu})}\right) dt + \Gamma_1(u)\,\overline{V_4(\tau_j)} + \Gamma_2(u)\,\overline{V_3(\tau_j)} +$$

$$\Gamma_3(u)\,\overline{V_2(\tau_j)} + \Gamma_4(u)\,\overline{V_1(\tau_j)}, \tag{20}$$

where $\Gamma_3(\cdot)$ and $\Gamma_4(\cdot)$ are linear forms [12], complementary linear forms $\Gamma_1(\cdot)$, $\Gamma_2(\cdot)$, up to a Dirichlet system of order 4. In [12], it is stated that the system of linear forms $\{V_1(\cdot), V_2(\cdot), V_3(\cdot), V_4(\cdot)\}$ is determined by $\{\Gamma_1(\cdot), \Gamma_2(\cdot), \Gamma_3(\cdot), \Gamma_4(\cdot)\}$ uniquely and forms a Dirichlet system of order 4.

Since the $\Gamma_1(u) = \Gamma_2(u) = 0$, $V_1(\tau_2) = V_2(\tau_1) = 0, V_1(\tau_1) = V_2(\tau_2) = \chi^*(\overline{\mu})$, then relation (20) takes the form

$$AF_j(\overline{\mu}) = \mu F_j(\overline{\mu}) + \Gamma_{5-j}(u) \cdot \overline{\chi^*(\overline{\mu})}, \quad j = 1, 2. \tag{21}$$

If μ_0 is an arbitrary zero of multiplicity m_0 of the characteristic function $\chi(\mu)$, then the last relation (21) implies the equalities

$$\begin{gathered} AF_j(\overline{\mu_0}) = \mu_0\, F_j(\overline{\mu_0}), \\ A\frac{dF_j(\overline{\mu_0})}{d\overline{\mu}} = \mu_0\, \frac{dF_j(\overline{\mu_0})}{d\overline{\mu}} + F_j(\overline{\mu_0}), \\ \dots \\ A\frac{d^{m_0-1}}{d\overline{\mu}^{m_0-1}}F_j(\overline{\mu_0}) = \mu_0\, \frac{d^{m_0-1}F_j(\overline{\mu_0})}{d\overline{\mu}^{m_0-1}} + \frac{d^{m_0-2}F_j(\overline{\mu_0})}{d\overline{\mu}^{m_0-2}}. \end{gathered} \tag{22}$$

Since $\mu_0 \overline{\in} \sigma(A)$, the relations (22) imply the equalities

$$\frac{d^s F_j(\overline{\mu_0})}{d\overline{\mu}^s} \equiv 0 \quad \text{at} \quad s = 0, 1, \dots, m_0 - 1. \tag{23}$$

Then for $j = 1, 2$ the relations $\dfrac{F_j(\overline{\mu})}{\chi^*(\overline{\mu})}$ are entire functions of $\overline{\mu}$, since at the point $\overline{\mu} = \overline{\mu}_0$ the relations $\dfrac{F_j(\overline{\mu})}{\chi^*(\overline{\mu})}$ have a removable singularity.

Now we pass to the second step of the proof. Since H is a separable Hilbert space, then there is a counting system of elements $v_1, v_2, \dots,$ whose linear span is dense in H.

We obtain the dot product of the function $F_j(\overline{\mu})$ and the element v_k and we denote them by

$$G_j^k(\overline{\mu}) \equiv \langle F_j(\overline{\mu}),\, v_k \rangle_H, \quad j = 1, 2, \quad k = 1, 2, \dots, \tag{24}$$

where $\langle \cdot, \cdot \rangle_H$ is the dot product in Hilbert space H.

Multiplicity of zeros of the functional $G_j^k(\mu)$ not less than multiplicities of zeros of functions $F_j(\mu)$. Therefore, the relationship

$$Q_j^k(\overline{\mu}) \equiv \frac{G_j^k(\overline{\mu})}{\chi^*(\overline{\mu})} \tag{25}$$

define entire functions from $\overline{\mu}$.

Further analysis of the entire functions $Q_j^k(\overline{\mu})$ is based on the technique of estimating the order of growth and the type of entire functions. Note that the entire function $Q_j^k(\overline{\mu})$ does not depend on the choice of the fundamental system of solutions of the homogeneous equation (11).

According to Lemma 2.7, there exists a constant M and a sequence of infinitely expanding closed contours Ω_n such that $\rho \in \Omega_n$ and

$$|Q_j^k(\overline{\rho}^2)| \leq M\,|\rho|\,|G_j^k(\overline{\rho}^2)|\,e^{-|Im\,\rho|T}$$

for all admissible index values k and j.

The last estimate and Corollary 2.6 imply the limit equalities

$$\lim_{n\to\infty} \max_{\rho\in\Omega_n} |\frac{1}{\rho} Q_j^k(\overline{\rho}^2)| = 0, \quad j = 1, 2, \quad \forall \geq 1.$$

It follows that the entire functions $Q_j^k(\overline{\rho}^2)$ at $\rho \to \infty$ grow slower than the first degree $|\rho|$. Then, by the Liouville theorem, we get that

$$Q_j^k(\overline{\rho}^2) \equiv f_j^k, \quad j = 1, 2, \quad \forall k \geq 1,$$

where f_j^k is some constants.

That's why

$$G_j^k(\overline{\mu}^2) = f_j^k \cdot \chi^*(\overline{\mu}), \quad \forall \mu \in \mathbb{C}. \tag{26}$$

Since

$$\tau_1(t,\overline{\mu}) = \begin{vmatrix} R_1(t,\overline{\mu}) & R_2(t,\overline{\mu}) \\ V_2(R_1) & V_2(R_2) \end{vmatrix}, \quad \tau_2(t,\overline{\mu}) = \begin{vmatrix} V_1(R_1) & V_1(R_2) \\ R_1(t,\overline{\mu}) & R_2(t,\overline{\mu}) \end{vmatrix},$$

then relations (26) take the form

$$\begin{cases} V_2(R_2)\alpha_1^k(\overline{\mu}) - V_2(R_1)\alpha_2^k(\overline{\mu}) = f_1^k\,\chi^*(\overline{\mu}), \\ -V_1(R_2)\alpha_1^k(\overline{\mu}) - V_1(R_1)\alpha_2^k(\overline{\mu}) = f_2^k\,\chi^*(\overline{\mu}), \end{cases} \tag{27}$$

where

$$\alpha_j^k(\overline{\mu}) = \langle \int_0^T \overline{R_j(t,\overline{\mu})}\,u(t)dt,\, v_k\rangle_H, \quad j = 1, 2.$$

Consequently, from the system (27) we obtain the equality

$$\{V_1(R_1)V_2(R_2) - V_1(R_2)V_2(R_1)\}\,\alpha_2^k(\overline{\mu}) = \chi^*(\overline{\mu})\,\{f_1^k V_1(R_2) + f_2^k V_2(R_2)\}.$$

Since

$$\chi^*(\overline{\mu}) = V_1(R_1)\,V_2(R_2) - V_1(R_2)\,V_2(R_1),$$

then

$$\alpha_2^k(\overline{\mu}) = f_1^k\, V_1(R_2) + f_2^k\, V_2(R_2).$$

We recall the definition (9) of the boundary forms V_1 and V_2 of the adjoint problem. Then we have the relation

$$\alpha_2^k(\overline{\mu}) = (f_1^k\, a_{12}^* + f_2^k\, a_{22}^*) + (f_1^k\, a_{13}^* + f_2^k\, a_{23}^*)\, R_2(T,\overline{\mu}) + \\ (f_1^k\, a_{14}^* + f_2^k\, a_{24}^*)\, R_2'(T,\overline{\mu}). \tag{28}$$

We consider the relation (28) for $\mu \to \pm\infty$. Recall [10] that the asymptotic formulas

$$R_2(T,\overline{\mu}) = \frac{\sin\sqrt{\overline{\mu}}T}{\sqrt{\overline{\mu}}} + \int_0^T K_1(\xi)\frac{\sin\sqrt{\overline{\mu}}\xi}{\sqrt{\overline{\mu}}}d\xi,$$

$$R_2'(T,\overline{\mu}) = \cos\sqrt{\overline{\mu}}T + \frac{1}{2}\int_0^T \overline{q(\xi)}d\xi \cdot \frac{\sin\sqrt{\overline{\mu}}T}{\sqrt{\overline{\mu}}} + \int_0^T K_2(\xi)\frac{\sin\sqrt{\overline{\mu}}\xi}{\sqrt{\overline{\mu}}}d\xi, \tag{29}$$

where $K_1(\cdot)$ and $K_1(\cdot)$ are some integrated functions.

Using Corollary 2.5 and the formulas (29), the relation (28) can be represented in the following form

$$\frac{\delta(\sqrt{\overline{\mu}})}{\sqrt{\overline{\mu}}} \equiv (f_1^k\, a_{12}^* + f_2^k\, a_{22}^*) + (f_1^k\, a_{13}^* + f_2^k\, a_{23}^*)\left(\frac{\sin\sqrt{\overline{\mu}}T}{\sqrt{\overline{\mu}}} + \frac{\varepsilon_1(\sqrt{\overline{\mu}}T)}{\sqrt{\overline{\mu}}}\right) + \\ (f_1^k\, a_{14}^* + f_2^k\, a_{24}^*)\left(\cos\sqrt{\overline{\mu}}T + \varepsilon_2(\sqrt{\overline{\mu}})\right), \tag{30}$$

where the functions δ, ε_1 and ε_2 tend to zero at $\mu \to \infty$.

This is possible if and only if the numbers

$$(f_1^k\, a_{12}^* + f_2^k\, a_{22}^*),\ (f_1^k\, a_{13}^* + f_2^k\, a_{23}^*),\ (f_1^k\, a_{14}^* + f_2^k\, a_{24}^*).$$

are identically equal to zero.

Consequently, from (28) it follows that $\alpha_2^k(\overline{\mu}) \equiv 0$.

From the last equality it follows that

$$\langle \int_0^T \overline{R_2(t,\overline{\mu})}\, u(t)dt,\ v_k\rangle_H = 0, \quad \forall k \geq 1.$$

Since the linear span of the system $\{v_k,\ \forall k \geq 1\}$ is dense in H, then we obtain the relation

$$\int_0^T \overline{R_2(t,\overline{\mu})}\, u(t)dt = 0, \quad \forall \mu \in \mathbb{C}.$$

The required equality follows from the last relation

$$u(t) = 0, \quad \forall t \in (0,T).$$

In order to verify this, it is necessary to repeat the arguments from ([10], page 42). Thus, Theorem 3.1 is proved. □

We give the following examples as applied to Theorem 3.1 for some operators A in equations (1).

Example 1. *Let $\Omega \subseteq \mathbb{R}^N$ be some bounded area with a smooth boundary $\partial\Omega$. In work [23] operator A is defined as $A(x, D) = \sum\limits_{|\alpha|\leq 2l} a_\alpha(x) D^\alpha$, which is a formally self-adjoint elliptic differential operator of order $2l$ with sufficiently smooth coefficients $a_\alpha(x)$, where $\alpha = (\alpha_1, \dots, \alpha_N)$ is a multi-index and $D = (D_1, \dots, D_N)$, $D_j = \frac{\partial}{\partial x_j}$.*

The domain of the operator A is given by the following boundary conditions in x:

$$B_j u(x,t) = \sum_{|\alpha|\leqslant m_j} b_{\alpha,j}(x) D^\alpha u(x,t) = 0, \quad 0 \leq m_j \leq 2l-1, \quad j = 1, 2, \dots, l, \quad x \in \partial\Omega. \tag{31}$$

where the coefficients $b_{\alpha,j}(x)$ are sufficiently smooth given functions. The following conclusion follows from Theorem 3.1.

Conclusion 1. *Let the operator B satisfy the requirements of Theorem 3.1. Then the homogeneous operator equation*

$$-\frac{\partial^2 u}{\partial t^2} + q(t)u = Au, \quad x \in \Omega, \quad t \in (0,T) \tag{32}$$

with initial-boundary conditions (2) and (31) has only a trivial solution $u \in D(B) \cap D(A)$ if and only if

$$\sigma(B) \cap \sigma(A) = \varnothing,$$

where $\sigma(B)$ and $\sigma(A)$ are spectra of the operators B and A, respectively.

This strengthens the main result of [23], since Conclusion 3.3 is valid for the operator B with non-degenerate boundary conditions. At the same time, in [23], the operator B was required to have the boundary conditions be strongly regular in the sense of Birkhoff [11]. The class of non-degenerate boundary conditions is wider than the class of strongly Birkhoff-regular boundary conditions.

Example 2. *The operator A is generated by the standard wave equation $Av(\cdot) = v_{xx}(\cdot) - v_{yy}(\cdot)$ in the two-dimensional region Ω bounded by the segment $OB : 0 \leq x \leq 1$ axes $y = 0$ and characteristics $OC : x + y = 0$, $BC : x - y = 1$.*

The domain of the operator A is given by the following boundary conditions with a shift along (x, y):

$$u(\theta, 0; t) = 0, \quad 0 \leq \theta \leq 1,$$
$$u\left(\frac{\theta}{2}, -\frac{\theta}{2}; t\right) = a\, u\left(\frac{\theta+1}{2}, \frac{\theta-1}{2}; t\right), \quad 0 \leq \theta \leq \frac{1}{2}, \quad 0 < t < T. \tag{33}$$

The following conclusion follows from Theorem 3.1.

Conclusion 2. *Let the conditions of Theorem 3.1 be satisfied for the operator B. Then the following homogeneous operator equation*

$$-\frac{\partial^2 u}{\partial t^2} + q(t)u = u_{xx}(x,y;t) - u_{yy}(x,y;t), \quad x \in \Omega, \quad t \in (0,T) \tag{34}$$

with initial-boundary conditions (2) and (33) has only a trivial solution $u \in D(B) \cap D(A)$ if and only if the spectra of these operators B and A do not intersect.

This strengthens the main result of the work [24].

Example 3. *The operator A is generated by the Tricomi equation. Let $\Omega \in \mathbb{R}^2$ be a finite domain bounded for $y > 0$ by the Lyapunov curve σ, ending in a neighborhood of points $O(0,0)$ and $B(1,0)$ small arcs of the "normal curve" σ_0, and for $y < 0$ by the characteristics $OC : x - \frac{2}{3}(-y)^{3/2} = 0$, $BC : x + \frac{2}{3}(-y)^{3/2} = 1$ equations*

$$Av(\cdot) = yv_{xx}(\cdot) + v_{yy}(\cdot).$$

The boundary conditions for the Tricomi operator are given by the Dirichlet condition on the elliptic part and the fractional derivative traces of the solution along the characteristics:

$$u(x,y;t)\big|_{\sigma_0} = 0, \quad \sigma_0 : \left(x - \frac{1}{2}\right)^2 + \frac{4}{9}y^3 = \frac{1}{4}, \tag{35}$$

$$x^{5/6} D_{0+}^{1/6}\left(u\left(\chi_0(x)\right)x^{-2/3}\right) + (1-x)^{5/6} D_{1-}^{1/6}\left(u\left(\chi_1(x)\right)(1-x)^{-2/3}\right) = 0, \tag{36}$$

where

$$u\left(\chi_0(x)\right) = u\left(x, -\left[\frac{3x}{2}\right]^{2/3}\right), \ 0 \le x \le \frac{1}{2},$$

$$u\left(\chi_1(x)\right) = u\left(x, -\left[\frac{3(1-x)}{2}\right]^{2/3}\right), \quad \frac{1}{2} \le x \le 1.$$

Application of Theorem 3.1 leads to the following conclusion.

Conclusion 3. *Let the conditions of Theorem 3.1 be satisfied for the operator B. Then the following homogeneous operator equation*

$$-\frac{\partial^2 u}{\partial t^2} + q(t)u = yu_{xx}(x,y;t) + u_{yy}(x,y;t), \quad x \in \Omega, \quad t \in (0,T) \tag{37}$$

with the initial-boundary conditions (2), (34), and (35) has only a trivial solution $u \in D(B) \cap D(A)$ if and only if the spectra of these operators B and A do not intersect.

This strengthens the main result of the work [25].

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Қанғожин Б. Е., Қошанов Б. Д. КЕЙБІР ДИФФЕРЕНЦИАЛДЫ-ОПЕРАТОРЛЫҚ ТЕҢДЕУ ҮШІН ШЕШІМІНІҢ БІРЕГЕЙЛІК КРИТЕРИЙІ.

Бұл мақалада екі коммутацияланатын A және B операторлардың айырымы түрінде берілген шекаралық есепке сәйкес келетін симметриялы L операторы қарастырылады. Егер A операторының спектрі B операторының спектрімен қиылыспайтын болса және B операторының анықталу облысы азғындалмаған шекаралық шарттармен берілсе, онда $Lu = (B - A)u = 0$ операторлық теңдеу шешімінің бірегейлігіне кепілдік беріледі. Бұл жұмыста тұжырымдалған шекаралық есептің бірегейлік критерийі қолданыстағы еңбектердің нәтижелерінен айырмашылығы B операторының түбірлік функциялар жүйесі сәйкес кеңістікте базис құрмаған кезде де қанағаттандырылады. Бұл ретте A сызықты операторының тұйықтығы ғана қажет.

Түйін сөздер: симметриялы операторлар, Штурм–Лиувилл теңдеуі, азғындалмаған шекаралық шарттар, шешімнің бірегейлігі, оператордың меншікті мәндері, толық ортогональды жүйелер, оператордың спектрі.

Кангужин Б. Е., Кошанов Б. Д. КРИТЕРИЙ ЕДИНСТВЕННОСТИ РЕШЕНИЯ ОДНОГО ДИФФЕРЕНЦИАЛЬНО-ОПЕРАТОРНОГО УРАВНЕНИЯ

В данной статье симметричный оператор L, соответствующий краевой задаче, представляется в виде разности двух коммутирующих операторов A и B. Единственность решения операторного уравнения $Lu = (B - A)u = 0$ гарантируется, если спектры операторов A и B не пересекаются и область определения оператора B задана невырожденными граничными условиями. В отличие от существующих результатов работ, сформулированный в данной статье критерий единственности краевой задачи выполняется даже в том случае, когда система корневых функций оператора B не образует базиса в соответствующем пространстве. При этом требуется лишь замкнутость линейного оператора A.

Ключевые слова: симметричные операторы, уравнение Штурма–Лиувилля, невырожденные граничные условия, единственность решения, собственные значения оператора, полные ортогональные системы, спектр оператора.