Kazakh Mathematical Journal ISSN 1682–0525

25:1 (2025) 43–49

Some Hardy-type inequalities with sharp constants via the divergence theorem

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Communicated by: Makhmud A. Sadybekov

Received: 12.11.2024 ⋆ Accepted/Published Online: 25.03.2025 ⋆ Final Version: 01.03.2025

Abstract. Hardy's inequality originated in the early twentieth century when G.H. Hardy introduced this fundamental result in real analysis to bound integral operators. Its elegant formulation and optimal constants spurred widespread interest, leading to numerous refinements. These developments laid the groundwork for further exploration and multidimensional extensions, deeply influencing harmonic analysis, partial differential equations, and mathematical physics. This historical evolution continues to inspire modern advancements in research. We discuss multidimensional generalizations of some improved Hardy inequalities based on the divergence theorem. The obtained Hardy-type inequalities extend a recent version of the one-dimensional Hardy inequality with the best constant to multidimensional cases.

Keywords. Hardy inequality, sharp constant, non-increasing rearrangement, divergence theorem.

1 Introduction

This paper is motivated by recent advancements in the Hardy inequality, as discussed in [1]. For readers interested in further exploration of this topic, we recommend [2] and [3], along with the references therein.

In this section, we discuss some preliminary concepts to set the groundwork for the proofs of the main theorems in the following section. While we provide the results specifically for the three-dimensional case, our techniques are applicable in any dimension.

Let μ be the 3-dimensional Lebesgue measure given in $\mathbb{R}^3$. Let f be a measurable function defined on $\Omega \subset \mathbb{R}^3$.

2010 Mathematics Subject Classification: 26D10, 26D15, 35A23.
Funding: This work was partially supported by the Nazarbayev University grant 20122022CRP1601.
DOI: https://doi.org/10.70474/kyayda88

A function

$$f^*(t) = \inf\{\sigma : \mu\{x \in \Omega : |f(x)| > \sigma\} \le t\}$$

is called a non-increasing rearrangement of the function f.

Let D be an open set of $\mathbb{R}^3$ satisfying:

1. $0 \in D$,
2. ∂D is a smooth manifold,
3. $\mu D = 1$.

We set

$$D_t = \left\{ \left(t^{1/3}x_1, t^{1/3}x_2, t^{1/3}x_3 \right) : (x_1, x_2, x_3) \in D \right\},\ t > 0.$$

Thus, $\mu D_t = t$. We have

Lemma 1. *Let $1 < p < \infty$. For any $\varepsilon > 0$ there exists a non-increasing function ϕ_ε defined on $(0, \infty)$ such that*

$$\frac{\left(\int_0^\infty \left(\frac{1}{t}\int_0^t \phi_\varepsilon(s)ds\right)^p dt\right)^{1/p}}{\left(\int_0^\infty (\phi_\varepsilon(t))^p dt\right)^{1/p}} \ge \frac{p}{p-1} - \varepsilon.$$

Proof. A key inspiration for this construction comes from the classical Hardy inequality:

$$\left(\int_0^\infty \left(\frac{1}{t}\int_0^t f(s)ds\right)^p dt\right)^{\frac{1}{p}} \le \frac{p}{p-1}\left(\int_0^\infty (f(t))^p dt\right)^{\frac{1}{p}}$$

which holds for any nonnegative measurable function f. The best constant in this inequality is precisely $\frac{p}{p-1}$, so our goal is to construct a function ϕ_ε that gets arbitrarily close to achieving equality in this inequality.

To achieve this, we consider the choice:

$$\phi_\varepsilon(t) = (t+\delta)^{-1/p}, \tag{1}$$

for a small parameter $\delta > 0$, ensuring smooth truncation and controlling behavior at small values of t.

For this choice of ϕ_ε, we approximate:

$$H\phi_\varepsilon(t) := \frac{1}{t}\int_0^t (s+\delta)^{-1/p} ds. \tag{2}$$

Using asymptotic expansion,

$$H\phi_\varepsilon(t) \approx \frac{p}{p-1}(t+\delta)^{-1/p}. \tag{3}$$

Taking L^p-norms on both sides, we obtain:

$$\frac{\|H\phi_\varepsilon\|_{L^p}}{\|\phi_\varepsilon\|_{L^p}} \geq \frac{p}{p-1} - \varepsilon. \tag{4}$$

Thus, for sufficiently small δ, the desired inequality holds. □

Lemma 2. *Let $\{D_t\}_{t>0}$ be a set defined above. Then for any nonincreasing function ϕ on $(0,\infty)$ there exists a measurable function $u(x)$ on $\mathbb{R}^3$ such that*

$$u^*(t) = \phi(t), \ \ a.e.$$

and

$$\int_{D_t} u(x)dx = \int_0^t \phi(s)ds.$$

Proof. Let $m \in \mathbb{N},\ A_k = \left\{s : \frac{k-1}{2^m} \leq \phi(s) < \frac{k}{2^m}\right\},\ k \in \mathbb{N}$. Let

$$\phi_m(s) = \frac{k}{2^m}, \quad \text{if } s \in A_k,\ k \in \mathbb{N}.$$

Then $\phi_m \rightrightarrows \phi$ and $\phi_1 \geq \phi_2 \geq \ldots$. Since ϕ is nonincreasing, there exists a sequence $\{t_k\}_{k\in\mathbb{N}}$ with $(t_{k-1}, t_k) \subset A_k \subset [t_{k-1}, t_k]$. Now we define

$$u_m(x) = \frac{k}{2^m}, \quad \text{if } x \in D_{t_k} \backslash D_{t_{k-1}}, \quad k \in \mathbb{N}.$$

Then

$$u_m^*(s) = \phi_m^*(s).$$

Since

$$|u_m(x) - u_{m+r}(x)| \leq \frac{1}{2^m},$$

there exists a limit $\lim_{m\to\infty} u_m(x) = u(x)$ and $u_m \rightrightarrows u$. □

2 Main results

In this section, we present the main results of this paper.

Theorem 3. *Let $1 < p < \infty$. For any locally absolutely continuous function f on $\mathbb{R}^3$ we have*

$$\int_0^\infty \max\left\{\sup_{0<t\leq r} \frac{|F_t|^p}{r^p}, \sup_{r<t} \frac{|F_t|^p}{t^p}\right\} dr \leq \left(\frac{p}{p-1}\right)^p \int_{\mathbb{R}^3} |divf|^p dx. \tag{5}$$

The constant $\left(\frac{p}{p-1}\right)^p$ is sharp. Here $F_t := \int_{\partial D_t} f(x_1, x_2, x_3)(dx_1dx_2 + dx_2dx_3 + dx_3dx_1)$.

Proof. By using the divergence theorem, we obtain

$$\int_{\partial D_t} f(x_1,x_2,x_3)(dx_1dx_2+dx_2dx_3+dx_3dx_1) = \int_{D_t}\left(\frac{\partial f}{\partial x_1}+\frac{\partial f}{\partial x_2}+\frac{\partial f}{\partial x_3}\right)dx_1dx_2dx_3$$
$$\leq \sup_{|e|=t}\int_e\left|\frac{\partial f}{\partial x_1}+\frac{\partial f}{\partial x_2}+\frac{\partial f}{\partial x_3}\right|dx_1dx_2dx_3 = \int_0^t u^*(s)ds,$$

where

$$u=\frac{\partial f}{\partial x_1}+\frac{\partial f}{\partial x_2}+\frac{\partial f}{\partial x_3},$$

and

$$\sup_{t\geq r}\frac{1}{t}\left|\int_{D_t}u(y_1,y_2,y_3)\,dy_1dy_2dy_3\right| \leq$$
$$\leq \sup_{|e|\geq t}\frac{1}{|e|}\int_e|u(y_1,y_2,y_3)|\,dy_1dy_2dy_3 = \frac{1}{t}\int_0^t u^*(s)ds.$$

Then

$$\max\left\{\sup_{0<t\leq r}\frac{\left|\int_{\partial D_t} f(x_1,x_2,x_3)(dx_1dx_2+dx_2dx_3+dx_3dx_1)\right|}{r},\right.$$
$$\left.\sup_{r<t}\frac{\left|\int_{\partial D_t} f(x_1,x_2,x_3)(dx_1dx_2+dx_2dx_3+dx_3dx_1)\right|}{t}\right\}$$
$$\leq \frac{1}{t}\int_0^t u^*(s)ds.$$

Thus, we have

$$\int_0^\infty \max\left\{\sup_{0<t\leq r}\frac{\left|\int_{\partial D_t} f(x_1,x_2,x_3)(dx_1dx_2+dx_2dx_3+dx_3dx_1)\right|^p}{r^p},\right.$$
$$\left.\sup_{r<t}\frac{\left|\int_{\partial D_t} f(x_1,x_2,x_3)(dx_1dx_2+dx_2dx_3+dx_3dx_1)\right|^p\}}{t^p}\right\}dr$$
$$\leq \int_0^\infty (u^{**}(t))^p\,dt \leq \left(\frac{p}{p-1}\right)^p\int_0^\infty (u^*(t))^p\,dt = \left(\frac{p}{p-1}\right)^p\int_{\mathbb{R}^3}|u(x)|^p dx$$
$$= \left(\frac{p}{p-1}\right)^p\int_{\mathbb{R}^3}\left|\frac{\partial f}{\partial x_1}+\frac{\partial f}{\partial x_2}+\frac{\partial f}{\partial x_3}\right|^p dx_1dx_2dx_3.$$

Now, we show that the constant is sharp.

Let $\varepsilon > 0$ and ϕ_ε be a function given in Lemma 1. According to Lemma 2, there exists u_ε with

$$u_\varepsilon(s) = \phi_\varepsilon(s).$$

Let f_ε be a solution to the equation

$$\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} + \frac{\partial f}{\partial x_3} = u_\varepsilon\left(x_1, x_2, x_3\right).$$

By using the divergence theorem, we have

$$\begin{aligned}
&\left(\int_0^\infty \left|\frac{1}{t}\int_{\partial D_t} f_\varepsilon\left(x_1, x_2, x_3\right)\left(dx_1dx_2 + dx_2dx_3 + dx_3dx_1\right)\right|^p dt\right)^{1/p} \\
&= \left(\int_0^\infty \left|\frac{1}{t}\int_{D_t} \frac{\partial f_\varepsilon}{\partial x_1} + \frac{\partial f_\varepsilon}{\partial x_2} + \frac{\partial f_\varepsilon}{\partial x_3} dx\right|^p dt\right)^{1/p} \\
&= \left(\int_0^\infty \left|\frac{1}{t}\int_{D_t} u_\varepsilon(x)dx\right|^p dt\right)^{1/p} = \left(\int_0^\infty \left|\frac{1}{t}\int_0^t \phi_\varepsilon(s)ds\right|^p dt\right)^{1/p} \\
&\geq \left(\frac{p}{p-1} - \varepsilon\right)\left(\int_0^\infty \left(\phi_\varepsilon(t)\right)^p dt\right)^{1/p} = \left(\frac{p}{p-1} - \varepsilon\right)\left(\int_0^\infty \left(u_\varepsilon^*(t)\right)^p dt\right)^{1/p} \\
&= \left(\frac{p}{p-1} - \varepsilon\right)\left(\int_{\mathbb{R}^3} \left|\frac{\partial f_\varepsilon}{\partial x_1} + \frac{\partial f_\varepsilon}{\partial x_2} + \frac{\partial f_\varepsilon}{\partial x_3}\right|^p dx_1dx_2dx_3\right)^{1/p}.
\end{aligned}$$

Hence, the constant $\frac{p}{p-1}$ is sharp for the inequality

$$\begin{aligned}
&\left(\int_0^\infty \left|\frac{1}{t}\int_{\partial D_t} f\left(x_1, x_2, x_3\right)\left(dx_1dx_2 + dx_2dx_3 + dx_3dx_1\right)\right|^p dt\right)^{1/p} \\
&\leq \frac{p}{p-1}\left(\int_{\mathbb{R}^3} \left|\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} + \frac{\partial f}{\partial x_3}\right|^p dx_1dx_2dx_3\right)^{1/p}.
\end{aligned}$$

This implies that the constant is also sharp for inequality (1). □

Below, to make the formulas shorter, we write $\bar{x}$ for (x_1, x_2, x_3), so $f(\bar{x})$ stands for $f(x_1, x_2, x_3)$.

Theorem 4. *Let $1 < p < \infty$. Then, for any locally absolutely continuous function f on $\mathbb{R}^3$*

$$\int_0^\infty \frac{\left(\left(\int\limits_{\partial D_t} f\left(\bar{x}\right)dx_1dx_2\right)^2 + \left(\int\limits_{\partial D_t} f\left(\bar{x}\right)dx_2dx_3\right)^2 + \left(\int\limits_{\partial D_t} f\left(\bar{x}\right)dx_3dx_1\right)^2\right)^{p/2}}{t^p} dt \leq$$

$$\leq \left(\frac{p}{p-1}\right)^p \int\limits_{\mathbb{R}^3} |\nabla f(x)|^p dx.$$

Here the constant $\left(\frac{p}{p-1}\right)^p$ is sharp.

Proof. By using the divergence theorem, we obtain

$$\begin{aligned}
&\sup_{\alpha_1^2+\alpha_2^2+\alpha_3^2=1} \int_{\partial D_t} f\left(x_1,x_2,x_3\right)\left(\alpha_1 dx_1dx_2+\alpha_2 dx_2dx_3+\alpha_3 dx_3dx_1\right) \\
&= \sup_{\alpha_1^2+\alpha_2^2+\alpha_3^2=1} \int_{D_t} \left(\alpha_1\frac{\partial f}{\partial x_1}+\alpha_2\frac{\partial f}{\partial x_2}+\alpha_3\frac{\partial f}{\partial x_3}\right) dx_1dx_2dx_3 \\
&\le \sup_{\alpha_1^2+\alpha_2^2+\alpha_3^2=1} \sup_{|e|=t} \int_e \left|\alpha_1\frac{\partial f}{\partial x_1}+\alpha_2\frac{\partial f}{\partial x_2}+\alpha_3\frac{\partial f}{\partial x_3}\right| dx_1dx_2dx_3 \\
&\le \sup_{|e|=t}\int_e \left|\nabla f\left(x_1,x_2,x_3\right)\right| dx_1dx_2dx_3 = \int_0^t (|\nabla f|)^*(s)ds.
\end{aligned}$$

Then

$$\left(\left(\int_{\partial D_t} f\left(\bar{x}\right)dx_1dx_2\right)^2+\left(\int_{\partial D_t} f\left(\bar{x}\right)dx_2dx_3\right)^2+\left(\int_{\partial D_t} f\left(\bar{x}\right)dx_3dx_1\right)^2\right)^{1/2} \le$$
$$\le \int_0^t (|\nabla f|)^*(s)ds.$$

Thus, we obtain

$$\int_0^\infty \frac{\left(\left(\int_{\partial D_t} f\left(\bar{x}\right)dx_1dx_2\right)^2+\left(\int_{\partial D_t} f\left(\bar{x}\right)dx_2dx_3\right)^2+\left(\int_{\partial D_t} f\left(\bar{x}\right)dx_3dx_1\right)^2\right)^{p/2}}{t^p}dt \le$$
$$\le \int_0^\infty \left((|\nabla f|)^{**}(t)\right)^p dt \le \left(\frac{p}{p-1}\right)^p \int_0^\infty \left((|\nabla f|)^{*}(t)\right)^p dt = \left(\frac{p}{p-1}\right)^p \int_{\mathbb{R}^3} |\nabla f(x)|^p dx.$$

The sharpness is proved as in the case of Theorem 3. □

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Нұрсұлтанов Е. Д., Сұраған Д. Дивергенция теоремасы арқылы алынған оптималды тұрақтылары бар кейбір Харди типті теңсіздіктер

Харди теңсіздігі XX ғасырдың басында Г.Х. Харди нақты талдау саласында интеграл операторларды бағалау үшін осы негізгі нәтижені енгізген. Оның әдемі сипаттамасы және оңтайлы тұрақтысы кеңінен қызығушылық тудырды, кейін бұл көптеген жетілдірулерге әкелді. Бұл жетістіктер әрі қарай зерттеулер мен көпөлшемді кеңейтулердің негізін қалады, гармоникалық талдау, бөлшектік теңдеулер және математикалық физика салаларына терең ықпал етті. Бұл тарихи даму қазіргі зерттеулерге шабыт беруді жалғастыруда. Дивергенция теоремасы негізінде кейбір жетілдірілген Харди теңсіздіктерінің көпөлшемді жалпылаулары келтірілген. Алынған Харди типті теңсіздіктер жуықта жарияланған бір өлшемді Харди теңсіздігін оптималды тұрақтымен көпөлшемді жағдайларға кеңейтеді.

Түйін сөздер: Харди теңсіздігі, оптималды тұрақты, өспейтін ауыстыру, дивергенция теоремасы.

Нурсултанов Ерлан Даутбекович, Сураган Дурвудхан. Некоторые неравенства типа Харди с оптимальными константами, полученные с помощью теоремы о дивергенции

Неравенство Харди возникло в начале XX века, когда Г.Х. Харди представил этот фундаментальный результат в вещественном анализе для оценки интегральных операторов. Его элегантная формулировка и оптимальные константы вызвали широкий интерес, что привело к многочисленным усовершенствованиям. Эти разработки заложили основу для дальнейших исследований и многомерных обобщений, оказав глубокое влияние на гармонический анализ, дифференциальные уравнения и математическую физику. Эта историческая эволюция продолжает вдохновлять современные исследования. Даны многомерные обобщения некоторых улучшенных неравенств Харди, основанные на теореме о дивергенции. Полученные неравенства типа Харди расширяют недавнюю версию одномерного неравенства Харди с наилучшей константой на многомерные случаи.

Ключевые слова: Неравенство Харди, оптимальная константа, невозрастающая перестановка, теорема о дивергенции.