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On one minimal hyperbolic Gellerstedt operator with internal degeneracy

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Abstract. The minimal operators generated by overdetermined boundary value problems for differential equations are extremely important in the description of regular boundary value problems for differential equations, and are also widely used in the study of local properties of solutions. The study of overdetermined boundary value problems is closely related to the theory of correct restrictions and extensions and the construction of minimal differential operators. In addition, for inverse problems of mathematical physics arising from applications, when determining unknown data, it is necessary to study problems with overdetermined boundary conditions, including minimal operators, which is reflected in the study of problems, including for hyperbolic equations and systems arising in physics, geophysics, seismic tomography, medicine and many other practical areas. Thus, the study of minimal operators is of both theoretical and applied interest. In this paper, a criterion for the invertibility of the minimal hyperbolic Gellerstedt operator with internal degeneracy is established. The proof is based on the Gellerstedt potential, properties of solutions to the Goursat problem, and properties of special functions. It should be noted that the Gellerstedt differential operator has numerous applications in transonic gas dynamics, the theory of infinitesimal surface bends, the instantaneous theory of shells with curvature of variable sign, magnetodynamics and hydrodynamics, and the conditions of invertibility of minimal operators imposed on the right side of the initial equation are widely used in the study of the so-called source problem, which arises in a variety of applications inverse problems of mathematical physics.

Keywords. minimal operator, Gellerstedt equation, criterion, internal degeneracy, invertibility.

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1 Introduction

The minimal operators generated by overdetermined boundary value problems for differential equations are used in the description of regular boundary value problems for differential equations, and are also widely used in establishing local properties of solutions.

Overdetermined boundary value problems and the minimal operators generated by them are closely related to the theory of restrictions and extensions of operators, in particular, all regular boundary value problems for linear differential equations are boundary conditions of regular extensions of the corresponding minimal differential operators.

The theory of regular extensions for elliptic equations was first constructed by M.I. Vishik in [1]. The development of M.I. Vishik's theory for arbitrary equations in Banach space, as well as the construction of invertible restrictions generated not necessarily by boundary conditions, was started with the work of M.O. Otelbaev and A.N. Shynybekov [2]. In the works of T.Sh. Kalmenov [3, 4], using regular extensions method, a description of a wide class of regular boundary value problems for the Lavrent'ev-Bitsadze and the Tricomi equations of mixed type is given.

The systematic study and finding of the boundary conditions of classical potentials was started with the work of T.Sh. Kalmenov and D. Suragan [5].

In addition to its theoretical significance in the description of regular boundary value problems for differential equations and the study of local properties of solutions, the study of minimal operators is also important from a practical point of view. For inverse problems of mathematical physics arising from applications, when determining unknown data, it is often necessary to study problems with overdetermined boundary conditions, including minimal operators.

Inverse problems, including problems for hyperbolic equations and systems, are currently widespread in physics, geophysics, medicine, ecology, economics and many other areas. The development of inverse problems of mathematical physics has been reflected in the works of a large number of authors, starting with the works of A.N. Tikhonov [6, 7], V.K. Ivanov [8], M.M. Lavrentiev [9].

This paper is devoted to the study of the minimal hyperbolic Gellerstedt operator with internal degeneracy. As is known, a deep study of mixed-type equations and degenerate equations was started with the work of F. Tricomi [10], and the problem he studied was called the Tricomi problem. S. Gellerstedt solves the Tricomi problem for a more general equation [11], which was later called the Gellerstedt equation. Significant results on regular boundary value problems for the Lavrent'ev-Bitsadze equation were obtained in the works of A.V. Bitsadze [12] and his followers.

It should be noted that differential equations of mixed type, in particular the Gellerstedt equation, have numerous applications in transonic gas dynamics, the theory of infinitesimal bending of surfaces, the instantaneous theory of shells with curvature of variable sign, magnetodynamics and hydrodynamics and are studied in the works of S.A. Chaplygin [13], F.I. Frankl [14], C.S. Moravetz [15, 16].

The study of minimal operators and the finding of conditions imposed on the right side of the equation so that the solution of the problem satisfies the specified properties are widespread in a wide variety of applications of inverse problems, and such an inverse problem itself is known as the source problem.

In this paper, a criterion for the invertibility of the minimal hyperbolic Gellerstedt operator with internal degeneracy is established. A distinctive feature of this work is the presence of internal degeneracy, that is, a change of the type of equation on the line located inside the area under consideration. The proof is based on the properties of the Gellerstedt potential, the properties of solutions to the Goursat problem, and the properties of special functions.

Note that the criterion of invertibility of the minimal hyperbolic Gellerstedt operator and the corresponding overdetermined problem in the case of a degeneracy line at the boundary of the domain were considered in our previous work [17].

2 Problem statement and the main idea of the proof

Let $\mathcal{D} \subset R^2$ be a characteristic quadrilateral bounded at $y<0$ by characteristics

$$AC:\ x-\frac{2}{m+2}(-y)^{\frac{m+2}{2}}=0 \text{ and } BC:\ x+\frac{2}{m+2}(-y)^{\frac{m+2}{2}}=1$$

and at $y>0$ by characteristics

$$AC^*:\ x-\frac{2}{m+2}y^{\frac{m+2}{2}}=0 \text{ and } BC^*:\ x+\frac{2}{m+2}y^{\frac{m+2}{2}}=1$$

of the Gellerstedt equation

$$Lu \equiv y^m u_{xx} - u_{yy} = f(x,y). \tag{1}$$

We consider the following overdetermined problem: to find a regular solution of Equation (1) in the domain $\mathcal{D}$, satisfying the conditions:

$$u\big|_{AC^*:\, x-\frac{2}{m+2}y^{\frac{m+2}{2}}=0}=0, \quad u\big|_{BC^*:\, x+\frac{2}{m+2}(-y)^{\frac{m+2}{2}}=1}=0. \tag{2}$$

$$u\big|_{AC:\, x-\frac{2}{m+2}(-y)^{\frac{m+2}{2}}=0}=0, \quad u\big|_{BC:\, x+\frac{2}{m+2}(-y)^{\frac{m+2}{2}}=1}=0. \tag{3}$$

In addition, the following matching conditions must be met on the line of degeneracy of the equation type $y=0$:

$$u(x,+0)=u(x,-0), \quad \frac{\partial u}{\partial y}(x,+0)=\frac{\partial u}{\partial y}(x,-0). \tag{4}$$

We denote by L_0 the closure in $L_2(\mathcal{D})$ of Operator (1) on a subset of functions $u \in C^2(\mathcal{D})$, satisfying Conditions (2)–(3).

We also denote $\mathcal{D}^+ = \mathcal{D} \cap y > 0$, $\mathcal{D}^- = \mathcal{D} \cap y < 0$, $f^+ = f$ at $y > 0$, $f^- = f$ at $y < 0$.

Since Problem (1), (2), (3) is overdetermined, it will be ill-posed in general case — for its well-posedness additional conditions must be imposed on the right side of equation (1) — the function $f(x, y)$.

Thus, we get the inverse problem: to determine the nature of the function $f(x, y)$ — the right side of Equation (1), that is, to establish the conditions that this function must satisfy in order for Problem (1), (2), (3) was regularly solvable.

The basic idea of establishing the minimality of the hyperbolic Gellerstedt operator with internal degeneracy is as follows. First, we solve the classical Goursat problem in the $\mathcal{D}^-$ and $\mathcal{D}^+$ domains (Problems (1), (3) and (1), (2)). Then, with the obtained solutions of the Goursat problems, we satisfy the conditions for matching the traces of the function and its normal derivative on the inner line of degeneracy of the equation type, thus obtaining the desired conditions for the right side of Equation (1).

3 The Goursat problem in $\mathcal{D}^-$ domain

Let us first consider the Goursat problem in $\mathcal{D}^-$ domain ((1), (3) problem — Γ^- problem). In characteristic variables $\xi = x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}}$, $\eta = x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}}$ the Riemann function of the Γ^- problem Goursat problem has the next form [18]:

$$R(\xi, \eta, \xi_1, \eta_1) = k \cdot \frac{(\eta_1 - \xi_1)^{2\beta}}{(\eta - \xi_1)^{\beta} \cdot (\eta_1 - \xi)^{\beta}} \cdot F(\beta, \beta; 1; \sigma), \tag{5}$$

where $F(a, b; c; z)$ is the hypergeometric function;

$$\sigma = \frac{(\xi_1 - \xi)(\eta_1 - \eta)}{(\xi_1 - \eta)(\eta_1 - \xi)}, \quad , k = \frac{\Gamma(\beta)}{\Gamma(1-\beta) \cdot \Gamma(2\beta)}, \quad \beta = \frac{1}{2(m+2)}, \tag{6}$$

where $\Gamma(z)$ is the gamma function.

In this case, the solution of the Γ^- Goursat problem ((1), (3) problem) is represented as:

$$L^1_{\Gamma^-} f^- = \int\limits_0^{\xi} d\xi_1 \int\limits_1^{\eta} R(\xi, \eta, \xi_1, \eta_1) \cdot f_1^-(\xi_1, \eta_1)\, d\eta_1, \tag{7}$$

where

$$f_1^-(\xi_1, \eta_1) = \left(\frac{4}{m+2}\right)^{4\beta} (\eta_1 - \xi_1)^{-4\beta} f^- \left(\frac{\xi_1 + \eta_1}{2}, -\left(\frac{m+2}{4}\right)^{1-2\beta} (\eta_1 - \xi_1)^{1-2\beta}\right). \tag{8}$$

Thus, Function (7) satisfies Equation (1) and Boundary conditions (3).

4 The Goursat problem in $\mathcal{D}^+$ domain

Let us consider the Goursat problem in $\mathcal{D}^+$ domain ((1), (2) problem — Γ^+ problem). Similar to area $\mathcal{D}^-$, in characteristic variables $\xi^* = x - \frac{2}{m+2}y^{\frac{m+2}{2}}$, $\eta^* = x + \frac{2}{m+2}y^{\frac{m+2}{2}}$ the Riemann function of the Γ^+ problem Goursat problem has the next form:

$$R^*\left(\xi^*, \eta^*, \xi_1, \eta_1\right) = k \cdot \frac{\left(\eta_1 - \xi_1\right)^{2\beta}}{\left(\eta^* - \xi_1\right)^{\beta} \cdot \left(\eta_1 - \xi^*\right)^{\beta}} \cdot F(\beta, \beta; 1; \sigma^*), \tag{9}$$

where

$$\sigma^* = \frac{\left(\xi_1 - \xi^*\right)\left(\eta_1 - \eta^*\right)}{\left(\xi_1 - \eta^*\right)\left(\eta_1 - \xi^*\right)}, \tag{10}$$

k, β are determined from the ratio (6).

In this case, the solution of the Γ^+ Goursat problem ((1), (2) problem) is represented in next form:

$$L^1_{\Gamma^+} f^+ = \int\limits_0^{\xi^*} d\xi_1 \int\limits_1^{\eta^*} R^*\left(\xi^*, \eta^*, \xi_1, \eta_1\right) \cdot f_1^+\left(\xi_1, \eta_1\right) d\eta_1, \tag{11}$$

where

$$f_1^+\left(\xi_1, \eta_1\right) = \left(\frac{4}{m+2}\right)^{4\beta} \left(\eta_1 - \xi_1\right)^{-4\beta} f^+\left(\frac{\xi_1 + \eta_1}{2}, -\left(\frac{m+2}{4}\right)^{1-2\beta} \left(\eta_1 - \xi_1\right)^{1-2\beta}\right). \tag{12}$$

Thus, Function (11) satisfies Equation (1) and Boundary conditions (2).

5 Matching conditions on the line of degeneracy of the equation type

Now we satisfy with Functions (7) and (11) the matching conditions on the line of degeneracy of the equation type $y = 0$ — Conditions (4). We have

$$\lim_{\eta \to \xi} \int\limits_0^{\xi} d\xi_1 \int\limits_1^{\eta} R\left(\xi, \eta, \xi_1, \eta_1\right) f_1^-\left(\xi_1, \eta_1\right) d\eta_1 =$$

$$= \lim_{\eta^* \to \xi^*} \int\limits_0^{\xi^*} d\xi_1 \int\limits_1^{\eta^*} R^*\left(\xi^*, \eta^*, \xi_1, \eta_1\right) f_1^+\left(\xi_1, \eta_1\right) d\eta_1 = 0, \tag{13}$$

$$\lim_{\eta \to \xi} \left(\frac{m+2}{2}\right)^{2\beta} (\eta - \xi)^{2\beta} \left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right) \int\limits_0^{\xi} d\xi_1 \int\limits_1^{\eta} R\left(\xi, \eta, \xi_1, \eta_1\right) f_1^-\left(\xi_1, \eta_1\right) d\eta_1 =$$

$$= \lim_{\eta^* \to \xi^*} \left(\frac{m+2}{2}\right)^{2\beta} (\eta^* - \xi^*)^{2\beta} \left(\frac{\partial}{\partial \eta^*} - \frac{\partial}{\partial \xi^*}\right) \int\limits_0^{\xi^*} d\xi_1 \int\limits_1^{\eta^*} R^* (\xi^*, \eta^*, \xi_1, \eta_1) f_1^+ (\xi_1, \eta_1) d\eta_1 = 0. \tag{14}$$

From the ratio (13), taking into account the formulas [19], pp. 556, 255:

$$F(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad c \neq 0, -1, -2, \ldots, \quad Re(c-a-b) > 0, \tag{15}$$

$$\Gamma(n+1) = n!, \tag{16}$$

we obtain, taking into account the relations (7) and (11) at $\eta \to \xi$ and $\eta^* \to \xi^*$:

$$\sigma|_{\xi=\eta} = \left.\frac{(\xi_1 - \xi)(\eta_1 - \eta)}{(\xi_1 - \eta)(\eta_1 - \xi)}\right|_{\xi=\eta} = \frac{(\xi_1 - \xi)(\eta_1 - \xi)}{(\xi_1 - \xi)(\eta_1 - \xi)} = 1;$$

$$\sigma^*|_{\xi^*=\eta^*} = \left.\frac{(\xi_1 - \xi^*)(\eta_1 - \eta^*)}{(\xi_1 - \eta^*)(\eta_1 - \xi^*)}\right|_{\xi^*=\eta^*} = \frac{(\xi_1 - \xi^*)(\eta_1 - \xi^*)}{(\xi_1 - \xi^*)(\eta_1 - \xi^*)} = 1;$$

$$F(\beta,\beta;1;\sigma)|_{\xi=\eta} = F(\beta,\beta;1;\sigma^*)|_{\xi^*=\eta^*} = F(\beta,\beta;1;1) =$$

$$= \frac{\Gamma(1)\Gamma(1-2\beta)}{\Gamma(1-\beta)\Gamma(1-\beta)} = \frac{\Gamma(1-2\beta)}{\Gamma(1-\beta)\Gamma(1-\beta)};$$

$$R(\xi,\eta,\xi_1,\eta_1)\Big|_{\xi=\eta} = k \cdot \frac{(\eta_1 - \xi_1)^{2\beta}}{(\eta - \xi_1)^{\beta} \cdot (\eta_1 - \xi)^{\beta}} \cdot \frac{\Gamma(1-2\beta)}{\Gamma(1-\beta)\Gamma(1-\beta)},$$

$$R^*(\xi^*,\eta^*,\xi_1,\eta_1)\Big|_{\xi^*=\eta^*} = k \cdot \frac{(\eta_1 - \xi_1)^{2\beta}}{(\eta^* - \xi_1)^{\beta} \cdot (\eta_1 - \xi^*)^{\beta}} \cdot \frac{\Gamma(1-2\beta)}{\Gamma(1-\beta)\Gamma(1-\beta)},$$

that is, at $\eta \to \xi$ and $\eta^* \to \xi^*$:

$$\int\limits_0^{\xi} d\xi_1 \int\limits_1^{\xi} k \cdot \frac{(\eta_1 - \xi_1)^{2\beta}}{(\xi - \xi_1)^{\beta} \cdot (\eta_1 - \xi)^{\beta}} \cdot \frac{\Gamma(1-2\beta)}{\Gamma(1-\beta)\Gamma(1-\beta)} f_1^- (\xi_1, \eta_1) d\eta_1 =$$

$$= \int\limits_0^{\xi^*} d\xi_1 \int\limits_1^{\xi^*} k \cdot \frac{(\eta_1 - \xi_1)^{2\beta}}{(\xi^* - \xi_1)^{\beta} \cdot (\eta_1 - \xi^*)^{\beta}} \cdot \frac{\Gamma(1-2\beta)}{\Gamma(1-\beta)\Gamma(1-\beta)} f_1^+ (\xi_1, \eta_1) d\eta_1.$$

Therefore, to fulfill the first of the initial Cauchy conditions (2), the function $f(x,y)$ must satisfy the condition

$$\int\limits_0^{\xi} d\xi_1 \int\limits_1^{\xi} \frac{(\eta_1 - \xi_1)^{2\beta}}{(\xi - \xi_1)^{\beta} \cdot (\eta_1 - \xi)^{\beta}} f_1^- (\xi_1, \eta_1) d\eta_1 =$$

$$= \int\limits_{0}^{\xi^*} d\xi_1 \int\limits_{1}^{\xi^*} \frac{(\eta_1-\xi_1)^{2\beta}}{(\xi^*-\xi_1)^{\beta}\cdot(\eta_1-\xi^*)^{\beta}} f_1^+(\xi_1,\eta_1)\,d\eta_1 = 0, \tag{17}$$

where the function $f_1^-(\xi_1,\eta_1)$ is defined by the formula (8), and the function $f_1^+(\xi_1,\eta_1)$ is defined by Formula (12).

Now let us to fulfill the second matching Condition (4), that is represented in the form (14). We take into account the formulas [19], pp. 557, 559, 556

$$\frac{d}{dz}F(a,b;c;z) = \frac{ab}{c}F(a+1,b+1;c+1;z), \tag{18}$$

$$F(a,b;c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b;a+b-c+1;1-z)+$$

$$+(1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a,c-b;c-a-b+1;1-z), \quad |arg(1-z)|<\pi, \tag{19}$$

$$F(a,b;c;0) = 1, \tag{20}$$

considering the formula

$$\frac{d}{dy}\int\limits_{\alpha(y)}^{\beta(y)} f(x,y)dx = \int\limits_{\alpha(y)}^{\beta(y)} f(x,y)dx + \beta'(y)f(\beta(y),y) - \alpha'(y)f(\alpha(y),y), \tag{21}$$

and also taking into account the ratios (14), (5), (9), (6), (10), (15), (16), we obtain using the method of work [17]:

$$\lim_{\eta\to\xi}\left(\frac{m+2}{2}\right)^{2\beta}(\eta-\xi)^{2\beta}\left(\frac{\partial}{\partial\eta}-\frac{\partial}{\partial\xi}\right)\int\limits_{0}^{\xi} d\xi_1 \int\limits_{1}^{\eta} R(\xi,\eta,\xi_1,\eta_1)\,f_1^-(\xi_1,\eta_1)\,d\eta_1 =$$

$$= 2k(1-2\beta)\left(\frac{m+2}{2}\right)^{2\beta}\frac{\Gamma(2\beta-1)}{\Gamma^2(\beta)}\int\limits_{0}^{\xi} d\xi_1 \int\limits_{1}^{\xi} \frac{\eta_1-\xi_1}{(\eta_1-\xi)^{1-\beta}(\xi-\xi_1)^{1-\beta}} f_1^-(\xi,\eta_1)\,d\eta_1;$$

$$\lim_{\eta^*\to\xi^*}\left(\frac{m+2}{2}\right)^{2\beta}(\eta^*-\xi^*)^{2\beta}\left(\frac{\partial}{\partial\eta^*}-\frac{\partial}{\partial\xi^*}\right)\int\limits_{0}^{\xi^*} d\xi_1 \int\limits_{1}^{\eta^*} R^*(\xi^*,\eta^*,\xi_1,\eta_1)\,f_1^+(\xi_1,\eta_1)\,d\eta_1 =$$

$$= 2k(1-2\beta)\left(\frac{m+2}{2}\right)^{2\beta}\frac{\Gamma(2\beta-1)}{\Gamma^2(\beta)}\int\limits_{0}^{\xi^*} d\xi_1 \int\limits_{1}^{\xi^*} \frac{\eta_1-\xi_1}{(\eta_1-\xi^*)^{1-\beta}(\xi^*-\xi_1)^{1-\beta}} f_1^+(\xi,\eta_1)\,d\eta_1.$$

As a result, to satisfy the second of the matching conditions (4) the function $f(x,y)$ must satisfy condition (14), that is, by virtue of the last relations, the condition

$$\int\limits_0^{\xi} d\xi_1 \int\limits_1^{\xi} \frac{\eta_1-\xi_1}{(\eta_1-\xi)^{1-\beta}(\xi-\xi_1)^{1-\beta}} f_1^-(\xi_1,\eta_1)\,d\eta_1 =$$

$$\int\limits_0^{\xi^*} d\xi_1 \int\limits_1^{\xi^*} \frac{\eta_1-\xi_1}{(\eta_1-\xi^*)^{1-\beta}(\xi^*-\xi_1)^{1-\beta}} f_1^+(\xi_1,\eta_1)\,d\eta_1, \tag{22}$$

where the function $f_1^-(\xi_1,\eta_1)$ is defined by the formula (8), and the function $f_1^+(\xi_1,\eta_1)$ is defined by the formula (12).

6 The criterion of invertibility of the minimal hyperbolic Gellerstedt operator with internal degeneracy

Thus, the following proposition is proved.

Proposition 1. *The minimal hyperbolic Gellerstedt operator is invertible in $L_2(\mathcal{D})$ if and only if conditions (17) and (22) are met, where the functions $f_1^-(\xi_1,\eta_1)$ and $f_1^+(\xi_1,\eta_1)$ are determined by a given function $f(x,y)$ from equation (1) according to formulas (8) and (12), respectively. When conditions (17) and (22) are met, the solution of the corresponding overdetermined problem is represented by formulas (7) and (11), where $R(\xi,\eta,\xi_1,\eta_1)$ and $R^*(\xi^*,\eta^*,\xi_1,\eta_1)$ are Riemann functions of the Γ^- and Γ^+ Goursat problems respectively, determined by formulas (5) and (9).*

Using the representation (8) of the function $f_1^-(\xi_1,\eta_1)$ and the representation (12) of the function $f_1^+(\xi_1,\eta_1)$, expressions (5) and (9) for the Riemann function and conditions (17), (22), we can formulate this proposition — the main result of this work — in the following form.

Theorem 2 (The criterion of invertibility of the minimal hyperbolic Gellerstedt operator with internal degeneracy)**.** *The minimal hyperbolic Gellerstedt operator with internal degeneracy is invertible in $L_2(\mathcal{D})$ (i.e. the problem (1), (2), (3), (4) is regularly solvable) if and only if the following conditions are met*

$$L_{\Gamma^+}^{-1}f^+\big|_{y=0+} = L_{\Gamma^-}^{-1}f^-\big|_{y=0-}, \quad \left.\frac{\partial L_{\Gamma^+}^{-1}f^+}{\partial y}\right|_{y=0+} = \left.\frac{\partial L_{\Gamma^-}^{-1}f^-}{\partial y}\right|_{y=0-}, \tag{23}$$

that is, the next conditions are met

$$\int\limits_0^{\xi} d\xi_1 \int\limits_1^{\xi} \frac{f^-\left(\frac{\xi_1+\eta_1}{2}, -\left(\frac{m+2}{4}\right)^{1-2\beta}(\eta_1-\xi_1)^{1-2\beta}\right)}{(\eta_1-\xi_1)^{2\beta}(\xi-\xi_1)^{\beta}(\eta_1-\xi)^{\beta}} d\eta_1 =$$

$$= \int\limits_0^{\xi^*} d\xi_1 \int\limits_1^{\xi^*} \frac{f^+\left(\frac{\xi_1+\eta_1}{2}, -\left(\frac{m+2}{4}\right)^{1-2\beta}(\eta_1-\xi_1)^{1-2\beta}\right)}{(\eta_1-\xi_1)^{2\beta}(\xi^*-\xi_1)^{\beta}(\eta_1-\xi^*)^{\beta}} d\eta_1, \tag{24}$$

$$\int\limits_0^{\xi} d\xi_1 \int\limits_1^{\xi} \frac{f^-\left(\frac{\xi_1+\eta_1}{2}, -\left(\frac{m+2}{4}\right)^{1-2\beta}(\eta_1-\xi_1)^{1-2\beta}\right)}{(\eta_1-\xi_1)^{4\beta-1}(\xi-\xi_1)^{1-\beta}(\eta_1-\xi)^{1-\beta}} d\eta_1 =$$

$$= \int\limits_0^{\xi^*} d\xi_1 \int\limits_1^{\xi^*} \frac{f^+\left(\frac{\xi_1+\eta_1}{2}, -\left(\frac{m+2}{4}\right)^{1-2\beta}(\eta_1-\xi_1)^{1-2\beta}\right)}{(\eta_1-\xi_1)^{4\beta-1}(\xi^*-\xi_1)^{1-\beta}(\eta_1-\xi^*)^{1-\beta}} d\eta_1. \tag{25}$$

When conditions (24) and (25) are met, the solution of Problem (1), (2), (3), and (4) is represented as:

$$y<0: \quad u^-(\xi,\eta) = \frac{1}{4}\left(\frac{4}{m+2}\right)^{4\beta} \frac{\Gamma(\beta)}{\Gamma(1-\beta)\cdot\Gamma(2\beta)} \times$$

$$\times \int\limits_0^{\xi} d\xi_1 \int\limits_1^{\eta} \frac{f^-\left(\frac{\xi_1+\eta_1}{2}, -\left(\frac{m+2}{4}\right)^{1-2\beta}(\eta_1-\xi_1)^{1-2\beta}\right)}{(\eta_1-\xi_1)^{2\beta}(\eta-\xi_1)^{\beta}(\eta_1-\xi)^{\beta}} F(\beta,\beta;1;\sigma) d\eta_1, \tag{26}$$

$$y>0: \quad u^+(\xi^*,\eta^*) = \frac{1}{4}\left(\frac{4}{m+2}\right)^{4\beta} \frac{\Gamma(\beta)}{\Gamma(1-\beta)\cdot\Gamma(2\beta)} \times$$

$$\times \int\limits_0^{\xi^*} d\xi_1 \int\limits_1^{\eta^*} \frac{f^+\left(\frac{\xi_1+\eta_1}{2}, -\left(\frac{m+2}{4}\right)^{1-2\beta}(\eta_1-\xi_1)^{1-2\beta}\right)}{(\eta_1-\xi_1)^{2\beta}(\eta^*-\xi_1)^{\beta}(\eta_1-\xi^*)^{\beta}} F(\beta,\beta;1;\sigma^*) d\eta_1, \tag{27}$$

where

$$\sigma = \frac{(\xi_1-\xi)(\eta_1-\eta)}{(\xi_1-\eta)(\eta_1-\xi)}, \quad \sigma^* = \frac{(\xi_1-\xi^*)(\eta_1-\eta^*)}{(\xi_1-\eta^*)(\eta_1-\xi^*)}, \quad \beta = \frac{1}{2(m+2)},$$

$$\xi = x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}}, \quad \eta = x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}},$$

$$\xi^* = x - \frac{2}{m+2}y^{\frac{m+2}{2}}, \quad \eta^* = x + \frac{2}{m+2}y^{\frac{m+2}{2}}.$$

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Роговой А. В., Кәлменов Т. Ш. ІШКІ АЗҒЫНДАЛУЫ БАР БІР МИНИМАЛДЫ ГИПЕРБОЛАЛЫҚ ГЕЛЛЕРСТЕДТ ОПЕРАТОРЫ ТУРАЛЫ

Дифференциалдық теңдеулер үшін артық анықталған шектік есептерінен туындаған минималды операторлар дифференциалдық теңдеулер үшін регулярлы шектік есептерін сипаттауда өте маңызды, сонымен қатар шешімдердің локальды қасиеттерін зерттеуде кеңінен қолданылады. Артық анықталған шеткі есептерді зерттеу корректілі тарылу және кеңею теориясымен және минималды дифференциалдық операторлардың құруымен тығыз байланысты. Сонымен қатар, қолданбалардан туындайтын математикалық физиканың кері есептері үшін белгісіз деректерді анықтау кезінде артық анықталған шекаралық шарттары бар есептерді, соның ішінде минималды операторларды зерттеу қажет, бұл есептерді зерттеуде, соның ішінде физикада, геофизикада, сейсмикалық томографияда, медицинада және т.б. практикалық салаларда туындайтын гиперболалық теңдеулер мен жүйелер көрініс тапты. Осылайша, минималды операторларды зерттеу теориялық және қолданбалы қызығушылық тудырады. Бұл жұмыста ішкі азғындалуы бар минималды Геллерстедт гиперболалық оператордың керілену критерийі құрылады. Дәлел Геллерстедт потенциалына, Гурса есептерінің шешімдерінің қасиеттеріне және арнайы функциялардың қасиеттеріне негізделген. Айта кету керек, Геллерстедт дифференциалдық операторы трансоникалық газодинамикасында, беттердің шексіз иілу теориясында, айнымалы таңбалы қисықтығы бар моментсіз қабық теориясында, магнитодинамикада және гидродинамикада көптеген қолданысқа ие және бастапқы теңдеудің оң жағына қойылған минималды операторлардың керілену шарттары әр түрлі математикалық физиканың кері есебінен туындайтын көз туралы есеп деп аталатын есепті зерттеуде көп қолданылады.

Түйін сөздер: минималды оператор, Геллерстедт теңдеуі, критерий, ішкі азғындалу, керілену.

Роговой А. В., Кальменов Т. Ш. ОБ ОДНОМ МИНИМАЛЬНОМ ГИПЕРБОЛИЧЕСКОМ ОПЕРАТОРЕ ГЕЛЛЕРСТЕДТА С ВНУТРЕННЕМ ВЫРОЖДЕНИЕМ

Минимальные операторы, порожденные переопределенными краевыми задачами для дифференциальных уравнений, крайне важны при описании регулярных краевых задач для дифференциальных уравнений, а также широко применяются при изучении локальных свойств решений. Исследование переопределенных краевых задач тесно связано с теорией корректных сужений и расширений и построением минимальных дифференциальных операторов. Помимо этого, для обратных задач математической физики, возникающих из приложений, при определении неизвестных данных необходимо изучить задачи с переопределенными граничными условиями, в том числе минимальные операторы, что нашло свое отражение при исследовании задач, в том числе для гиперболических уравнений и систем, возникающих в физике, геофизике, сейсмической томографии, медицине и многих других практических областях. Тем самым, изучение минимальных операторов представляет и теоретический, и прикладной интерес. В настоящей работе установлен критерий обратимости минимального гиперболического оператора Геллерстедта с внутренним вырождением. Доказательство основано на потенциале Геллерстедта, свойствах решений задачи Гурса и свойствах специальных функций. Следует отметить, что дифференциальный оператор Геллерстедта имеет многочисленные приложения в трансзвуковой газодинамике, теории бесконечно малых изгибаний поверхностей, безмоментной теории оболочек с кривизной переменного знака, магнитодинамике и гидродинамике, а условия обратимости минимальных операторов, налагаемые на правую часть исходного уравнения, широко используются при исследовании так называемой задачи об источнике, возникающей в самых разных приложениях обратных задач математической физики.

Ключевые слова: минимальный оператор, уравнение Геллерстедта, критерий, внутреннее вырождение, обратимость.