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Remark on o-stable pure ordered groups

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Abstract. In this article, we describe pure ordered groups, whose elementary theory is o-stable or o-superstable. The notion of o-stability is a natural combination of stability and o-minimality, namely, an ordered structure is called o-stable if any of its cuts are consistent with a few complete one-types.

Keywords. ordered group, NIP theory, o-minimal theory, o-stable theory, o-superstable theory.

1 Preliminaries

During the last 40 years, model theory has focused on investigating ordered structures. Starting from o-minimality, specialists in Model Theory have developed many generalizations, such as weak o-minimality, quasi-o-minimality, weak quasi-o-minimality, inp-minimality, dp-minimality, and o-stability. Among theories which includes the symbol '$<$' and axioms saying that this relation is reflexive, antisymmetric, and transitive, the class of o-stable theories is the most wide. This paper investigates pure ordered groups whose elementary theory is either o-superstable or o-stable. The main results are the following. In Theorem 20 we classify o-superstable pure ordered groups. In Theorem 21 we give an answer to the question by E. Palyutin on orderable Abelian torsion-free superstable groups. In Theorem 22 we give an example of an orderable stable Abelian group with two orderings: one gives an o-stable group, while the other one gives an o-unstable group.

In the next section, we give all necessary definitions and theorems concerning o-stability. In Section 3, we make an introduction to dp-minimal theory, including definitions and basic facts which we use in the last section, where we show the main results.

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2 Introduction

Let $\mathcal{M} = (M, <, \dots)$ be a totally ordered structure, a an element of M, and let A, B be subsets of M. As usual, we write

$$a < A, \quad \text{if } a < b \text{ for any } b \in A,$$
$$A < B, \quad \text{if } a < b \text{ for any } a \in A \text{ and } b \in B.$$

A partition $\langle C, D \rangle$ of M is called a *cut* if $C < D$. Given a cut $\langle C, D \rangle$ one can construct a partial type $\{c < x < d : c \in C,\ d \in D\}$, which we also call a cut and use the same notation $\langle C, D \rangle$.

A subset A of a totally ordered set M is called *convex* if for any a and $b \in A$ the interval $[a, b]$ is a subset of A. A *convex component* of a set A is a maximal convex subset of A. The *convex hull* A^c of of a set A is defined as

$$A^c = \big\{ b \in M : \exists a_1, a_2 \in A \, (a_1 \le b \le a_2) \big\},$$

that is it is the least convex set containing the set A.

Let P be some property. We say that the property P holds *eventually in* A if there is an element $a \in M$ such that $a < \sup A$ and the property P holds on the intersection $(a, \infty) \cap A$. If $A = M$, we just write that the property P *holds eventually.* If sets B and C are eventually equal in A we denote this by $B \overset{\infty}{=}_A C$.

Let T be an $\mathcal{L}$-theory and $\phi(\bar{x}, \bar{y})$ a formula. The formula ϕ has the *independence property* (relative to T) if for all $n < \omega$ there is a model $\mathcal{M} \models T$ and two sequences $(\bar{a}_i : i < n)$ and $(\bar{b}_J : J \subseteq n)$ in M such that $\mathcal{M} \models \phi(\bar{a}_i, \bar{b}_J)$ if and only if $i \in J$. A theory T has the independence property if some formula has the independence property relative to T.

Notation 1. *Let s be a partial n-type, A a set. Then*

$$S^n_s(A) \triangleq \big\{ p \in S^n(A) : p \cup s \quad \textit{is consistent} \big\}.$$

Note, s need not be a partial type over the set A.

Definition 2. 1. An ordered structure $\mathcal{M}$ is *o-stable in* λ if for any $A \subseteq M$ with $|A| \le \lambda$ and for any cut $\langle C, D \rangle$ in $\mathcal{M}$ there are at most λ 1-types over A which are consistent with the cut $\langle C, D \rangle$, i.e.

$$\big| S^1_{\langle C, D \rangle}(A) \big| \le \lambda.$$

2. A theory T is *o-stable in* λ if every model of T is. Sometimes we write T is o-λ-stable.

3. A theory T is *o-stable* if there exists an infinite cardinal λ in which T is o-stable.

4. A theory T is *o-superstable* if there exists a cardinal λ such that T is o-stable in all $\mu \ge \lambda$.

5. A theory T is *strongly o-stable* if in addition to its o-stability any definable cut in any model $\mathcal{M}$ of T is definable in the language of pure ordering, or, equivalently, if $\sup A \in M$ for any definable subset A of $\mathcal{M}$.

Definition 3. A totally ordered structure $\mathcal{M}=(M,<,\dots)$ has *the strict order property insider a cut*, if there is a formula $\phi(x,\bar{y})$ such that for any natural number n there are tuples $\bar{a}_1$, …, $\bar{a}_n$ and a cut $\langle C_n, D_n\rangle$ in $\mathcal{M}$ such that in some $|M|^+$-saturated elementary extension $\mathcal{N}$ of $\mathcal{M}$ the following holds:

$$\phi(\mathcal{N},\bar{a}_1)\cap\langle C_n,D_n\rangle(\mathcal{N})\subset\phi(\mathcal{N},\bar{a}_2)\cap\langle C_n,D_n\rangle(\mathcal{N})\\ \subset\cdots\subset\phi(\mathcal{N},\bar{a}_n)\cap\langle C_n,D_n\rangle(\mathcal{N}).$$

A theory has *the strict order property inside a cut* if some of its models has.

The following lemma is an analog of the fact that a stable theory does not have the strict order property.

Lemma 4 (The strict order property inside a cut, [10])**.** *Any o-stable theory does not have the strict order property inside a cut.*

It is well-known that any stable theory does not have the independence property, as it was proved in [2] so do o-stable theories. Thus, we can conclude the following.

Theorem 5 (V. Verbovskiy, [10])**.** *Let a language L contain binary symbol '$<$' and a theory T of L include axioms saying that '$<$' is a total order. The theory T is o-stable if and only if it does not have the independence property and the strict order property inside a cut.*

Let H be a convex subgroup of an ordered group G (not necessarily definable). In the following, we work in the cut $\sup H = \langle H^-\cup H, H^+\rangle$. As in the theory of stable groups for any formula $\varphi(x,\bar{y})$ there is a natural number n such that each chain $K_1\cap H\subset K_2\cap H\subset\cdots\subset K_m\cap H$ has length at most n, provided that K_i is definable by the formula $\varphi(x,\bar{a}_i)$ for some $\bar{a}_i$ and that each subgroup K_i is not bounded in the convex subgroup H. I call this *trivial chain condition* for H. Here I use only the fact that K_i are subsets of G, in other words, the trivial chain condition follows from absence of the strict order property inside a cut. For two subsets A and B of G denote

$$A\overrightarrow{\sqcap}B\triangleq\begin{cases}A\cap B, & \text{if } A\cap B \text{ is not bounded in } B,\\ \{e\} & \text{otherwise.}\end{cases}$$

Note that here $A\overrightarrow{\sqcap}B$ is not necessarily equal to $B\overrightarrow{\sqcap}A$. Using this notation one can rewrite the trivial chain condition as for any formula $\varphi(x,\bar{y})$ there is a natural number n such that each chain $K_1\overrightarrow{\sqcap}H\subset K_2\overrightarrow{\sqcap}H\subset\cdots\subset K_m\overrightarrow{\sqcap}H$ has length at most n, provided that $K_i=\varphi(G,\bar{a}_i)$.

The proofs of the following three lemmata are similar to the proofs of the corresponding facts for stable groups. One can see any textbook on stable groups for details, for instance, [6]. The first lemma is a modification of Baldwin-Saxl condition for stable groups [3].

Lemma 6 (Baldwin–Saxl condition, [10])**.** *For any formula $\varphi(x, \bar{y})$ and any convex subgroup H there is a natural number n such that the intersection of a family of subgroups of the form $K_i \overrightarrow{\sqcap} H$, where $K_i = \varphi(G, \bar{a}_i)$, is the intersection of just n of them. Consequently, subgroups which are finite or infinite intersections of K_i, form an almost uniform family in that sense that for any set of indices I there are $\bar{b}_0, \ldots, \bar{b}_{n-1}$ such that $\bigcap_{i\in I}(K_i \overrightarrow{\sqcap} H) = \bigcap_{j<n}(K_j \overrightarrow{\sqcap} H)$. So it is possible to apply the trivial chain condition.*

Lemma 7 (O-superstable chain condition, [10])**.** *In an o-superstable ordered group G for any convex subgroup H there is no infinite decreasing sequence*

$$K_0 \overrightarrow{\sqcap} H \supset K_1 \overrightarrow{\sqcap} H \supset \cdots \supset K_n \overrightarrow{\sqcap} H \supset \cdots,$$

where K_i are definable subgroups of the group G, such that

$$\left|K_n \overrightarrow{\sqcap} H : K_{n+1} \overrightarrow{\sqcap} H\right| = \infty \quad \text{for each } n.$$

Lemma 8 (O-ω-stable chain condition, [10])**.** *In an o-ω-stable ordered group G for any convex subgroup H there is no an infinite decreasing sequence*

$$K_0 \overrightarrow{\sqcap} H \supset K_1 \overrightarrow{\sqcap} H \supset \cdots \supset K_n \overrightarrow{\sqcap} H \supset \cdots,$$

where K_i are definable subgroups of G.

In [10], the author proved that any ordered group with an o-stable elementary theory is Abelian.

Lemma 9 (V. Verbovskiy, [10])**.** *Let K be a definable unbounded subgroup of a linearly ordered group G with an o-stable theory. Then the elementary theory of G/K with the full induced structure is stable.*

Observe that similarly, it is just routine to prove the following lemma.

Lemma 10. *Let K be a definable unbounded subgroup of a linearly ordered group G with an o-superstable theory. Then the elementary theory of G/K with the full induced structure is superstable.*

A more exact formulation is the following: if $Th(G)$ is o-stable in λ, then $Th(G/K)$ is stable in λ.

Let Σ be the family of all convex (not necessarily definable) subgroups of G. A pair of convex subgroups A and $B \in \Sigma$ are called a *jump*, if A is a proper subgroup of B and there is no subgroup $C \in \Sigma$ such that $A < C < B$.

Based on the lemmata 6, 7, and 8 the following three lemmata have been proved.

Lemma 11 (V. Verbovskiy, [10])**.** *For each natural $n \geq 2$ the number of jumps $A < B$ such that B/A is not n-divisible, is finite.*

Lemma 12 (V. Verbovskiy, [10])**.** *If the elementary theory of G is o-superstable, then $|G : nG| < \infty$ for any positive integer n.*

Lemma 13 (V. Verbovskiy, [10])**.** *If the elementary theory of G is o-ω-stable, then G as a pure ordered group $\big($i. e. in the language $\{<, +, 0\}\big)$ is elementarily equivalent to the ordered group of rationals.*

In [10], the following questions have been posed.

1. Is it true, that totally ordered groups with o-stable theory satisfy the following description: a totally ordered group G is o-stable if and only if G it is Abelian and for any natural number $n \geq 2$ the number of jumps $A < B$ such that the quotient group B/A is not n-divisible, is finite.

2. Is it true, that totally ordered groups with o-superstable theory satisfy the following description: a totally ordered group G is o-superstable if and only if G it is Abelian and $|G : nG| < \infty$ for any positive integer n.

3 Introduction to dp-minimality

Definition 14. 1. An *inp-pattern of depth* κ is a family $\{\varphi_i(\overline{x}; \overline{a}_{i,j}) : i < \kappa, j < \omega\}$ of formulas over parameters $\overline{a}_{i,j}$ and a collection $\{k_i : i < \kappa\}$ of positive integers k_i such that:

   (a) for each $i < \kappa$, the collection of formulas $\{\varphi_i(\overline{x}; \overline{a}_{i,j}) : j < \omega\}$ is k_i-inconsistent (that is, the conjunction of any k_i of them is inconsistent); and

   (b) for each function $\eta : \kappa \to \omega$, the partial type $\{\varphi_i(\overline{x}; \overline{a}_{i,\eta(i)}) : i < \kappa\}$ is consistent.

2. An *ict-pattern of depth* κ is a family $\{\varphi_i(\overline{x}; \overline{a}_{i,j}) : i < \kappa, j < \omega\}$ of formulas over parameters $\overline{a}_{i,j}$ such that for every $\eta : \kappa \to \omega$, the partial type

$$\{\varphi_i^{if j=\eta(i)}(\overline{x}; \overline{a}_{i,j}) : i < \kappa\}$$

   is consistent, where the exponent "if $j = \eta(i)$" signifies that we take a positive instance of $\varphi(\overline{x}; \overline{a}_{i,j})$ if $j = \eta(i)$, and its negation otherwise.

3. Let κ_{inp} be the supremum of the depths of all inp-patterns in T in which the tuple $\overline{x}$ in the definition above consists of a single variable x (or "∞" if the depths are not bounded), and similarly let κ_{ict} be the supremum of the depths of all ict-patterns in T in a single variable.

4. T is *dp-minimal* if $\kappa_{ict} = 1$, and T is *inp-minimal* if $\kappa_{inp} = 1$.

From the proof of Proposition 10 of [1], we find:

Theorem 15 (H. Adler, [1])**.** *The theory T is NIP if and only if $\kappa_{ict} < \infty$, and if T is NIP, then $\kappa_{ict} = \kappa_{inp}$. In particular, "dp-minimal" is equivalent to "NIP and inp-minimal."*

An Abelian group G is said to be *non-singular* if G/pG is finite for all primes p. We use the following characterization of dp-minimal ordered groups.

Theorem 16 (F. Jahnke, P. Simon and E. Walsberg, [5])**.** *The following are equivalent:*

1. *$(G, +, \leq)$ is non-singular.*
2. *$(G, +, \leq)$ is dp-minimal.*
3. *There is a definitional expansion of $(G, +, \leq)$ by countably many formulas which is weakly quasi-o-minimal.*

Lemma 17 (P. Simon, [7])**.** *Let $\phi(x; \bar{y})$ be a formula with parameters in some densely ordered structure $\mathcal{M}_0$ whose elementary theory is dp-minimal. Then there are $\bar{b}_1, \ldots, \bar{b}_n$ such that for every $\bar{b}$, there is $\alpha \in M_0$ and k such that the sets defined by $\phi(x; \bar{b}) \wedge \alpha < x$ and $\phi(x; \bar{b}_k) \wedge \alpha < x$ are equal.*

Lemma 18 (P. Simon, [7])**.** *Let $\phi(x; \bar{y})$ be a formula with parameters in some densely ordered structure $\mathcal{M}_0$ whose elementary theory is dp-minimal. Then there exists n such that for any point a, there are $\bar{b}_1, \ldots, \bar{b}_n$ such that for all $\bar{b}$, there are $\alpha, \beta \in M_0$ and k such that $\alpha < a < \beta$ and the sets defined by $\phi(x; \bar{b}) \wedge \alpha < x < \beta$ and $\phi(x; \bar{b}_k) \wedge \alpha < x < \beta$ are equal.*

In [9], it has been proved that if the elementary theory T of an ordered structure is dp-minimal, then T is o-stable. Here we give a stronger result.

Lemma 19. *Let the elementary theory T of an ordered structure of a language $\mathcal{L}$ be dp-minimal. Then T is o-superstable.*

Proof. Let $\mathcal{M} \vdash T$, and let s be a cut in $\mathcal{M}$. By Lemmata 17 and 18 for any formula $\phi(x; \bar{y})$ the number of ϕ-types which are consistent with s is finite. Then there exist at most $2^{|\mathcal{L}|}$ complete types over M which are consistent with s. Thus T is o-superstable. □

4 Main results

Theorem 20. *Let $(G, <, +)$ be a pure ordered Abelian group. Then the following is equivalent:*

1. *$(G, <, +)$ is non-singular.*

2. $(G, <, +)$ *is dp-minimal.*

3. *There is a definitional expansion of* $(G, +, \leq)$ *by countably many formulas which is weakly quasi-o-minimal.*

4. $(G, <, +)$ *is o-superstable.*

Proof. The equivalence of the first three items follows from Theorem 16. The last condition implies the first one by Lemma 12. The second item implies the last one by Lemma 19. □

E. Palyutin asked the following question. Let G be an orderable Abelian group with a superstable theory. Let $(G, <, +)$ be a such expansion of G that this group is ordered. Does it follow that $Th(G, <, +)$ is o-superstable? Here we give a positive answer to this question.

Recall that a group is said to be *orderable*, if there exists an expansion of this group by a linear order which makes this group ordered. It is well known that an Abelian group is orderable if and only if it is torsion-free.

Theorem 21. *Let G be an Abelian torsion-free group with a superstable theory. Then any its expansion by a linear order which makes this group ordered has an o-superstable theory.*

Proof. Let $(G, +)$ be an Abelian torsion-free group with a superstable theory. Then it is non-singular. Then $(G, <, +)$ is non-singular, because this notion depends only on the pure group structure. If $(G, <, +)$ is an ordered group, then by Theorem 20 it has an o-superstable theory. □

A similar question can be asked for Abelian torsion-free groups with a stable theory. But here we have a negative answer.

Theorem 22. *There exists an Abelian torsion-free group G such that one its ordering makes this group o-stable while another ordering makes this group o-unstable.*

Proof. Let G be the direct power $(\mathbb{Z}, +)^\omega$. Let p_i be the i-th prime and G_R be the direct sum of $\sqrt{p_i} \cdot \mathbb{Z}$. Then there exists an isomorphism τ from G to G_R. Since G_R is a subset of $\mathbb{R}$ we can consider $(G_R, <)$ where the ordering $<$ comes from the natural ordering of $\mathbb{R}$. We define the first ordering $<_1$ on G as follows: $g_1 <_1 g_2$ if and only if $\tau(g_1) < \tau(g_2)$. In [8], it has been proved that $Th(\mathbb{R}, <, +, H)$ with an augmented dense subgroup H admits quantifier elimination in the language

$$\{<, +, -, H, H_q, c_{p,q,i}\}_{p,a \in \mathbb{Q}^+, i < n_{p,q}},$$

where $H_q = q \cdot H$ for any $q \in \mathbb{Q}^+$, $n_{p,q}$ is equal $[H_q : H_p]$ whenever $H_p \leq H_q$, and $c_{p,q,i}$ are representatives of all cosets of H_p in H_q. Obviously, if we consider the elementary theory T_H of restricting this structure to the definable subset H, then it also admits quantifier elimination.

Thus, $T_G \triangleq Th(G, <_1, +)$ admits the elimination of the quantifier in an appropriate expansion. Assume that T_G is not o-stable. Then T_G has either the independence property or the strict order property within a cut. Due to the theorem by Y. Gurevich and Peter H. Schmitt, [4] the elementary theory of any Abelian ordered group does not have the independence property. That is why, for some formula $\varphi(x;\bar{y})$ and some cut $\langle C, D\rangle$ in some elementary extension of $(G, <_1, +)$, we obtain that $\varphi(x;\bar{y})$ has the strict order property inside the cut $\langle C, D\rangle$. By the quantifier elimination theorem, this formula φ is a Boolean combination of intervals and formulae in the language of a pure group. We can rewrite this Boolean combination in a disjunctive normal form. Since the intersection of intervals is an interval, we can assume that φ is the conjunction of an interval and a formula in the language of a pure group. Since we consider the strict order property inside a cut, we can remove the interval from φ. Hence, we obtain a formula φ with the strict order property in the language of a pure group, that is, in $(G, +)$. But it is well known that any Abelian group has a stable theory, so, it does not have the strict order property, for a contradiction. So, $Th(G, <_1, +)$ is o-stable.

Let $<_2$ be a lexicographical ordering of G. Then it is easy to see that $Th(G, <_2, +)$ is not o-stable, because the formula

$$\varphi(x, y) \triangleq 0 <_2 y \wedge \exists z(0 \leq_2 z \leq_2 y \wedge \exists t(t + t = z + x))$$

has the strict order property inside the cut $+\infty$. □

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Вербовский В. В. О-ТҰРАҚТЫ ТАЗА РЕТТЕЛГЕН ТОПТАРҒА ЕСКЕРТУ

Бұл мақалада біз қарапайым теориясы о-тұрақты немесе о-өте тұрақты болып табылатын таза реттелген топтарды сипаттаймыз. о-тұрақтылық түсінігі тұрақтылық пен о-минималдылықтың табиғи қосындысы болып табылады, атап айтқанда, реттелген құрылым, егер оның кез келген кесінділері бірнеше толық бір типке сәйкес келсе, о-тұрақты деп аталады.

Түйін сөздер: реттелген топ, NIP теория, о-минималды теория, о-тұрақты теория, о-супертұрақты теория.

Вербовский В. В. ЗАМЕЧАНИЕ ОБ О-СТАБИЛЬНЫХ ЧИСТЫХ УПОРЯДОЧЕННЫХ ГРУППАХ

В этой статье мы описываем чистые упорядоченные группы, элементарная теория которых является о-стабильной или о-суперстабильной. Понятие о-стабильности является естественным сочетанием стабильности и о-минимальности, а именно, упорядоченная структура называется о-стабильной, если любое ее сечение совместно с малым числом полных один-типов.

Ключевые слова: упорядоченная группа, NIP теория, о-минимальная теория, о-стабильная теория, о-суперстабильная теория.