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Solution of the heat equation with a discontinuous coefficient with nonlocal boundary conditions by the Fourier method

Umbetkul K. Koilyshov[1], Nurmukhamed T. Abdyrakhimov[2]

[1,2]Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan

[1,2]Al-Farabi Kazakh National University, Almaty, Kazakhstan

[1]koylyshov@math.kz, [2]nabdyrakhimov@mail.ru

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Abstract. This paper substantiates the solution by the method of separation of variables of the initial-boundary value problem for the heat equation with a discontinuous coefficient, under periodic or anti-periodic boundary conditions. Using the Fourier method, this problem is reduced to the corresponding spectral problem. The eigenvalues and eigenfunctions of this spectral problem are found. It is shown that the spectral problem is non-self-adjoint and a conjugate spectral problem of this original spectral problem is constructed. Further, it is proved that the system of eigenfunctions forms a Riesz basis. For this purpose, a self-adjoint spectral problem is constructed and its eigenvalues and eigenfunctions are found. In conclusion, using biorthogonality, the main theorem on the existence and uniqueness of a classical solution to the problem is proven.

Keywords. Heat equation with discontinuous coefficients, spectral problem, non-self-adjoint problem, Riesz basis, classical solution, Fourier method.

1 Introduction
Problem statement and research methods

We consider an initial boundary value problem for the heat equation with a piecewise constant coefficient

$$\frac{\partial u}{\partial t} = k_i^2 \frac{\partial^2 u}{\partial x^2} + f(x,t) \tag{1}$$

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in the domain $\Omega = \Omega_1 \cup \Omega_2$, where

$$\Omega_1 = \{(x,t) : 0 < x < x_0,\ 0 < t < T\},\ \Omega_2 = \{(x,t) : x_0 < x < l,\ 0 < t < T\}$$

with the initial condition

$$u(x,0) = \varphi(x), \quad 0 \le x \le l, \tag{2}$$

the boundary conditions of the form

$$\begin{cases} u(0,t) + (-1)^m u(l,t) = 0, \\ k_1 \frac{\partial u(0,t)}{\partial x} + (-1)^m k_2 \frac{\partial u(l,t)}{\partial x} = 0, \end{cases} \qquad 0 \le t \le T, \tag{3}$$

and with the conjugation conditions

$$\begin{cases} u(x_0-0,t) = u(x_0+0,t), \\ k_1 \frac{\partial u(x_0-0,t)}{\partial x} = k_2 \frac{\partial u(x_0+0,t)}{\partial x}, \end{cases} \tag{4}$$

where the point x_0 is a strictly internal point of the interval $(0,l)$, that is, $0 < x_0 < l$. The coefficients $k_i > 0$, $(i = 1,2)$, $m = 1,2$.

Parabolic type equations with discontinuous coefficients have been studied quite well [1–5]. In these works, the correctness of various initial boundary value problems for a parabolic type equation with discontinuous coefficients was proven using the Green's function and thermal potentials method. In the case without a discontinuity, the spectral theory of these problems is constructed almost completely [6–12]. In [13–14], some properties of the eigenfunctions of the Sturm-Liouville operator with discontinuous coefficients were studied. In the case of a discontinuous coefficient, the spectral theory of such problems is considered in the works [15–17].

First, we consider the case $m = 1$. We look for a solution to Problem (1)–(4) in the form $u(x,t) = v(x,t) + w(x,t)$, where $v(x,t)$ is a solution to the following problem A:

$$\frac{\partial v}{\partial t} = k_i^2 \frac{\partial^2 v}{\partial x^2}$$

$$v(x,0) = \varphi(x), \quad 0 \le x \le l,$$

$$\begin{cases} v(0,t) - v(l,t) = 0, \\ k_1 \frac{\partial v(0,t)}{\partial x} - k_2 \frac{\partial v(l,t)}{\partial x} = 0, \end{cases} \qquad 0 \le t \le T,$$

$$\begin{cases} v(x_0-0,t) = v(x_0+0,t), \\ k_1 \frac{\partial v(x_0-0,t)}{\partial x} = k_2 \frac{\partial v(x_0+0,t)}{\partial x}, \end{cases}$$

where $w(x,t)$ is a solution to the following problem B:

$$\frac{\partial w}{\partial t} = k_i^2 \frac{\partial^2 w}{\partial x^2} + f(x,t)$$

$$w(x,0) = 0, \quad 0 \le x \le l,$$

$$\begin{cases} w(0,t) - w(l,t) = 0, \\ k_1 \frac{\partial w(0,t)}{\partial x} - k_2 \frac{\partial w(l,t)}{\partial x} = 0, \end{cases} \qquad 0 \le t \le T,$$

$$\begin{cases} w(x_0 - 0, t) = w(x_0 + 0, t), \\ k_1 \frac{\partial w(x_0 - 0,t)}{\partial x} = k_2 \frac{\partial w(x_0+0,t)}{\partial x}, \end{cases}$$

Let W denote the linear variety of functions from the class

$$u(x,t) \in C(\overline{\Omega}) \cup C^{2,1}(\overline{\Omega_1}) \cup C^{2,1}(\overline{\Omega_2})$$

which satisfy all conditions (2)–(4).

We call a function $u(x,t)$ from the class $u(x,t) \in W$ a *classical solution* to Problem (1)–(4), if

1) it is continuous in the domain $\overline{\Omega}$;

2) has in the domain continuous derivatives of the first order with respect to t and continuous derivatives of the second order with respect to x;

3) satisfies Equation (1) and all Conditions (2)–(4) in the usual, continuous sense.

We look for a solution to problem A using the Fourier method: $v_j(x,t) = X_j(x) \cdot T(t) \neq 0$. Substituting the boundary conditions and the pairing conditions into the equations, and separating the variables, we obtain the following spectral problem

$$LX(x) = \begin{Bmatrix} -k_1^2 X''(x), & 0 < x < x_0 \\ -k_2^2 X''(x), & x_0 < x < l \end{Bmatrix} = \lambda X(x) \tag{5}$$

$$\begin{cases} X_1(0) - X_2(l) = 0 \\ k_1 X_1'(0) - k_2 X_2'(l) = 0 \end{cases} \tag{6}$$

$$X_1(x_0 - 0) = X_2(x_0 + 0), \quad k_1 X_1'(x_0 - 0) = k_2 X_2'(x_0 + 0), \tag{7}$$

The function $T(t)$ is a solution to the equation

$$T'(t) + \lambda T(t) = 0.$$

Now we need to find the eigenvalues and eigenfunctions of Problem (5)–(7). The general solution to Equation (5) has the form

$$\begin{cases} X(x) = c_1 \cos \mu_1 x + c_2 \sin \mu_1 x, & 0 < x < x_0, \\ X(x) = d_1 \cos \mu_2 x + d_2 \sin \mu_2 x, & x_0 < x < l, \end{cases} \tag{8}$$

where $\mu_i = \dfrac{\sqrt{\lambda}}{k_i}, \quad (i = 1, 2).$

Substituting the general solution (8) into the boundary conditions (6) and the conjugation conditions (7), and taking into account that $\mu_1 k_1 = \mu_2 k_2 = \sqrt{\lambda}$ we obtain

$$\begin{cases} c_1 = d_1\cos(\mu_2 l) + d_2\sin(\mu_2 l) \\ c_2 = -d_1\sin(\mu_2 l) + d_2\cos(\mu_2 l) \\ c_1\cos(\mu_1 x_0) + c_2\sin(\mu_1 x_0) - d_1\cos(\mu_2 x_0) - d_2\sin(\mu_2 x_0) = 0 \\ -c_1\sin(\mu_1 x_0) + c_2\sin(\mu_1 x_0) + d_1\sin(\mu_2 x_0) - d_2\cos(\mu_2 x_0) = 0 \end{cases} \tag{9}$$

We find the characteristic determinant of the system (9):

$$\Delta(\lambda) = 2\Big(1 - cos\Big(\frac{\sqrt{\lambda}}{r}\Big)\Big) = 4sin^2\Big(\frac{\sqrt{\lambda}}{2r}\Big) = 0, \tag{10}$$

where

$$r = \frac{1}{\frac{x_0}{k_1} + \frac{l-x_0}{k_2}}. \tag{11}$$

From Equation (11) one can find the eigenvalues (they are twofold)

$$\lambda_n = (2\pi n r)^2, \quad \text{where } n = 0,\ 1,\ 2, \ldots \tag{12}$$

These eigenvalues correspond to the eigenfunctions

$$X_n(x) = C\begin{cases} sin(\frac{2\pi n r}{k_1}x), & 0 < x < x_0, \\ sin(\frac{2\pi n r}{k_2}(x-l)), & x_0 < x < l, \end{cases} \tag{13}$$

$$\tilde{X}_n(x) = C\begin{cases} cos(\frac{2\pi n r}{k_1}x), & 0 < x < x_0, \\ cos(\frac{2\pi n r}{k_2}(x-l)), & x_0 < x < l, \end{cases} \tag{14}$$

where r determined by the formula (11).

Lemma 1. *Spectral problem (5)–(7) is non-self-adjoint. The adjoint problem to problem (5)–(7) has the following form:*

$$LY(x) = \begin{Bmatrix} -k_1^2 Y''(x), & 0 < x < x_0 \\ -k_2^2 Y''(x), & x_0 < x < l \end{Bmatrix} = \lambda Y(x) \tag{15}$$

$$\begin{cases} k_1 Y_1(0) - k_2 Y_2(l) = 0 \\ k_1^2 Y_1'(0) - k_2^2 Y_2'(l) = 0 \end{cases} \tag{16}$$

$$k_1 Y_1(x_0 - 0) = k_2 Y_2(x_0 + 0), \quad k_1^2 Y_1'(x_0 - 0) = k_2^2 Y_2'(x_0 + 0), \tag{17}$$

Proof. We find the conjugate problem to Problem (5)–(7). Given the following formula

$$-X''(x)Y(x) = (Y'(x)X(x) - Y(x)X'(x))' - Y''(x)X(x)$$

we obtain

$$\int_0^l Y(x)LX(x)\,dx = -\int_0^{x_0} Y(x)k_1^2X''(x)\,dx - \int_{x_0}^l Y(x)k_2^2X''(x)\,dx =$$

$$= -k_1^2Y(x_0-0)X'(x_0-0) + k_1^2Y(0)X'(0) + k_1^2Y'(x_0-0)X(x_0-0) +$$
$$+k_1^2Y'(0)X(0) - k_2^2Y(l)X'(l) + k_2^2Y(x_0+0)X'(x_0+0) + k_2^2Y'(l)X(l) -$$
$$-k_2^2Y'(x_0+0)X(x_0+0) + \int_0^l X(x)LY(x)\,dx.$$

Using boundary conditions (6) and pairing conditions (7) we have

$$\int_0^l Y(x)LX(x)dx = X(x_0+0)\Big(k_1^2Y'(x_0-0) - k_2^2Y'(x_0+0)\Big) +$$
$$+k_1X'(x_0-0)\Big(k_2Y(x_0+0) - k_1Y(x_0-0)\Big) +$$
$$+k_1X'(0)\Big(k_1Y(0) - k_2Y(l)\Big) + X(0)\Big(k_2^2Y'(l) - k_1^2Y'(0)\Big) + \int_0^l X(x)LY(x)dx.$$

From the last equality it follows that the formula

$$\int_0^l Y(x)LX(x)dx = \int_0^l X(x)LY(x)dx$$

is executed only if Conditions (16)–(17). It follows that Problem (5)–(7) is not self-adjoint. □

Lemma 2. *The following problem is self-adjoint.*

$$LZ(x) = \begin{Bmatrix} -k_1^2Z''(x), & 0 < x < x_0 \\ -k_2^2Z''(x), & x_0 < x < l \end{Bmatrix} = \lambda Z(x) \tag{18}$$

$$\begin{cases} \sqrt{k_1}Z_1(0) - \sqrt{k_2}Z_2(l) = 0 \\ k_1^{\frac{3}{2}}Z_1'(0) - k_2^{\frac{3}{2}}Z_2'(l) = 0 \end{cases} \tag{19}$$

$$\sqrt{k_1}Z_1(x_0-0) = \sqrt{k_2}Z_2(x_0+0), \quad k_1^{\frac{3}{2}}Z_1'(x_0-0) = k_2^{\frac{3}{2}}Z_2'(x_0+0), \tag{120}$$

The proof of this lemma is similar to the proof of the previous one lemma 1. The eigenvalues of the spectral problem (18)–(20) are equal to $\lambda_n = (2\pi nr)^2$, where $(n = 0, 1, 2, ...)$, and two-fold, i.e. coincide with the eigenvalues of problem (5)–(7). The eigenfunctions are equal

$$Z_n(x) = C\begin{cases} \frac{1}{\sqrt{k_1}} sin(\frac{2\pi nr}{k_1}x), & 0 < x < x_0, \\ \frac{1}{\sqrt{k_2}} sin(\frac{2\pi nr}{k_2}(x-l)), & x_0 < x < l, \end{cases}$$

$$\tilde{Z}_n(x) = C\begin{cases} \frac{1}{\sqrt{k_1}} cos(\frac{2\pi nr}{k_1}x), & 0 < x < x_0, \\ \frac{1}{\sqrt{k_2}} cos(\frac{2\pi nr}{k_2}(x-l)), & x_0 < x < l, \end{cases}$$

From the normalization condition we find $C = \sqrt{2r}$, where r is determined by formula (11). Then finally, the eigenfunctions of Problem (18)-+(20) have the form:

$$Z_n(x) = \sqrt{2r}\begin{cases} \frac{1}{\sqrt{k_1}} sin(\frac{2\pi nr}{k_1}x), & 0 < x < x_0, \\ \frac{1}{\sqrt{k_2}} sin(\frac{2\pi nr}{k_2}(x-l)), & x_0 < x < l, \end{cases} \tag{21}$$

$$\tilde{Z}_n(x) = \sqrt{2r}\begin{cases} \frac{1}{\sqrt{k_1}} cos(\frac{2\pi nr}{k_1}x), & 0 < x < x_0, \\ \frac{1}{\sqrt{k_2}} cos(\frac{2\pi nr}{k_2}(x-l)), & x_0 < x < l, \end{cases} \tag{22}$$

Lemma 3. *The system of the eigenfunctions (13)–(14) forms the Riesz basis.*

Proof. From the formulas (13)–(14) and (21)–(22) it is easy to notice that the eigenvalues of Problem (18)–(20) and Problem (5)–(7) coincide, while the eigenfunctions differ by a piecewise constant factor. From the formulas (13)–(14) and (21)–(22) it is clear that the eigenfunctions of Problem (5)–(7) and (18)–(20) are related by the following equality:

$$\begin{pmatrix} Z_n(x) \\ \tilde{Z}_n(x) \end{pmatrix} = \alpha(x)\begin{pmatrix} X_n(x) \\ \tilde{X}_n(x) \end{pmatrix}, \quad where \quad \alpha(x) = \begin{cases} \frac{1}{\sqrt{k_1}}, & 0 < x < x_0, \\ \frac{1}{\sqrt{k_2}}, & x_0 < x < l \end{cases} \tag{23}$$

Sinse $Z_n(x)$ and $\tilde{Z}_n(x)$ are the eigenfunctions of the self-adjoint problem (18)–(20), the system $Z_n(x)$, $\tilde{Z}_n(x)$ of eigenfunctions forms an $L_2(0, l)$ orthonormal basis [18]. We rewrite formula (23) in the following form:

$$\begin{pmatrix} Z_n(x) \\ \tilde{Z}_n(x) \end{pmatrix} = A\begin{pmatrix} X_n(x) \\ \tilde{X}_n(x) \end{pmatrix}, \quad where \quad A\begin{pmatrix} X_n(x) \\ \tilde{X}_n(x) \end{pmatrix} = \alpha(x)\begin{pmatrix} X_n(x) \\ \tilde{X}_n(x) \end{pmatrix},$$

$A : L_2(0, l) \to L_2(0, l)$ is a bounded operator and there exists A^{-1} that is also bounded. It follows that the system of the eigenfunctions $X_n(x)$, $\tilde{X}_n(x)$ forms a Riesz basis.

Now we find the eigenvalues and the eigenfunctions of the conjugate problem (15)–(17). The eigenvalues of the conjugate problem are not difficult to find, they are equal $\lambda_n = (2\pi nr)^2$, where $(n = 0, 1, 2, ...)$, and they are also twofold and coincide with the eigenvalues of Problem (5)–(7). The eigenfunctions are defined as follows:

$$Y_n(x) = C \begin{cases} \frac{1}{k_1} sin(\frac{2\pi nr}{k_1}x), & 0 < x < x_0, \\ \frac{1}{k_2} sin(\frac{2\pi nr}{k_2}(x-l)), & x_0 < x < l, \end{cases} \tag{24}$$

$$\tilde{Y}_n(x) = C \begin{cases} \frac{1}{k_1} cos(\frac{2\pi nr}{k_1}x), & 0 < x < x_0, \\ \frac{1}{k_2} cos(\frac{2\pi nr}{k_2}(x-l)), & x_0 < x < l, \end{cases} \tag{25}$$

It follows from the general theory that the system of eigenfunctions $X_n(x)$, $\tilde{X}_n(x)$ and $Y_n(x)$, $\tilde{Y}_n(x)$ is biorthogonal, i.e.

$$\int_0^l X_n(x)\tilde{Y}_m(x)dx = 0 \;\text{ and }\; \int_0^l \tilde{X}_n(x)Y_m(x)dx = 0,$$

for any $(n, m = 1, 2, ...)$, and

$$\int_0^l X_n(x)Y_m(x)dx = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases} \quad \text{and } \int_0^l \tilde{X}_n(x)\tilde{Y}_m(x)dx = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases}$$

From the normalization condition we find $C = \sqrt{2r}$. □

Now we prove the main theorem.

Theorem 4. *Let $\varphi(x)$ be a continuously differentiable function satisfying the conditions $\varphi(0) = \varphi(l)$, $k_1\varphi'(0) = k_2\varphi'(l)$, $\varphi(x_0 - 0) = \varphi(x_0 + 0)$, $k_1\varphi'(x_0 - 0) = k_2\varphi'(x_0 + 0)$.*

Then the function

$$v(x,t) = \sum_{n=1}^{\infty} \left(\varphi_n X_n(x) + \widetilde{\varphi}_n \widetilde{X}_n(x)\right) e^{-\lambda_n t} \tag{26}$$

where the coefficients $\varphi_n, \tilde{\varphi}_n$ are determined by the formulas

$$\varphi_n = \int_0^l \varphi(x)Y_n(x)dx, \tilde{\varphi}_n = \int_0^l \varphi(x)\tilde{Y}_n(x)dx \tag{27}$$

is the only classical solution to problem A.

Proof. First, we prove the existence of solution (26). Since $X_n(x)$, $\tilde{X}_n(x)$ are the eigenfunctions and the eigenvalues of Problem (5)–(7), then it is easy to verify that the function $v(x,t)$ determined by formula (26) satisfies the equation, initial condition, boundary conditions and pairing conditions of problem A. Series (26) is the sum of functions

$$v_n(x,t) = \left(\varphi_n X_n(x) + \widetilde{\varphi}_n \widetilde{X}_n(x)\right) e^{-\lambda_n t} \tag{28}$$

We show that when $t \geq \varepsilon > 0$ (here, ε is an arbitrary positive number) the series

$$\sum_{n=1}^{\infty} v_n(x,t), \quad \sum_{n=1}^{\infty} \frac{\partial v_n}{\partial t}, \quad \sum_{n=1}^{\infty} \frac{\partial^2 v_n}{\partial x^2}$$

converges uniformly. Obviously, $|\varphi| \leq M_1$. Then from the formula (27) it follows that $\left\{|\varphi_n|, |\widetilde{\varphi}_n|\right\} \leq M_2$. Then from Equality (28) and from the following equalities

$$\frac{\partial v_n}{\partial t} = \left(-\lambda_n X_n(x)\varphi_n - \lambda_n \widetilde{X}_n(x)\widetilde{\varphi}_n\right) e^{-\lambda_n t}, \quad \frac{\partial^2 v_n}{\partial x^2} = \frac{\lambda_n}{k_j^2} \left(-X_n(x)\varphi_n - \widetilde{X}_n(x)\widetilde{\varphi}_n\right) e^{-\lambda_n t},$$

we obtain

$$|v_n(x,t)| \leq M_3 e^{-\lambda_n \varepsilon}, \quad \left\{\left|\frac{\partial v_n}{\partial t}\right|, \left|\frac{\partial^2 v_n}{\partial x^2}\right|\right\} \leq M_4 \lambda_n e^{-\lambda_n \varepsilon},$$

where the constants M_3 and M_4 are positive and does not depend on n. Thus

$$\left\{\sum_{n=1}^{\infty}\Big|v_n(x,t)\Big|, \sum_{n=1}^{\infty}\left|\frac{\partial v_n}{\partial t}\right|, \sum_{n=1}^{\infty}\left|\frac{\partial^2 v_n}{\partial x^2}\right|\right\} \leq \sum_{n=1}^{\infty} M n^2 e^{-\left(2\pi n r\right)^2 \varepsilon},$$

where $M > 0$ and does not depend on n. Since the series

$$\sum_{n=1}^{\infty} M n^2 e^{-\left(2\pi n r\right)^2 \varepsilon}$$

is an absolutely convergent series, therefore, according to Weierstrass's test, the series

$$\left\{\sum_{n=1}^{\infty}\Big|v_n(x,t)\Big|, \sum_{n=1}^{\infty}\left|\frac{\partial v_n}{\partial t}\right|, \sum_{n=1}^{\infty}\left|\frac{\partial^2 v_n}{\partial x^2}\right|\right\}$$

converge uniformly for $t \geq \varepsilon$ and the functions $v(x,t)$, $\dfrac{\partial v(x,t)}{\partial t}$, $\dfrac{\partial^2 v(x,t)}{\partial x^2}$ are continuous for $t \geq \varepsilon$.

Now we need to prove that series (26) converges uniformly everywhere in $\overline{\Omega}$. Note that the n-th term of the series (26) is dominated by the sum $|\varphi_n| + |\widetilde{\varphi}_n|$. Integrating by parts the integral in formula (27), we obtain

$$|\varphi_n| \leq \frac{C_1}{2\pi r} \cdot \frac{|\alpha_n|}{n}, \quad |\widetilde{\varphi}_n| \leq \frac{C_1}{2\pi r} \cdot \frac{|\widetilde{\alpha}_n|}{n}, \quad C_1 = \max\left(\sqrt{k_1}, \sqrt{k_2}\right),$$

where

$$\alpha_n = \int_0^l \varphi'(x)\tilde{Z}_n(x)dx \;\; \text{and} \;\; \widetilde{\alpha}_n = \int_0^l \varphi'(x) Z_n(x)dx$$

are Fourier coefficients of the function $\varphi'(x)$ with respect to the eigenfunctions $Z_n(x)$, $\tilde{Z}_n(x)$ orthonormal on an interval $[0,l]$, determined by the formulas (21)–(22). It is known that the eigenfunctions $Z_n(x)$, $\tilde{Z}_n(x)$ form an orthonormal basis. (See Lemma 2). Taking into account the inequality $ab \leq \frac{1}{2}(a^2+b^2)$ we have

$$|\varphi_n| + |\widetilde{\varphi}_n| \leq \frac{C_1}{4\pi r} \cdot \left(\alpha_n^2 + \widetilde{\alpha}_n^2 + \frac{2}{n^2}\right).$$

Using the Bessel inequality

$$\sum_{n=1}^{\infty} \left(\alpha_n^2 + \widetilde{\alpha}_n^2\right) \leq \|\varphi'\|^2,$$

and the well-known equality $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, we obtain

$$\sum_{n=1}^{\infty} (|\varphi_n| + |\widetilde{\varphi}_n|) \leq C.$$

Thus, the majorizing series is absolutely convergent, this means series (26) converges uniformly in $\overline{\Omega}$ and defines a continuous function $v(x,t)$ in $\overline{\Omega}$. Thus, we proved the existence of a solution. Now we prove its uniqueness. We assume that there are two solutions $\widetilde{v}(x,t)$ and $\widehat{v}(x,t)$. Then for the function $v(x,t) = \widetilde{v}(x,t) - \widehat{v}(x,t)$ we have the following *problem* C:

$$\frac{\partial v}{\partial t} = k_j^2 \frac{\partial^2 v}{\partial x^2},$$

$$v(x,0) = 0, \quad 0 \leq x \leq l,$$

$$\begin{cases} v(0,t) - v(l,t) = 0, \\ k_1 \dfrac{\partial v(0,t)}{\partial x} - k_2 \dfrac{\partial v(l,t)}{\partial x} = 0, \end{cases} \qquad 0 \leq t \leq T,$$

$$\begin{cases} v(x_0 - 0, t) = v(x_0 + 0, t), \\ k_1 \dfrac{\partial v(x_0 - 0, t)}{\partial x} = k_2 \dfrac{\partial v(x_0 + 0, t)}{\partial x} \end{cases}.$$

The solution to this *problem* C can be represented in the form of an expansion in terms of the basis $\left\{X_n(x), \widetilde{X}_n(x)\right\}$ and it has the form:

$$v(x,t) = \sum_{n=1}^{\infty} \left(A_n(t) X_n(x) + \widetilde{A}_n(t) \widetilde{X}_n(x)\right). \tag{29}$$

The coefficients $A_n(t)$ and $\widetilde{A}_n(t)$ are easy to find if we multiply both sides of equality (29) respectively by the functions $Y_n(x)$ and $\widetilde{Y}_n(x)$, and integrate the resulting relationship from 0 to l and take into account the biorthogonality of the sequences $\left\{X_n(x), \widetilde{X}_n(x)\right\}$ and $\left\{Y_n(x), \widetilde{Y}_n(x)\right\}$ Then we obtain

$$A_n(t) = \int_0^l v(x,t)Y_n(x)dx, \quad \widetilde{A}_n = \int_0^l v(x,t)\widetilde{Y}_n(x)dx. \tag{30}$$

First we transform the first equality in formula (30). Differentiating with respect to the variable t we obtain

$$A_n'(t) = \int_0^l \frac{\partial v(x,t)}{\partial t} Y_n(x)dx =$$

$$= k_1 \int_0^{x_0} \frac{\partial^2 v(x,t)}{\partial x^2} \sin\left(\frac{2\pi nr}{k_1}x\right)dx + k_2 \int_{x_0}^l \frac{\partial^2 v(x,t)}{\partial x^2} \sin\left(\frac{2\pi nr}{k_2}(x-l)\right)dx$$

Integrating by parts twice and using the boundary conditions and conjugation conditions, we have

$$A_n'(t) = -\frac{(2\pi nr)^2}{k_1} \int_0^{x_0} v(x,t) \sin\left(\frac{2\pi nr}{k_1}x\right)dx -$$

$$-\frac{(2\pi nr)^2}{k_2} \int_{x_0}^l v(x,t) \sin\left(\frac{2\pi nr}{k_2}(x-l)\right)dx = -\lambda_n \int_0^l v(x,t)Y_n(x)dx = -\lambda_n A_n(t),$$

Therefore $A_n(t) = c_n e^{-\lambda_n t}, (n = 1, 2, ...)$. Transforming in a similar way we obtain for the coefficient $\widetilde{A}_n(t)$ the following:

$$\widetilde{A}_n'(t) = -\lambda_n \widetilde{A}_n(t) \quad \Rightarrow \quad \widetilde{A}_n(t) = \widetilde{c}_n e^{-\lambda_n t}.$$

Substituting the found $A_n(t)$ and $\widetilde{A}_n(t)$ into formula (30) we obtain

$$\int_0^l v(x,t)Y_n(x)dx = c_n e^{-\lambda_n t}, \quad \int_0^l v(x,t)\widetilde{Y}_n(x)dx = \widetilde{c}_n e^{-\lambda_n t}. \tag{31}$$

Passing to the limit $t \to 0$ in equality (31) what is possible due to continuity $v(x,t)$ in $\overline{\Omega}$, we obtain

$$\lim_{t\to 0} \int_0^l v(x,t)Y_n(x)dx = 0 = A_n(0), \quad \lim_{t\to 0} \int_0^l v(x,t)\widetilde{Y}_n(x)dx = 0 = \widetilde{A}_n(0),$$

therefore $c_n = 0, \quad \widetilde{c}_n = 0, \quad (n = 1, 2, ...)$.

Then from Formula (29) we obtain $v(x,t) = 0$. It follows from this that $\widetilde{v}(x,t) = \widehat{v}(x,t)$. The theorem is proved. □

Knowing the solution to problem A, it is not difficult to obtain a solution to problem B. This solution is given by the formula

$$w(x,t) = \sum_{n=1}^{\infty} \left(\int_0^t f_n e^{-\lambda_n(t-\tau)} X_n(x) + \int_0^t \widetilde{f}_n e^{-\lambda_n(t-\tau)} \widetilde{X}_n(x) \right), \tag{32}$$

where

$$f_n(\tau) = \int_0^l f(x,\tau) Y_n(x) dx, \quad \widetilde{f}_n(\tau) = \int_0^l f(x,\tau) \widetilde{Y}_n(x) dx.$$

Adding (26) and (32) we obtain a solution to Problem (1)–(4).

Now consider the case $m = 2$. Then, after applying the method of separation of variables, we obtain the following spectral problem

$$LX(x) = \begin{Bmatrix} -k_1^2 X''(x), & 0 < x < x_0 \\ -k_2^2 X''(x), & x_0 < x < l \end{Bmatrix} = \lambda X(x) \tag{33}$$

$$\begin{cases} X_1(0) + X_2(l) = 0 \\ k_1 X_1'(0) + k_2 X_2'(l) = 0 \end{cases} \tag{34}$$

$$X_1(x_0 - 0) = X_2(x_0 + 0), \quad k_1 X_1'(x_0 - 0) = k_2 X_2'(x_0 + 0), \tag{35}$$

The eigenvalues of Problem (33)–(35) have the form: $\lambda_n = ((2n+1)\pi r)^2, (n = 0, 1, 2, ...)$. The following eigenfunctions correspond to these eigenvalues.

$$X_n(x) = C \begin{cases} sin(\frac{(2n+1)\pi r}{k_1} x), & 0 < x < x_0, \\ sin(\frac{(2n+1)\pi r}{k_2}(l - x)), & x_0 < x < l, \end{cases}$$

$$\tilde{X}_n(x) = C \begin{cases} cos(\frac{(2n+1)\pi r}{k_1} x), & 0 < x < x_0, \\ -cos(\frac{(2n+1)\pi r}{k_2}(l - x)), & x_0 < x < l, \end{cases}$$

where r is determined by the formula (11).

All other calculations, including the proof of the theorem, are carried out in a similar way.

2 Conclusion

The method proposed in this article can be used in the case of n break points, where $n \geq 3$, and for the more general case of the conjugation condition (in this work, the ideal contact condition is considered).

3 Author Contributions and Conflict of Interest

All authors contributed equally to this work. The authors declare no conflict of interest.

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Қойлышов Ү.Қ., Абдырахимов Н.Т. КОЭФФИЦИЕНТІ ҮЗІЛІСТІ БЕЙЛОКАЛЬДЫ ШЕКАРАЛЫҚ ШАРТТАРМЕН БЕРІЛГЕН ЖЫЛУӨТКІЗГІШТІК ТЕҢДЕУДІ ФУРЬЕ ӘДІСІМЕН ШЕШУ

Мақалада коэффициенті үзілісті жылуөткізгіштік теңдеу үшін периодтық немесе антипериодтық шарттармен берілген бастапқы-шеттік есепті айнымалыларды ажырату әдісімен шешу негізделген Фурье әдісін қолдану арқылы бұл есеп сәйкес спектрлік есепке келтірілген. Берілген спектрлік есептің меншікті мәндері мен меншікті функциялары табылған. Спектрлік есептің өзіне-өзі түйіндес емес екені көрсетілген және берілген спектрлік есепке түйіндес есеп құрылған. Берілген есептің меншікті функциялар жүйесі Рисс базисін құрайтыны дәлелденген. Өзіне-өзі түйіндес спектрлік есеп құрылған және оның меншікті мәндері мен меншікті функциялары табылған. Қорытындылай келе, биортогональдықты пайдалана отырып, қойылған есептің классикалық шешімінің бар және жалғыздығы туралы негізгі теорема дәлелденді.

Түйін сөздер: коэффициенті үзілісті жылуөткізгіштік теңдеу, спектрлік есеп, өзіне-өзі түйіндес емес есеп, Рисс базисі, классикалық шешім, Фурье әдісі.

Койлышов У.К., Абдырахимов Н.Т. РЕШЕНИЕ УРАВНЕНИЯ ТЕПЛОПРОВОДНОСТИ С РАЗРЫВНЫМ КОЭФФИЦИЕНТОМ С НЕЛОКАЛЬНЫМИ КРАЕВЫМИ УСЛОВИЯМИ МЕТОДОМ ФУРЬЕ

В данной работе обосновано решение методом разделения переменных начально-краевой задачи для уравнения теплопроводности с разрывным коэффициентом, при периодических или антипериодических граничных условиях. Методом Фурье данная задача сведена к соответствующей спектральной задаче. Найдены собственные значения и собственные функции данной спектральной задачи. Показана, что спектральная задача несамосопряженная и построена сопряженная спектральная задача данной первоначальной спектральной задачи. Далее, доказывается, что система собственных функций образует базис Рисса. Для этого построена самосопряженная спектральная задача и найдены ее собственные значения и собственные функции. В заключении, используя биортогональность доказана основная теорема о существовании и единственности классического решения поставленной задачи.

Ключевые слова: Уравнение теплопроводности с разрывными коэффициентами, спектральная задача, несамосопряженная задача, базис Рисса, классическое решение, метод Фурье.