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On the solvability of the Dirichlet problem for the viscous Burgers equation

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Abstract. In this work, we study a Dirichlet problem for the viscous Burgers equation in a domain with moving boundaries that degenerates at the initial moment. The primary method of investigation is the Galerkin method, for which we construct an orthonormal basis suitable for domains with moving boundaries. Uniform a priori estimates are obtained, and based on these, theorems on the unique solvability of the problem are proven using methods of functional analysis. The viscous Burgers equation serves as a simplified model for studying fundamental aspects of nonlinear systems. It bridges the gap between purely theoretical nonlinear equations (like the inviscid Burgers equation) and more complex systems like the Navier-Stokes equations, making it a valuable tool in mathematical and physical research.

Keywords. Burgers equation, a priori estimates, Galerkin method.

1 Introduction

Let $\Omega = \{x,t \,|\, \varphi_1(t) < x < \varphi_2(t),\ 0 < t < T < \infty\}$ be a domain that degenerates into a point. The functions $\varphi_1(t)$ and $\varphi_2(t)$ are defined on $[0,T]$ and are strictly monotonically decreasing and increasing functions, respectively, which belong to $C^1(0,T)$ with $\varphi_1(0) = \varphi_2(0)$ and $\Omega_t = (\varphi_1(t), \varphi_2(t))$ for $t \in (0,T)$.

The study of solvability issues for initial-boundary value problems in domains with moving boundaries, namely, in domains whose boundaries change over time, has been the focus of numerous works; we note only a few of them [1, 2, 3, 4, 5]. In these works, we observed that the lack of a suitable basis applicable to such domains necessitates transforming these domains into ones with stationary boundaries. This transformation leads to the need to study several auxiliary problems, significantly complicating the research process. Previously,

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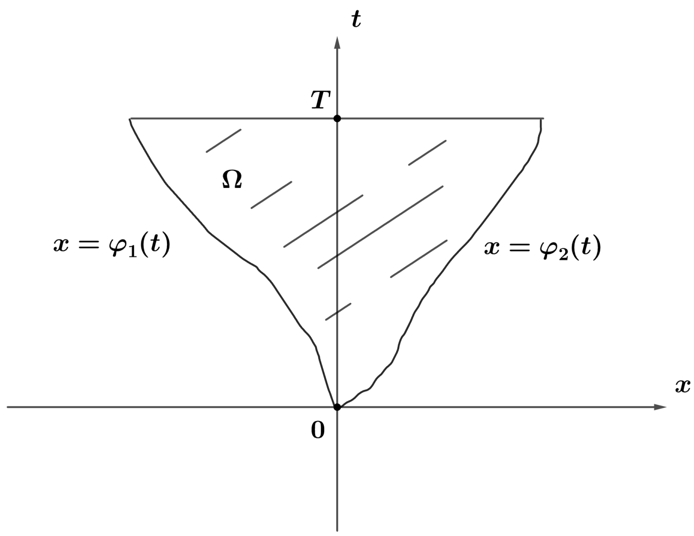

Figure 1: The degenerating domain Ω.

in work [6], we constructed an orthonormal basis and demonstrated its application to solving initial-boundary value problems in degenerate domains.

In this paper, in the domain Ω we are studying the solvability issues of the following boundary value problem for viscous Burgers equation:

$$\partial_t u(x,t) + u(x,t)\partial_x u(x,t) - \nu \partial_x^2 u(x,t) + \partial_x u(x,t) = f(x,t), \ \ (x,t) \in \Omega, \tag{1}$$

with homogeneous boundary conditions

$$u(\varphi_1(t),t) = u(\varphi_2(t),t) = 0, \ \ t \in (0,T). \tag{2}$$

We look for some conditions for functions $\varphi_1(t)$ and $\varphi_2(t)$ such that the problem (1)–(2) admits a unique solution. So, to establish the unique solvability of the problem (1)–(2) we suppose that

$$|\varphi'(t)| \leq \gamma \text{ for all } t \in [0,T], \ \varphi(t) = \varphi_2(t) - \varphi_1(t), \, \gamma = \text{const} > 0. \tag{3}$$

Here is our main result on the problem(1)–(2):

Theorem 1. *Let $f(x,t) \in L^2(\Omega)$ and conditions* (3) *be satisfied. Then boundary value problem* (1)–(2) *has a unique solution*

$$u \in H_0^{2,1}(Q) \equiv \left\{ L^2(0,T; H_0^2(\Omega_t)) \cap H^1(0,T; L^2(\Omega_t)) \right\}.$$

In work [7], the homogeneous version of problem (1)–(2) was previously studied in a non-degenerate domain, where theoretical and numerical results were obtained by the authors. In works [8, 9], the authors investigated the existence of solutions to boundary value problems for the Burgers equation in both degenerate and non-degenerate domains.

The paper is divided as follows: Section 2 investigates one auxiliary initial boundary value problem for the Burgers equation in the non-degenerate domain, where $\varphi_1(1/n) \neq \varphi_2(1/n)$. In Section 3, we obtain the necessary a priori estimates. In Section 4, we solve one spectral problem and construct the necessary orthonormal basis, then, based on the obtained basis, we introduce an approximate solution. In this section, we also prove the solvability of the Cauchy problem for the coefficients of the approximate solution. The unique solvability of the auxiliary problem is given in Section 5. Section 6 is devoted to the proof of the main result. A brief conclusion completes the work.

2 Statement of auxiliary problem

We introduce the family of domains $\Omega^n = \{x, t \,|\, \varphi_1(t) < x < \varphi_2(t),\, 1/n < t < T\}$, $n \in \mathbb{N}^*$, $n \in \mathbb{N}^* \equiv \{n \in \mathbb{N} \,|n \geq n_1,\, 1/n_1 < T\}$. These domains Ω^n are “curvilinear” trapezoids for which $\varphi_1(1/n) \neq \varphi_2(1/n)$ holds and now the domains do not degenerate at the point $t = 1/n$. We aslo note that between the initial domain Ω and domains Ω^n there are strict embeddings $\Omega^n \subset \Omega^{n+1} \subset ... \subset \Omega$ and, obviously, that $\lim\limits_{n\to\infty} \Omega^n = \Omega$.

In the domains Ω^n, we will consider the following initial boundary value problems for the Burgers equation with respect to the functions $u_n(x,t)$:

$$\partial_t u_n(x,t) + u_n(x,t)\partial_x u_n(x,t) - \nu\partial_x^2 u_n(x,t) + \partial_x u_n(x,t) = f_n(x,t), \tag{4}$$

with homogeneous boundary

$$u_n(\varphi_1(t),t) = u_n(\varphi_2(t),t) = 0, \;\; t \in (1/n, T), \tag{5}$$

and initial conditions

$$u_n(x, 1/n) = 0, \;\;\; x \in \Omega_{1/n} = (\varphi_1(1/n), \varphi_2(1/n)). \tag{6}$$

Obviously, if $f(x,t) \in L^2(\Omega)$, then $f_n(x,t) \in L^2(\Omega^n)$, where $f_n(x,t)$ is the restriction of function $f(x,t) \in L^2(\Omega)$ to domains Ω^n.

For the problem (4)–(6) we have the following

Theorem 2. *For every fixed $n \in \mathbb{N}^*$ the initial-boundary value problem (4)–(6) is uniquely solvable in the space $u_n(x,t) \in H_0^{2,1}(\Omega^n)$.*

3 A priori estimates

Lemma 3. *There is a positive, independent of* n*, constants* K_1*,* K_2 *and* K_3*, such that for all* $t \in [1/n, T]$ *we have estimates*

$$\|u_n(x,t)\|^2_{L^2(\Omega_t)} + \int\limits_{1/n}^{t} \|\partial_x u_n(x,\tau)\|^2_{L^2(\Omega_\tau)} d\tau \le K_1 \|f_n(x,t)\|^2_{L^2(\Omega^n)}, \tag{7}$$

$$\|\partial_x u_n(x,t)\|^2_{L^2(\Omega_t)} + \int\limits_{1/n}^{t} \|\partial_x^2 u_n(x,\tau)\|^2_{L^2(\Omega_\tau)} d\tau \le K_2 \|f_n(x,t)\|^2_{L^2(\Omega^n)}, \tag{8}$$

$$\|\partial_t u_n(x,t)\|^2_{L^2(\Omega^n)} \le K_3 \|f_n(x,t)\|^2_{L^2(\Omega^n)}. \tag{9}$$

Proof. We start with the proof of the first a priori estimate. Multiplying the equation (4) by the function $u_n(x,t)$ scalarly in $L^2(\Omega_t)$ and using the ε-Cauchy inequlity we get

$$\frac{d}{dt}\|u_n(x,t)\|^2_{L^2(\Omega_t)} + 2\nu\|\partial_x u_n(x,t)\|^2_{L^2(\Omega_t)} \le \|f_n(x,t)\|^2_{L^2(\Omega_t)} + \|u_n(x,t)\|^2_{L^2(\Omega_t)}. \tag{10}$$

By applying the Gronwall inequality to (10), we obtain the estimate (7).

Let us proceed to the proof of the second a priori estimate. Multiplying the equation (4) by $-\partial_x^2 u_n(x,t)$ scalarly in $L^2(\Omega_t)$ we get

$$\frac{d}{dt}\|\partial_x u_n(x,t)\|^2_{L^2(\Omega_t)} + 2\nu\|\partial_x^2 u_n(x,t)\|^2_{L^2(\Omega_t)} \le 2\Bigg|\int\limits_{\varphi_1(t)}^{\varphi_2(t)} u_n(x,t)\partial_x u_n(x,t)\partial_x^2 u_n(x,t)dx\Bigg|$$

$$+2\Bigg|\int\limits_{\varphi_1(t)}^{\varphi_2(t)} f_n(x,t)\partial_x^2 u_n(x,t)dx\Bigg| + 2\Bigg|\int\limits_{\varphi_1(t)}^{\varphi_2(t)} \partial_x u(x,t)\partial_x^2 u(x,t)dx\Bigg|$$

$$+\gamma\left(|\partial_x u_n(\varphi_2(t),t)|^2 + |\partial_x u_n(\varphi_1(t),t)|^2\right). \tag{11}$$

To estimate the nonlinear term in the right-hand side of (11) we use the following inequality ([10], Theorems 5.8–5.9, p.140–141)

$$\|\partial_x u_n(x,t)\|_{L^4(\Omega_t)} \le K\|\partial_x u_n(x,t)\|^{1/2}_{H^1(\Omega_t)}\|\partial_x u_n(x,t)\|^{1/2}_{L^2(\Omega_t)}, \ \forall\, \partial_x u_n(x,t) \in H^1(\Omega_t),$$

Young's inequality $(r^{-1} + s^{-1} = 1)$:

$$|UV| = \left|\left(\Theta^{1/r}U\right)\left(\Theta^{1/s}\frac{V}{\Theta}\right)\right| \le \frac{\Theta}{r}|U|^r + \frac{\Theta}{s\Theta^s}|V|^s,$$

with $\Theta = \nu/6,\ r = 4/3,\ s = 4,$

$$U = \|\partial_x u_n(x,t)\|^{3/2}_{H^1(\Omega_t)},\ \ V = K\, \|u_n(x,t)\|_{L^4(\Omega_t)}\, \|\partial_x u_n(x,t)\|^{1/2}_{L^2(\Omega_t)}.$$

After which we will get

$$\Big|\int\limits_{\varphi_1(t)}^{\varphi_2(t)} u_n(x,t)\partial_x u_n(x,t)\partial_x^2 u_n(x,t)dx\Big| \le \frac{\nu}{8}\|\partial_x^2 u_n(x,t)\|^2_{L^2(\Omega_t)}$$

$$+\left[\frac{\nu}{8}+\frac{54}{\nu^3}K^4\|u_n(x,t)\|^4_{L^4(\Omega_t)}\right]\|\partial_x u_n(x,t)\|^2_{L^2(\Omega_t)}. \tag{12}$$

For the remaining terms in (11) we have

$$\Big|\int\limits_{\varphi_1(t)}^{\varphi_2(t)} f_n(x,t)\partial_x^2 u_n(x,t)dx\Big| \le \frac{\nu}{8}\|\partial_x^2 u_n(x,t)\|^2_{L^2(\Omega_t)} + \frac{2}{\nu}\|f_n(x,t)\|^2_{L^2(\Omega_t)}, \tag{13}$$

$$\Big|\int\limits_{\varphi_1(t)}^{\varphi_2(t)} \partial_x u(x,t)\partial_x^2 u(x,t)dx\Big| \le \frac{2}{\nu}\|\partial_x u_n(x,t)\|^2_{L^2(\Omega_t)} + \frac{\nu}{8}\|\partial_x^2 u_n(x,t)\|^2_{L^2(\Omega_t)}, \tag{14}$$

$$\gamma|\partial_x u_n(\varphi_i(t),t)|^2 \le \gamma\|\partial_x u_n(x,t)\|^2_{L^\infty(\Omega_t)} \le K^2\gamma\|\partial_x u_n(x,t)\|_{H^1(\Omega_t)}\|\partial_x u_n(x,t)\|_{L^2(\Omega_t)}$$
$$= K^2\gamma\|\partial_x u_n(x,t)\|_{L^2(\Omega_t)}\left[\|\partial_x u_n(x,t)\|_{L^2(\Omega_t)} + \|\partial_x^2 u_n(x,t)\|_{L^2(\Omega_t)}\right]$$
$$\le \frac{\nu}{8}\|\partial_x^2 u_n(x,t)\|^2_{L^2(\Omega_t)} + \left[K^2\gamma + \frac{K^4\gamma^2}{2\nu}\right]\|\partial_x u_n(x,t)\|^2_{L^2(\Omega_t)},\ i = 1,2. \tag{15}$$

Based on inequalities (11)–(15) we obtain:

$$\frac{d}{d\,t}\|\partial_x u_n(x,t)\|^2_{L^2(\Omega_t)} + \nu\|\partial_x^2 u_n(x,t)\|^2_{L^2(\Omega_t)} \le C_1\|f_n(x,t)\|^2_{L^2(\Omega_t)} + C_2\|\partial_x u_n(x,t)\|^2_{L^2(\Omega_t)}, \tag{16}$$

where $C_1 = \frac{4}{\nu},\ C_2 = \frac{\nu}{4} + \frac{4}{\nu} + \frac{108K^4}{\nu^3}C_5^4 + \frac{\gamma K^2\nu + 2\gamma^2 K^4}{\nu}$, since

$$\|u_n(x,t)\|_{L^4(\Omega_t)} \le \max_{1/n\le t\le T}\sqrt[4]{\varphi(t)}\|u_n(x,t)\|_{L^\infty(\Omega_t)} \le C_4\|u_n(x,t)\|_{H^1(\Omega_t)} \le C_5,$$

where

$$C_4 = \max_{1/n\le t\le T}\sqrt[4]{\varphi(t)}C_3.$$

From inequality (16) similarly as in the proof of the first a priori estimate we obtain the required estimate (8).

Now, let us proceed to the proof of the final a priori estimate. From equation (4) we have

$$\|\partial_t u_n(x,t)\|_{L^2(\Omega^n)} \leq \nu\|\partial_x^2 u_n(x,t)\|_{L^2(\Omega^n)} + \|f_n(x,t)\|_{L^2(\Omega^n)}$$

$$+\|\partial_x u_n(x,t)\|_{L^2(\Omega^n)} + \|u_n(x,t)\partial_x u_n(x,t)\|_{L^2(\Omega^n)}. \tag{17}$$

According to (8) we need to estimate the last term in (17) only. Using the embedding $H^1(\Omega_t) \hookrightarrow L^\infty(\Omega_t)$ and estimates (7) and (8) we have

$$\|u_n(x,t)\partial_x u_n(x,t)\|^2_{L^2(\Omega^n)} \leq C_6 \int\limits_{1/n}^{T} \|u_n(x,t)\|^2_{H^1(\Omega_t)} \|\partial_x u_n(x,t)\|^2_{L^2(\Omega_t)} dt$$

$$\leq C_6 \|u_n(x,t)\|^2_{L^\infty(1/n,T;H^1(\Omega_t))} \|\partial_x u_n(x,t)\|^2_{L^2(Q^n)} \leq C_7 \|f_n(x,t)\|^2_{L^2(\Omega^n)}, \tag{18}$$

where $C_7 = K_1 K_2 C_6 T$, and K_1, K_2 are the constants from (7) and (8).

Based on inequalities (17)–(18) we establish the estimate (9). This completes the proof of Lemma 3. □

4 Spectral problem and approximate solution

4.1 Spectral problem

To apply the Faedo-Galerkin approach, it is necessary to resolve the corresponding spectral problem

$$-\partial_x^2 Y_k(x,t) = \lambda_k^2(t) Y_k(x,t),\ (x,t) \in \Omega^n,\ k \in \mathbb{N}_0, \tag{19}$$

$$Y_k(\varphi_1(t),t) = Y_k(\varphi_2(t),t) = 0. \tag{20}$$

The solution to this problem is sought in the form

$$Y_k(x,t) = A_k(t)\cos(\lambda_k(t)x) + B_k(t)\sin(\lambda_k(t)x). \tag{21}$$

Using the conditions (20) from (21) we get:

$$\begin{cases} A_k(t)\cos(\lambda_k(t)\varphi_1(t)) + B_k(t)\sin(\lambda_k(t)\varphi_1(t)) = 0, \\ A_k(t)\cos(\lambda_k(t)\varphi_2(t)) + B_k(t)\sin(\lambda_k(t)\varphi_2(t)) = 0. \end{cases} \tag{22}$$

For the system (22) to admit a nontrivial solution, the following condition must hold:

$$\begin{vmatrix} \cos(\lambda_k(t)\varphi_1(t)) & \sin(\lambda_k(t)\varphi_1(t)) \\ \sin(\lambda_k(t)\varphi_2(t)) & \cos(\lambda_k(t)\varphi_2(t)) \end{vmatrix} = 0,$$

From where we obtain

$$\sin\left(\lambda_k(t)\varphi(t)\right) = 0, \;\; k \in \mathbb{N}_0,$$

hence

$$\lambda_k(t) = \frac{\pi k}{\varphi(t)}, \;\; k \in \mathbb{N}_0. \tag{23}$$

From (22) we also obtain

$$A_k(t) = -B_k(t)\frac{\sin\lambda_k(t)\varphi_1(t)}{\cos\lambda_k(t)\varphi_1(t)}. \tag{24}$$

Substituting (24) into (21) and choosing

$$B_k(t) = \frac{\sqrt{2}\cos\lambda_k(t)\varphi_1(t)}{\sqrt{\varphi(t)}},$$

we have

$$Y_k(x,t) = \frac{\sqrt{2}}{\sqrt{\varphi(t)}}\sin\left(\lambda_k(t)(x-\varphi_1(t))\right), \;\; \lambda_k^2(t) = \left(\frac{\pi k}{\varphi(t)}\right)^2, \;\; k \in \mathbb{N}_0. \tag{25}$$

4.2 Approximate solution

The following approximate solution

$$u_n^N(x,t) = \sum_{j=1}^{N} c_j(t)Y_j(x,t), \;\; u_n^N(x,1/n) = \sum_{j=1}^{N} c_j(1/n)Y_j(x,1/n) = 0, \tag{26}$$

is introduced and utilized to solve the problem (4)–(6):

$$\int\limits_{\varphi_1(t)}^{\varphi_2(t)} \left[\partial_t u_n^N(x,t)dx + u_n^N(x,t)\partial_x u_n^N(x,t) - \nu\partial_x^2 u_n^N(x,t) + \partial_x u_n^N(x,t)\right] Y_k(x,t)dx$$

$$= \int\limits_{\varphi_1(t)}^{\varphi_2(t)} f_n(x,t)Y_k(x,t)dx, \tag{27}$$

$$u_n^N(x,1/n) = 0, \;\; x \in \Omega_{1/n}, \tag{28}$$

for all $k = 1,\ldots,N$ and $t \in [1/n, T]$.

Lemma 4. *The problem (27)–(28) has a unique solution* $C(t) = \{c_j(t)\}_{j=1}^N$.

Proof. Given that the system of functions $\{Y_k(x,t)\}_{k\in\mathbb{N}_0}$ forms an orthonormal basis in $L^2(\Omega_t)$ for $\mathbb{N}_0 = \{0,1,2,\dots\}$, it follows that for any finite N:

$$\int\limits_{\varphi_1(t)}^{\varphi_2(t)} \partial_t u_n^N(x,t)Y_k(x,t)dx = \sum_{j=1}^{N} c_j'(t) \int\limits_{\varphi_1(t)}^{\varphi_2(t)} Y_j(x,t)Y_k(x,t)dx$$

$$+\sum_{j=1}^{N} c_j(t) \int\limits_{\varphi_1(t)}^{\varphi_2(t)} Y_j'(x,t)Y_k(x,t)dx = c_j'(t) + S_1(t)c_j(t),$$

where for all $k = 1,\dots,N$

$$S_1(t)c_j(t) = (I_1(t) + I_2(t) + I_3(t))\, c_j(t),$$

$$I_1(t)c_j(t) = -\frac{\varphi'(t)}{\varphi^2(t)} \sum_{j=1}^{N} c_j(t) \int\limits_{\varphi_1(t)}^{\varphi_2(t)} \sin\left(\lambda_j(t)(x-\varphi_1(t))\right) \sin\left(\lambda_k(t)(x-\varphi_1(t))\right) dx,$$

$$I_2(t)c_j(t) = \left(\frac{2\pi}{\varphi^2(t)}\left(\frac{\varphi'(t)\varphi_1(t)}{\varphi(t)} - \varphi_1'(t)\right)\right) \sum_{j=1}^{N} jc_j(t)$$

$$\cdot \int\limits_{\varphi_1(t)}^{\varphi_2(t)} \cos\left(\lambda_j(t)(x-\varphi_1(t))\right) \sin\left(\lambda_k(t)(x-\varphi_1(t))\right) dx,$$

$$I_3(t)c_j(t) = -\frac{2\pi\varphi'(t)}{\varphi^3(t)} \sum_{j=1}^{N} jc_j(t) \int\limits_{\varphi_1(t)}^{\varphi_2(t)} x\cos\left(\lambda_j(t)(x-\varphi_1(t))\right) \sin\left(\lambda_k(t)(x-\varphi_1(t))\right) dx.$$

From (19) we have $\partial_x^2 u_n^N(x,t) = -\sum\limits_{j=1}^{N} \lambda_j^2(t)c_j(t)Y_j(x,t)$. Then, for all $t \in [1/n,\, T]$,

$$-\int\limits_{\varphi_1(t)}^{\varphi_2(t)} \partial_x^2 u_n^N(x,t)Y_k(x,t)dx = \sum_{j=1}^{N} \lambda_j^2(t)c_j(t) \int\limits_{\varphi_1(t)}^{\varphi_2(t)} Y_j(x,t)Y_k(x,t)dx = \lambda_j^2(t)c_j(t).$$

For the nonlinear term we have

$$\int\limits_{\varphi_1(t)}^{\varphi_2(t)} u_n^N(x,t)\partial_x u_n^N(x,t)Y_k(x,t)dx$$

$$= \int\limits_{\varphi_1(t)}^{\varphi_2(t)} \sum_{l=1}^{N} c_l(t) Y_l(x,t) \sum_{m=1}^{N} c_m(t) \partial_x Y_m(x,t) Y_k(x,t) dx = S_2(t) c_{lm}(t).$$

For the last term we have

$$\int\limits_{\varphi_1(t)}^{\varphi_2(t)} \partial_x u_n^N(x,t) Y_k(x,t) dx = \sum_{j=1}^{N} c_j(t) \int\limits_{\varphi_1(t)}^{\varphi_2(t)} \partial_x Y_j(x,t) Y_k(x,t) dx = S_3(t) c_j(t)$$

For $j \in \mathbb{N}$, the problem (27)–(28) can be reformulated as the following Cauchy problem for a finite system of nonlinear ordinary differential equations:

$$c_j'(t) = \left(-S_1(t) - \nu\lambda_j^2(t) - S_3(t)\right) c_j(t) - S_2(t) c_{lm}(t) + g_j(t), \;\; c_j(1/n) = 0, \tag{29}$$

where

$$g_j(t) = \int\limits_{\varphi_1(t)}^{\varphi_2(t)} f_n(x,t) Y_j(x,t) dx, \;\; j \in \mathbb{N}.$$

Since $f(x,t) \in L^2(Q)$, it follows that $g_k(t)$ is a square-integrable function, and function $S_1(t)$ is well defined. Consequently, the Cauchy problem (29) has a unique solution on some interval $[1/n, T']$, where $T' \leq T$. Moreover, due to the a priori estimates provided in Lemma 3 in Section 3, the solution $C(t)$ can be extended up to the finite time T.

Thus, for any fixed finite N, the functions $C(t) = \{c_j(t)\}_{j=1}^N$ are determined as the solution to the Cauchy problem (29). Along with these, the unique approximate solution $u_n^N(x,t)$ to problem (27)–(28) is obtained. This concludes the proof of Lemma 4. □

5 Solvatility of auxiliary problem

5.1 Proof of Theorem 2. Existence

By virtue of Lemma 3 we can extract weakly convergent subsequences from bounded sequences $\{u_n^N(x,t),\, \partial_t u_n^N(x,t)\, N = 1, 2, ...\}$:

$$u_n^N(x,t) \to u_n(x,t) \;\text{ weakly in } L^2(1/n, T; H_0^2(\Omega_t)) \cap H^1(1/n, T; L^2(\Omega_t)), \tag{30}$$

$$u_n^N(x,t) \to u_n(x,t) \;\text{ strongly in } L^2(1/n, T; L^2(\Omega_t)) \;\text{ and a.e. in } \Omega^n. \tag{31}$$

We introduce the new function $w_j(x,t) = \psi(t) Y_j(x,t)$, where $Y_j(x,t) \in H_0^2(\Omega_t)$ and $\psi(t) \in C^1([1/n, T])$. Next, we multiply the identity (27) by $\psi(t) \in C^1([1/n, T])$ and after that we integrate the resulting expression with respect to t over the interval $[1/n, T]$ for

$j = 1, \dots, N$ and use the fact that the set of all linear combinations of $\{w_j(x,t)\}$ is dense in $L^2(1/n, T; H_0^2(\Omega_t))$. Thus, we obtain:

$$\int\limits_{1/n}^{T}\int\limits_{\varphi_1(t)}^{\varphi_2(t)} \big[\partial_t u_n^N(x,t) + u_n^N(x,t)\partial_x u_n^N(x,t) - \nu\partial_x^2 u_n^N(x,t) + \partial_x u_n^N(x,t) - f(x,t)\big] w(x,t)dxdt = 0,$$

$$\forall\, w(x,t) \in L^2(1/n, T; H_0^2(\Omega_t)). \tag{32}$$

In the identity (32) we take the limit as $N \to \infty$. For the linear terms in equation (4), the passage to the limit is performed using the relations (30) and (31). Regarding the nonlinear term, as $N \to \infty$ we arrive at the following result:

$$\int\limits_{1/n}^{T}\int\limits_{\varphi_1(t)}^{\varphi_2(t)} [u_n^N(x,t) - u_n(x,t)]\partial_x u_n^N(x,t) w(x,t)\, dx\, dt$$

$$+ \int\limits_{1/n}^{T}\int\limits_{\varphi_1(t)}^{\varphi_2(t)} u_n(x,t)\partial_x u_n^N(x,t) w(x,t)\, dx\, dt \to \int\limits_{1/n}^{T}\int\limits_{\varphi_1(t)}^{\varphi_2(t)} u_n(x,t)\partial_x u_n(x,t) w(x,t)\, dx\, dt, \tag{33}$$

since according to (30)–(31) there exists a limiting relationship

$$\int\limits_{1/n}^{T}\int\limits_{\varphi_1(t)}^{\varphi_2(t)} [u_n^N(x,t) - u_n(x,t)]\partial_x u_n^N(x,t) w(x,t)\, dx\, dt \to 0.$$

Thus, by passing to the limit as $N \to \infty$ in the identity (32), and taking into account the limiting relation (33) along with the initial condition (28), we obtain:

$$\int\limits_{1/n}^{T}\int\limits_{\varphi_1(t)}^{\varphi_2(t)} \big[\partial_t u_n(x,t) + u_n(x,t)\partial_x u_n(x,t) - \nu\partial_x^2 u_n(x,t) + \partial_x u_n(x,t) - f(x,t)\big]\, w(x,t)dxdt = 0,$$

$$\forall\, w(x,t) \in L^2(1/n, T; H_0^2(\Omega_t)), \tag{34}$$

$$\int\limits_{\varphi_1(1/n)}^{\varphi_2(1/n)} u_n(x, 1/n)\theta(x)\, dx = 0, \ \ \forall\, \theta(x) \in L^2(\Omega_{1/n}). \tag{35}$$

Thus, from (34)–(35), it follows that the limiting function $u_n(x,t)$ satisfies equation (4) along with the boundary and initial conditions (5)–(6).

5.2 Proof of Theorem 2. Uniqueness

We suppose that the initial boundary value problem (4)–(6) has two distinct solutions, denoted by $u_n^{(1)}(x,t)$ and $u_n^{(2)}(x,t)$. Then, their difference, given by $u_n(x,t) = u_n^{(1)}(x,t) - u_n^{(2)}(x,t)$, fulfills the following problem:

$$\partial_t u_n(x,t) + u_n(x,t)\partial_x u_n^{(1)}(x,t) + u_n^{(2)}(x,t)\partial_x u_n(x,t) - \nu\partial_x^2 u_n(x,t) + \partial_x u_n(x,t) = 0, \quad (36)$$

$$u_n(\varphi_1(t),t) = u_n(\varphi_2(t),t) = 0, \;\; t \in (1/n, T), \quad (37)$$

$$u_n(x, 1/n) = 0, \;\; x \in \Omega_{1/n}. \quad (38)$$

By Lemma 3, it follows that

$$u_n^{(k)}(x,t) \in L^\infty(1/n, T; H^1(\Omega_t)) \cap L^2(1/n, T; H_0^2(\Omega_t)), \; k = 1, 2. \quad (39)$$

Consider equality

$$\frac{1}{2}\frac{d}{dt}\|u_n(x,t)\|^2_{L^2(\Omega_t)} + \nu\,\|\partial_x u_n(x,t)\|^2_{L^2(\Omega_t)} =$$

$$-\int\limits_{\varphi_1(t)}^{\varphi_2(t)} \left[u_n(x,t)\partial_x u_n^{(1)}(x,t)u_n(x,t) + u_n^{(2)}(x,t)\partial_x u_n(x,t)u_n(x,t)\right] dx, \quad (40)$$

derived by taking the scalar product of equation (36) with the function $u_n(x,t)$ in the space $L^2(\Omega_t)$.

From (39), we derive an estimate for the right-hand side of (40):

$$\int\limits_{\varphi_1(t)}^{\varphi_2(t)} \left[|u_n(x,t)|^2\partial_x u_n^{(1)}(x,t) + u_n^{(2)}(x,t)\partial_x u_n(x,t)u_n(x,t)\right] dx$$

$$= \int\limits_{\varphi_1(t)}^{\varphi_2(t)} \left[-2u_n^1(x,t)u_n(x,t)\partial_x u_n(x,t) + u_n^{(2)}(x,t)\partial_x u_n(x,t)u_n(x,t)\right] dx$$

$$\leq \frac{1}{2\nu}\left[2\|u_n^{(1)}(x,t)\|_{L^\infty(1/n,T;(\Omega_t))} + \|u_n^{(2)}(x,t)\|_{L^\infty(1/n,T;(\Omega_t))}\right]^2 \|u_n(x,t)\|^2_{L^2(\Omega_t)}$$
$$+\frac{\nu}{2}\|\partial_x u_n(x,t)\|^2_{L^2(\Omega_t)} = C_8\|u_n(x,t)\|^2_{L^2(\Omega_t)} + \frac{\nu}{2}\|\partial_x u_n(x,t)\|^2_{L^2(\Omega_t)}, \quad (41)$$

where

$$C_8 = \frac{1}{2\nu}\left[2\|u_n^{(1)}(x,t)\|_{L^\infty(1/n,T;(\Omega_t))} + \|u_n^{(2)}(x,t)\|_{L^\infty(1/n,T;(\Omega_t))}\right]^2.$$

Using relation (40), we deduce:

$$\frac{d}{dt}\|u_n(x,t)\|^2_{L^2(\Omega_t)} + \nu\,\|\partial_x u_n(x,t)\|^2_{L^2(\Omega_t)} \le C_9\|u(x,t)\|^2_{L^2(\Omega_t)},\ \ \forall\, t\in(1/n,T], \tag{42}$$

where $C_9 = 2C_8$. From (42), by applying Gronwall's inequality, we obtain:

$$\|u_n(x,t)\|^2_{L^2(\Omega_t)} \equiv 0,\ \forall\, t\in(1/n,T].$$

This implies that $u_n^{(1)}(x,t)\equiv u_n^{(2)}(x,t)$ in $L^2(\Omega^n)$, meaning the solution to the initial boundary value problem (4)–(6) is unique. Hence, the uniqueness has been established, and Theorem 2 is proven.

6 Proof of the main result

6.1 Proof of Theorem 1. Existence

In the boundary value problems (4)–(6), we extend each element of the sequence $\{u_n(x,t) : (x,t)\in\Omega^n,\ n\in\mathbb{N}^*\}$ by zero to the entire domain Ω. As a result, we obtain a bounded sequence of functions $\left\{\widetilde{u_n(x,t)},\ n\in\mathbb{N}^*\right\}$, from which a convergent subsequence can be extracted (retaining n as the index for this subsequence), i.e.

$$\widetilde{u_n(x,t)} \to u(x,t)\ \text{ weakly in }\ H_0^{2,1}(\Omega), \tag{43}$$

$$\widetilde{u_n(x,t)} \to u(x,t)\ \text{ strongly in }\ L^2(\Omega). \tag{44}$$

Then, based on (43)–(44), we can pass to the limit as $n\to\infty$ in the following integral identity for all $\psi(x,t)\in L^2(\Omega)$

$$\int\limits_Q \Big[\partial_t\widetilde{u_n(x,t)} + \widetilde{u_n(x,t)}\partial_x\widetilde{u_n(x,t)} - \nu\partial_x^2\widetilde{u_n(x,t)} + \partial_x\widetilde{u_n(x,t)} - f_n(x,t)\Big]\,\psi(x,t)\,dx\,dt \to$$

$$\to \int\limits_Q \left[\partial_t u(x,t) + u(x,t)\partial_x u(x,t) - \nu\partial_x^2 u(x,t) + \partial_x u(x,t) - f(x,t)\right]\psi(x,t)\,dx\,dt = 0, \tag{45}$$

and $\widetilde{u_n(x,1/n)}\to 0,$ as $n\to\infty$. Thus, it has been shown that the boundary value problem (1)–(2) possesses a solution $u(x,t)\in H_0^{2,1}(\Omega)$, as defined by the integral identity (45). This proves the existence of a solution, thereby confirming Theorem 1.

6.2 Proof of Theorem 1. Uniqueness

Suppose that the boundary value problem (1)–(2) has two distinct solutions, denoted $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$. Then, their difference $u(x,t) = u^{(1)}(x,t) - u^{(2)}(x,t)$ will fulfill the following problem:

$$\partial_t u(x,t) + u(x,t)\partial_x u^{(1)}(x,t) + u^{(2)}(x,t)\partial_x u(x,t) - \nu\partial_x^2 u(x,t) + \partial_x u(x,t) = 0, \tag{46}$$

$$u(\varphi_1(t),t) = u(\varphi_2(t),t) = 0, \ t \in (0,T). \tag{47}$$

By similar reasoning as in Lemma 3, the following inequality can be established:

$$\|u^{(k)}(x,t)\|_{L^\infty(0,T;H^1(\Omega_t))} \le M = K_2\|f(x,t)\|_{L^2(\Omega)}, \, k = 1,2. \tag{48}$$

Consider the equality

$$\frac{1}{2}\frac{d}{dt}\|u(x,t)\|^2_{L^2(\Omega_t)} + \nu\,\|\partial_x u(x,t)\|^2_{L^2(\Omega_t)} =$$

$$-\int\limits_{\varphi_1(t)}^{\varphi_2(t)} \Big[|u(x,t)|^2\partial_x u^{(1)}(x,t) + u^{(2)}(x,t)\partial_x u(x,t)u(x,t)\Big]\,dx, \tag{49}$$

which is obtained by multiplying the equation (46) by function $u(x,t)$ scalarly in $L^2(\Omega_t)$.

From (48), we obtain an estimate for the right side of (49)

$$\int\limits_{\varphi_1(t)}^{\varphi_2(t)} \Big[|u(x,t)|^2\partial_x u^{(1)}(x,t) + u^{(2)}(x,t)\partial_x u(x,t)u(x,t)\Big]\,dx$$

$$\le C_{10}\|u(x,t)\|^2_{L^2(\Omega_t)} + \frac{\nu}{2}\|\partial_x u(x,t)\|^2_{L^2(\Omega_t)}, \tag{50}$$

where

$$\frac{1}{2\nu}\Big[2\|u^{(1)}(x,t)\|_{L^\infty(0,T;\,(\Omega_t))} + \|u^{(2)}(x,t)\|_{L^\infty(0,T;(\Omega_t))}\Big]^2 \le \frac{9M^2}{2\nu} = C_{10},$$

and M is the constant from (48).

Based on relations (49)–(50) we obtain

$$\frac{d}{dt}\|u(x,t)\|^2_{L^2(\Omega_t)} + \nu\,\|\partial_x u(x,t)\|^2_{L^2(\Omega_t)} \le C_{11}\|u(x,t)\|^2_{L^2(\Omega_t)}, \ \ \forall\, t \in (0,T], \tag{51}$$

where $C_{11} = 2C_{10}$. From (51), applying the Gronwall inequality, we obtain that

$$\|u(x,t)\|^2_{L^2(\Omega_t)} \equiv 0, \ \ \forall\, t \in (0,T].$$

This implies that $u^{(1)}(x,t) \equiv u^{(2)}(x,t)$ in $L^2(\Omega)$, i.e. solution to the boundary value problem (1)–(2) is unique. Thus, we have proved the main result of the work, namely, Theorem 1.

7 Conclusion

In this work, we studied a Dirichlet problem for the Burgers equation in a domain with moving boundaries that degenerates at the initial moment. An orthonormal basis suitable for domains with moving boundaries was constructed. Uniform a priori estimates were obtained, based on which theorems on the unique solvability of the considered problem were proven using methods of functional analysis.

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Сарыбай Т.А., Ерғалиев М.Ғ., Жақсыбай Ы.К. ТҰТҚЫР БЮРГЕРС ТЕҢДЕУІ ҮШІН ҚОЙЫЛҒАН ДИРИХЛЕ ЕСЕБІНІҢ ШЕШІМДІЛІГІ ТУРАЛЫ

Жұмыста біз уақыттың бастапқы мезетінде жойылмалы және шекаралары қозғалмалы облыста Бюргерс теңдеуі үшін қойылған Дирихле есебін зерттейміз. Зерттеудің негізгі әдісі — Гарекин әдісі болғандықтан, біз шекаралары қозғалмалы облыстар үшін қолдануға болатын ортонормаланған базис құрылады. Бірқалыпты априорлы бағалаулар алынып, олардың негізінде қарастырылып отырған есептің бірмәнді шешімділігі туралы теоремалар функционалдық талдау әдістері көмегімен дәлелденді. Тұтқыр Бургерс теңдеуі сызықты емес жүйелердің іргелі аспектілерін зерттеу үшін жеңілдетілген үлгі ретінде қызмет етеді. Ол таза теориялық сызықты емес теңдеулер (мысалы, бұрыс Бургерс теңдеуі) мен Навье-Стокс теңдеулері сияқты күрделі жүйелер арасындағы алшақтықты жояды, бұл оны математикалық және физикалық зерттеулерде құнды құрал етеді.

Түйін сөздер: Бюргерс теңдеуі, априорлы бағалаулар, Галеркин әдісі.

Сарыбай Т.А., Ергалиев М.Г., Жаксыбай Ы.К. О РАЗРЕШИМОСТИ ЗАДАЧИ ДИРИХЛЕ ДЛЯ ВЯЗКОГО УРАВНЕНИЯ БЮРГЕРСА

В работе нами исследуется одна задача Дирихле для уравнения Бюргерса в области с подвижными границами, которая вырождается в начальный момент времени. Основным методом исследования является метод Галеркина, для применения которого нами в работе строится ортонормированный базис, применимый для областей с подвижными границами. Получены равномерные априорные оценки на основе которых методами функционального анализа доказаны теоремы однозначной разрешимости рассматриваемой задачи. Вязкое уравнение Бюргерса служит упрощенной моделью для изучения фундаментальных аспектов нелинейных систем. Оно заполняет пробел между чисто теоретическими нелинейными уравнениями (такими как невязкое уравнение Бюргерса) и более сложными системами, такими как уравнения Навье-Стокса, что делает его ценным инструментом в математических и физических исследованиях.

Ключевые слова: уравнение Бюргерса, априорные оценки, метод Галеркина.