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Uniform convexity and uniform smoothness of $\ell^p(\widehat{G})$ spaces based on the Hilbert-Schmidt ideal.

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Abstract. In this paper, we study uniform convexity and uniform smoothness properties of ℓ^p-spaces associated with the unitary dual of a compact group based on the Hilbert-Schmidt ideal.

Keywords. Compact group, ℓ^p-spaces associated with a compact groups, uniformly smooth Banach space, uniformly convex Banach space

1 Introduction

In [2], Clarkson introduced the notion of uniformly convex Banach space. Namely, a Banach space X is said to be uniformly convex if for each $0 < \varepsilon \le 2$ there exists $\delta(\varepsilon) > 0$ such that

$$\left\|\frac{x+y}{2}\right\| \le 1 - \delta(\varepsilon)$$

whenever

$$\|x\| = \|y\| = 1, \quad \|x - y\| = \varepsilon.$$

In geometrical terms the above definition can be reformulated in the following way: the mid-point of an arbitrary chord of the unit sphere of the space cannot approach to the surface of that sphere unless the length of the chord goes to zero [2].

Note that any finite dimensional Euclidean space $\mathbb{R}^n$, $n \ge 1$, and any Hilbert space $\mathcal{H}$ is clearly uniformly convex due to the parallelogram identity

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2). \tag{1}$$

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In the same paper [2, Section 3], Clarkson proved that the classical Lebesgue spaces $L^p(\mu)$ and ℓ^p, for $1 < p < \infty$, satisfy this property too, i.e., they are uniformly convex.

The dual notion to uniformly convex Banach space is the notion of uniformly smooth Banach space. A Banach space X is said to be uniformly smooth if the expression

$$\sup\left\{1 - \left\|\frac{x+y}{2}\right\| : \quad \|x\| = \|y\| = 1, \quad \|x-y\| \leq 2\tau\right\}$$

equals to $o(\tau)$ as $\tau \to 0$.

The duality is proven in [10, Propositition 1.e.2] (see also [3, 9]). Namely, a Banach space X is uniformly convex if and only if X^* is uniformly smooth, where X^* is the dual space. Moreover, in the same proposition, authors proved the identity connecting the modulus of convexity (see Definition 3) of X and the modulus of smoothness (see Definition 3) of its dual X^*. The notion of a uniformly smooth Banach space is also closely related to uniform Frechet differentiability of a norm of the given Banach space [4, Section 2.4].

Equivalent definitions of uniform convexity and uniform smoothness via the modulus of convexity and the modulus of smoothness, respectively, can be found in Definition 4.

In general, geometric properties including uniform convexity and uniform smoothness properties of ℓ^p-spaces associated with the unitary dual of a compact group G based on the Schatten-von Neumann ideals were studied recently in [1].

Using the same approach, in this paper we investigate the uniform convexity and uniform smoothness of non-commutative ℓ^p-spaces on unitary dual of a compact group G based on the Hilbert-Schmidt ideals (see Theorem 5), simply denoted as $\ell^p(\widehat{G})$ [6, Section 2.1.4] (see also [5, Section 2.14.2], [8]). As a consequence of the Milman-Pettis's theorem, we state that these spaces are also reflexive for $1 < p < \infty$. These $\ell^p(\widehat{G})$ spaces based on Hilbert-Schmidt ideal are the generalization of ℓ^p-spaces over the compact group G, denoted as $\ell^p(G)$ (see [5], [6], [8], [11]). One of the known applications of $\ell^p(\widehat{G})$ spaces is the Hausdorff-Young theorem for all compact groups [5, Section 2.14.1].

2 Preliminaries

In this section, we recall the necessary preliminaries and the basics of the main object of this paper, the noncommutative ℓ^p-spaces, denoted by $\ell^p(\widehat{G})$ [11, Section 10.3.3], associated with a compact group based on the Hilbert-Schmidt ideal. We will follow the nomenclature and notation of [1] and we also refer to [1] (or [11]) for any unexplained terminology. Throughout this paper, by $\big(\mathcal{L}(\mathcal{H}), \|\cdot\|_{\mathcal{L}(\mathcal{H})}\big)$ and $\big(\mathcal{S}^2(\mathcal{H}), \|\cdot\|_{\mathcal{S}^2(\mathcal{H})}\big)$, we denote the $*$-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$ and Hilbert-Schmidt ideal of compact operators on $\mathcal{H}$, respectively (see, for example, [12, Chapter 2] or [7, Chapter 3]).

Let G be a compact group. By $\widehat{G}$, we denote the unitary dual of G, i.e., the set of all equivalence classes of irreducible unitary representations from $\mathrm{Rep}(G)$ (see [11, Definition

7.5.7 and 10.2.1]). Let $[\xi] \in \widehat{G}$ denote the equivalence class of a strongly continuous irreducible unitary representation $\xi : G \to \mathcal{U}(\mathcal{H}_\xi)$, where $\mathcal{H}_\xi$ is a representation space and note that $\mathcal{H}_\xi$ is finite dimensional since G is compact. We also set $\dim(\xi) = \dim(\mathcal{H}_\xi)$.

The space $\mathcal{M}(\widehat{G})$ consists of all mappings

$$F : \widehat{G} \to \bigcup_{[\xi]\in\widehat{G}} \mathcal{L}(\mathcal{H}_\xi) \subset \bigcup_{m=1}^{\infty} \mathbb{C}^{m\times m},$$

satisfying $F([\xi]) \in \mathcal{L}(\mathcal{H}_\xi)$ for every $[\xi] \in \widehat{G}$. Note that in matrix representations, one can view $F([\xi]) \in \mathbb{C}^{\dim(\xi)\times\dim(\xi)}$ as a $\dim(\xi) \times \dim(\xi)$ matrix.

Let $\langle\xi\rangle := \sqrt{1 + \lambda_{[\xi]}^2}$, where $\lambda_{[\xi]}, [\xi] \in \widehat{G}$, denotes the corresponding eigenvalue of the positive Laplacian (in a bijective manner) indexed by an equivalence class $[\xi] \in \widehat{G}$ (for more details, see [11, Definition 10.3.18]). The space $S'(\widehat{G})$ of slowly increasing or tempered distributions on the unitary dual $\widehat{G}$ is defined as the space of all $H \in \mathcal{M}(\widehat{G})$ for which there exists some $k \in \mathbb{N}$ such that

$$\sum_{[\xi]\in\widehat{G}} \dim(\xi)\langle\xi\rangle^{-k} \|H(\xi)\|_{\mathcal{S}^2(\mathcal{H}_\xi)} < \infty,$$

where $\|\cdot\|_{\mathcal{S}^2(\mathcal{H}_\xi)} := \|\cdot\|_{\mathcal{S}^2}$ is a Hilbert-Schmidt norm. The convergence in $S'(\widehat{G})$ is defined as follows: the sequence $H_j \in S'(\widehat{G})$ is said to be converging to $H \in S'(\widehat{G})$ in $S'(\widehat{G})$ as $j \to \infty$, if there exists some $k \in \mathbb{N}$ such that

$$\sum_{[\xi]\in\widehat{G}} \dim(\xi)\langle\xi\rangle^{-k} \|H_j(\xi) - H(\xi)\|_{\mathcal{S}^2(\mathcal{H}_\xi)} \to 0, \quad j \to \infty.$$

We now define ℓ^p-spaces over the unitary dual $\widehat{G}$ of a compact group G based on the Hilbert-Schmidt ideal.

Definition 1. [11, Definition 10.3.36] For $1 \le p < \infty$, the space $\ell^p(\widehat{G}) \equiv \ell^p\left(\widehat{G}, \dim(\xi)^{p(\frac{2}{p}-\frac{1}{2})}\right)$ is given as the space of all $H \in S'(\widehat{G})$ such that

$$\|H\|_{\ell^p(\widehat{G})} := \left(\sum_{[\xi]\in\widehat{G}} (\dim(\xi))^{p\left(\frac{2}{p}-\frac{1}{2}\right)} \|H(\xi)\|^p_{\mathcal{S}^2(\mathcal{H}_\xi)}\right)^{1/p} < \infty.$$

For $p = \infty$, the space $\ell^\infty(\widehat{G})$ is given as the space of all $H \in S'(\widehat{G})$ such that

$$\|H\|_{\ell^\infty(\widehat{G})} := \sup_{[\xi]\in\widehat{G}} (\dim(\xi))^{-1/2} \|H(\xi)\|_{\mathcal{S}^2(\mathcal{H}_\xi)} < \infty.$$

For the noncommutative spaces $\ell^p(\widehat{G})$, $1 < p < \infty$, the following Clarkson type inequalities are known from [13, Theorem 3]. The proof can also be found in [13].

Proposition 2. *Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then, for any $H_1, H_2 \in \ell^p(\widehat{G})$, one has the following inequalities:*

1. If $1 < p \le 2$, then

$$\left(\left\| \frac{H_1 + H_2}{2} \right\|^q_{\ell^p(\widehat{G})} + \left\| \frac{H_1 - H_2}{2} \right\|^q_{\ell^p(\widehat{G})} \right)^{1/q} \le \left(\frac{1}{2} \left(\|H_1\|^p_{\ell^p(\widehat{G})} + \|H_2\|^p_{\ell^p(\widehat{G})} \right) \right)^{1/p};$$

2. If $2 \le p < \infty$, then

$$\left(\left\| \frac{H_1 + H_2}{2} \right\|^p_{\ell^p(\widehat{G})} + \left\| \frac{H_1 - H_2}{2} \right\|^p_{\ell^p(\widehat{G})} \right)^{1/p} \le \left(\frac{1}{2} \left(\|H_1\|^q_{\ell^p(\widehat{G})} + \|H_2\|^q_{\ell^p(\widehat{G})} \right) \right)^{1/q}.$$

In general, for a given Banach space, one can define the notions of its modulus of convexity and modulus of smoothness.

Definition 3. Let $(X, \|\cdot\|_X)$ be a Banach space. Its modulus of convexity and modulus of smoothness are defined by

$$\delta_X(\varepsilon) := \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\|_X \mid x, y \in X, \quad \|x\|_X = \|y\|_X = 1, \quad \|x - y\|_X = \varepsilon \right\},$$

for $0 < \varepsilon \le 2$, and

$$\rho_X(t) := \sup \left\{ \frac{\|x + ty\|_X + \|x - ty\|_X}{2} - 1 \mid x, y \in X, \quad \|x\|_X = \|y\|_X = 1 \right\},$$

for $t > 0$, respectively.

These notions are helpful to classify uniformly convex and uniformly smooth Banach spaces, respectively.

Definition 4. Let X be a Banach space, and $\delta_X(\varepsilon)$, $0 < \varepsilon < 2$, and $\rho_X(t)$, $t > 0$, be its modulus of convexity and modulus of smoothness, respectively. Then, the Banach space X is said to be uniformly convex if $\delta_X(\varepsilon) > 0$ for every $\varepsilon > 0$, and uniformly smooth if $\lim\limits_{t \to 0} \frac{\rho_X(t)}{t} = 0$.

3 Main results

The main result of this paper is the following one from [1, Theorem 4.1]. In contrast to [1], we now present its full proof here.

Theorem 5. *The space $\ell^p(\widehat{G})$ is uniformly convex and uniformly smooth for $1 < p < \infty$.*

To prove Theorem 5, we first present the following lemma.

Lemma 6. *Let $1 < p < \infty$, $0 < \varepsilon \leq 2$ and $t > 0$. Let q be a conjugate index of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$.*

1. If $1 < p \leq 2$, then

$$\delta_{\ell^p(\widehat{G})}(\varepsilon) \geq \frac{\varepsilon^q}{q \cdot 2^q}, \quad \rho_{\ell^p(\widehat{G})}(t) \leq \frac{t^p}{p}.$$

2. If $2 < p < \infty$, then

$$\delta_{\ell^p(\widehat{G})}(\varepsilon) \geq \frac{\varepsilon^p}{p \cdot 2^p}, \quad \rho_{\ell^p(\widehat{G})}(t) \leq \frac{t^q}{q}.$$

Proof. 1. Let $1 < p \leq 2$ be given. We begin by proving the inequality for the modulus of convexity. Assume that $H_1, H_2 \in \ell^p(\widehat{G})$ satisfy both $\|H_1\|_{\ell^p(\widehat{G})} = \|H_2\|_{\ell^p(\widehat{G})} = 1$ and $\|H_1 - H_2\|_{\ell^p(\widehat{G})} = \varepsilon$. By Proposition 2, we then have the following:

$$\left(\left\|\frac{H_1 + H_2}{2}\right\|_{\ell^p(\widehat{G})}^q + \left(\frac{\varepsilon}{2}\right)^q\right)^{1/q} = \left(\left\|\frac{H_1 + H_2}{2}\right\|_{\ell^p(\widehat{G})}^q + \left\|\frac{H_1 - H_2}{2}\right\|_{\ell^p(\widehat{G})}^q\right)^{1/q}$$

$$\leq \left(\frac{1}{2}\left(\|H_1\|_{\ell^p(\widehat{G})}^p + \|H_2\|_{\ell^p(\widehat{G})}^p\right)\right)^{1/p} = 1.$$

Thus, we obtain the inequality:

$$\left\|\frac{H_1 + H_2}{2}\right\|_{\ell^p(\widehat{G})}^q \leq 1 - \left(\frac{\varepsilon}{2}\right)^q,$$

which implies that

$$\left\|\frac{H_1 + H_2}{2}\right\|_{\ell^p(\widehat{G})} \leq \left(1 - \left(\frac{\varepsilon}{2}\right)^q\right)^{1/q} \leq 1 - \frac{\varepsilon^q}{q2^q},$$

where the last inequality follows from Bernoulli's inequality, $(1 + x)^t \leq 1 + tx$ for all real numbers $0 \leq t \leq 1$ and $x \geq -1$. Finally, rewriting the expression as

$$\frac{\varepsilon^q}{q2^q} \leq 1 - \left\|\frac{H_1 + H_2}{2}\right\|_{\ell^p(\widehat{G})},$$

and taking the infimum over all $H_1, H_2 \in \ell^p(\widehat{G})$ that satisfy the initial conditions, we conclude that

$$\delta_{\ell^p(\widehat{G})}(\varepsilon) \geq \frac{\varepsilon^q}{q2^q}, \quad 0 < \varepsilon \leq 2.$$

We now demonstrate the inequality for the modulus of smoothness. Let $H_1, H_2 \in \ell^p(\widehat{G})$ satisfy $\|H_1\|_{\ell^p(\widehat{G})} = \|H_2\|_{\ell^p(\widehat{G})} = 1$. It is important to note that the function $f(x) = x^q$ for $x > 0$, where $q \geq 2$, is convex. Now, define

$$x_1 = \|H_1 + tH_2\|_{\ell^p(\widehat{G})}, \quad x_2 = \|H_1 - tH_2\|_{\ell^p(\widehat{G})}.$$

By the definition of convexity, we have

$$f\left(\frac{x_1 + x_2}{2}\right) = \left(\frac{\|H_1 + tH_2\|_{\ell^p(\widehat{G})} + \|H_1 - tH_2\|_{\ell^p(\widehat{G})}}{2}\right)^q$$
$$\leq \frac{f(x_1)}{2} + \frac{f(x_2)}{2} = \frac{\|H_1 + tH_2\|^q_{\ell^p(\widehat{G})}}{2} + \frac{\|H_1 - tH_2\|^q_{\ell^p(\widehat{G})}}{2}.$$

The right hand side simplifies to

$$\frac{2^q}{2}\left(\left\|\frac{H_1 + tH_2}{2}\right\|^q_{\ell^p(\widehat{G})} + \left\|\frac{H_1 - tH_2}{2}\right\|^q_{\ell^p(\widehat{G})}\right).$$

Combining this with the last inequality, we have

$$\frac{\|H_1 + tH_2\|_{\ell^p(\widehat{G})} + \|H_1 - tH_2\|_{\ell^p(\widehat{G})}}{2} \leq \frac{2}{2^{1/q}}\left(\left\|\frac{H_1 + tH_2}{2}\right\|^q_{\ell^p(\widehat{G})} + \left\|\frac{H_1 - tH_2}{2}\right\|^q_{\ell^p(\widehat{G})}\right)^{1/q}.$$

Next, applying Proposition 2 to the right-hand side, we obtain

$$\frac{\|H_1 + tH_2\|_{\ell^p(\widehat{G})} + \|H_1 - tH_2\|_{\ell^p(\widehat{G})}}{2} \leq \frac{2}{2^{1/q}}\left(\frac{1}{2}\left(\|H_1\|^p_{\ell^p(\widehat{G})} + \|tH_2\|^p_{\ell^p(\widehat{G})}\right)\right)^{1/p} = (1 + t^p)^{1/p}.$$

Thus, we have

$$\frac{\|H_1 + tH_2\|_{\ell^p(\widehat{G})} + \|H_1 - tH_2\|_{\ell^p(\widehat{G})}}{2} - 1 \leq (1 + t^p)^{1/p} - 1 \leq \frac{t^p}{p},$$

where the final inequality follows from Bernoulli's inequality. Finally, taking the supremum over all $H_1, H_2 \in \ell^p(\widehat{G})$ that satisfy the initial conditions, we conclude

$$\rho_{\ell^p(\widehat{G})} \leq \frac{t^p}{p}, \quad t > 0.$$

2. The proof of the second part follows a similar approach to part (i) and utilizes the inequality from Proposition 2, so we omit the details. □

Proof of Theorem 5. Let $1 < p \leq 2$. By Lemma 6, we have

$$\delta_{\ell^p(\widehat{G})}(\varepsilon) \leq \frac{\varepsilon^q}{q \cdot 2^q}, \quad \forall \varepsilon > 0,$$

and

$$\lim_{t \to 0} \frac{\rho_{\ell^p(\widehat{G})}}{t} \leq \lim_{t \to 0} \frac{t^{p-1}}{p} = 0.$$

Therefore, $\ell^p(\widehat{G})$ is uniformly convex and uniformly smooth for $1 < p \leq 2$. The case for $2 < p < \infty$ follows similarly from Lemma 6, so we omit the details. □

The Milman-Pettis theorem states that every uniformly convex and every uniformly smooth Banach space is reflexive [10, Proposition 1.e.3]. Hence, as a consequence of Theorem 5, we easily derive the following fact.

Corollary 7. *The space $\ell^p(\widehat{G})$ is reflexive for* $1 < p < \infty$.

Another proof of this result is given in [13, Theorem 2].

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Ахымбек М.Е., Туленов К.С. ГИЛЬБЕРТ-ШМИДТ ИДЕАЛЫНА НЕГІЗДЕЛГЕН $\ell^p(\widehat{G})$ КЕҢІСТІКТЕРІНІҢ БІРҚАЛЫПТЫ ДӨҢЕСТІГІ МЕН БІРҚАЛЫПТЫ ТЕГІСТІГІ.

Бұл жұмыста біз Гильберт-Шмидт идеалына негізделген ықшам топтың унитарлы дуалымен байланысты ℓ^p-кеңістіктерінің бірқалыпты дөңестігі мен бірқалыпты тегістігін зерттейміз.

Түйін сөздер: Компакт топ, компакт топтармен қауымдастырылған ℓ^p-кеңістіктері, бірқалыпты тегіс Банах кеңістігі, бірқалыпты дөңес Банас кеңістігі.

Ахымбек М.Е., Туленов К.С. РАВНОМЕРНАЯ ВЫПУКЛОСТЬ И РАВНОМЕРНАЯ ГЛАДКОСТЬ ПРОСТРАНСТВ $\ell^p(\widehat{G})$ НА ОСНОВЕ ИДЕАЛА ГИЛЬБЕРТА-ШМИДТА.

В данной работе изучаются свойства равномерной выпуклости и равномерной гладкости ℓ^p-пространств, связанных с унитарным сопряженным к компактной группе, основанные на идеале Гильберта-Шмидта.

Ключевые слова: Компактная группа, ℓ^p-пространства ассоциированные с компактной группой, равномерно гладкое банахово пространство, равномерно выпуклое банахово пространство.