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3-nil alternative, pre-Lie, and assosymmetric operads

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Abstract. Alternative algebras are vital for studying and modeling systems that deviate from strict associativity but maintain enough structure to be useful. Indeed, alternative algebras generalize associative algebras by relaxing the strict associativity condition. Alternative algebras naturally include the octonions, which are a key example of a non-associative division algebra. The octonions are part of the Cayley-Dickson construction and play a critical role in geometry, topology, and theoretical physics, especially in string theory and exceptional Lie groups. The origin of alternative algebras lies in the historical exploration of division algebras and their applications extend to various mathematical and physical disciplines, especially in understanding non-associative algebraic structures. In this paper, we consider free alternative algebra with the additional identity $x^3 = 0$. For motivation, we refer to the dual operad of the alternative operad. Also, we obtain pre-Lie algebra with the identity $x^3 = 0$ from binary perm algebra. Finally, we consider assosymmetric algebra with identity $x^3 = 0$.

Keywords. alternative algebra, pre-Lie algebra, assosymmetric algebra, polynomial identities.

1 Introduction

An algebra is called alternative if it satisfies the following identities:

$$(ab)c - a(bc) = -(ac)b + a(cb), \tag{1}$$

$$(ab)c - a(bc) = -(ba)c + b(ac). \tag{2}$$

A natural source of alternative algebras is Artin's theorem, which states that its every two-generated subalgebra of alternative algebra is associative [11]. Let us demonstrate some works related to the subvarieties of the variety of alternative algebras. In [9], the authors

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constructed a basis of the free alternative algebra with identity $[a,b][c,d]=0$ and proved that every metabelian Malcev algebra can be embedded into appropriate alternative algebra under commutator. In [5], the authors considered a variety of alternative algebras with the identity

$$(ab)c+(cb)a=(ac)b+(ca)b, \tag{3}$$

which coincides with a variety of binary perm algebras. There is given a basis of the free alternative algebra with identity (3) and described a complete list of identities of algebras that appear under commutator and anti-commutator.

The variety of alternative algebras is a natural generalization of the variety of associative algebras. On the other side, the dual operad of the alternative operad is an associative operad with additional identity $x^3=0$. So, we obtain

$$\mathcal{A}lt^! = \mathcal{A}s + \{x^3=0\}.$$

Also, we obtain the following trivial result which immediately follows from the definitions given above:

Theorem 1. *Let $\mathcal{A}lt_3$ be a variety of alternative algebras defined by identity $x^3=0$. Then every two-generated algebra from $\mathcal{A}lt_3$ lies in $\mathcal{A}lt^!$, i.e., $\mathcal{A}lt+\{x^3=0\}=\mathcal{A}lt^!{}_2$.*

All described motivations can be illustrated as inclusions of the varieties as follows:

$$\begin{array}{ccc} \mathcal{A}s & \supset & \mathcal{A}s+\{x^3=0\} = \mathcal{A}lt^! \\ \cap & & \cap \qquad\qquad\quad \cap \\ \mathcal{A}lt & \supset & \mathcal{A}lt+\{x^3=0\} = \mathcal{A}lt^!{}_2 \end{array}$$

Also, we consider Koszul dual operad $\mathcal{P}_2{}^!$, where $\mathcal{P}_2$ is a variety of binary perm algebra, i.e., it is an alternative operad with additional identity (3). It turns out that $\mathcal{P}_2{}^!$ is a variety of pre-Lie algebras with two additional independent identities, where one of them is $x^3=0$. In addition, it is observed the fact that an algebra from $\mathcal{P}_2{}^!$ is a Lie algebra with an additional independent identity of degree 5. The situation looks like for the Novikov algebras under commutator [3]. For Novikov algebras, there occurs a standard identity of degree 5

$$\sum_{\sigma\in S_4}(-1)^{\sigma}[x_{\sigma(1)},[x_{\sigma(2)},[x_{\sigma(3)},[x_{\sigma(4)},x_5]]]]=0.$$

Finally, we consider assosymmetric algebra with identities generated by 1-dimensional invariant basis vectors which are described in [7]. These identities are

$$\sum_{\sigma\in S_3}(-1)^{\sigma}x_{\sigma(1)}(x_{\sigma(2)}x_{\sigma(3)}) \ \text{ and } \ \sum_{\sigma\in S_3}x_{\sigma(1)}(x_{\sigma(2)}x_{\sigma(3)}).$$

Indeed, considered alternative and pre-Lie algebras with identity $x^3 = 0$ is equivalent to the alternative and pre-Lie algebras with identity generated by 1-dimensional invariant basis vectors of identities space of degree 3, see [7]. For more details on assosymmetric and pre-Lie algebras, see [1, 6, 8, 10].

We consider all algebras over a field K of characteristic 0.

2 Some properties of algebras with identity $x^3 = 0$

Definition 2. An alternative algebra with additional identity $x^3 = 0$ is called a 3-nil alternative algebra. We denote by Alt_3 and $Alt_3\langle X\rangle$ the variety of 3-nil alternative algebras and free algebra if the variety Alt_3, respectively.

In characteristic 0, the identity $x^3 = 0$ comes to

$$(xy)z + (yx)z + (xz)y + (zx)y + (yz)x + (zy)x = 0. \tag{4}$$

By using (1) and (2), the identity (4) can be rewritten as

$$x(yz) + x(zy) + y(xz) + y(zx) + z(xy) + z(yx) = 0.$$

Both identities that are obtained from $x^3 = 0$ give

$$\{\{a, b\}, c\} + \{\{b, c\}, a\} + \{\{c, a\}, b\} = 0. \tag{5}$$

Proposition 3. *The polarization of 3-nil alternative algebra gives*

$$[x, \{y, z\}] = \{[x, y], z\} + \{[x, z], y\}$$

and

$$\{\{x, y\}, z\} = 1/3([x, [y, z]] - [[x, z], y]).$$

Proof. It can be stated by straightforward calculations. □

Indeed, the identity (4) and its consequence in alternative algebra can be rewritten as

$$\sum_{\sigma\in S_3} (x_{\sigma(1)}x_{\sigma(2)})x_{\sigma(3)} = 0$$

and

$$\sum_{\sigma\in S_3} x_{\sigma(1)}(x_{\sigma(2)}x_{\sigma(3)}) = 0.$$

These identities correspond to the 1-dimensional invariant basis vectors.

Theorem 4. *An operad $\mathcal{A}lt_3$ is self-dual.*

Proof. Firstly, let us fix a multilinear basis of algebra $\mathcal{A}lt_3$ of degree 3. That is

$$c(ba) = -(ac)b + (cb)a + a(cb),$$

$$c(ab) = (ca)b + (ac)b - a(cb),$$
$$b(ca) = (ac)b + (bc)a - a(cb),$$
$$(ab)c = -(ac)b + a(cb) + a(bc),$$
$$b(ac) = -(ca)b - (ac)b - (cb)a - (bc)a - a(bc)$$

and

$$(ba)c = -(ca)b - (cb)a - (bc)a - a(cb) - a(bc).$$

The Lie-admissibility condition for $S \otimes U$ gives the defining identities of the operad $\mathcal{A}lt_3^!$, where S is a 3-nil alternative algebra. Then

$$[[a \otimes u, b \otimes v], c \otimes w] = (ab)c \otimes (uv)w - (ba)c \otimes (vu)w - c(ab) \otimes w(uv) + c(ba) \otimes w(vu) =$$
$$(-(ac)b + a(cb) + a(bc)) \otimes (uv)w - (-(ca)b - (cb)a - (bc)a - a(cb) - a(bc)) \otimes (vu)w$$
$$- ((ca)b + (ac)b - a(cb)) \otimes w(uv) + (-(ac)b + (cb)a + a(cb)) \otimes w(vu).$$

Also, we obtain

$$[[b \otimes v, c \otimes w], a \otimes u] = (bc)a \otimes (vw)u - (cb)a \otimes (wv)u - a(bc) \otimes u(vw) + a(cb) \otimes u(wv).$$

and

$$[[c \otimes w, a \otimes u], b \otimes v] = (ca)b \otimes (wu)v - (ac)b \otimes (uw)v - b(ca) \otimes v(wu) + b(ac) \otimes v(uw) =$$
$$(ca)b \otimes (wu)v - (ac)b \otimes (uw)v - ((ac)b + (bc)a - a(cb)) \otimes v(wu)$$
$$+ (-(ca)b - (ac)b - (cb)a - (bc)a - a(bc)) \otimes v(uw).$$

Calculating the sum and collecting the same basis monomials, we obtain

$$[[a \otimes u, b \otimes v], c \otimes w] + [[b \otimes v, c \otimes w], a \otimes u] + [[c \otimes w, a \otimes u], b \otimes v] =$$
$$(ac)b \otimes (-(uv)w - w(uv) - w(vu) - (uw)v - v(wu) - v(uw))$$
$$+ a(cb) \otimes ((uv)w + (vu)w + w(uv) + w(vu) + u(wv) + v(wu))$$
$$+ a(bc) \otimes ((uv)w + (vu)w - u(vw) - v(uw)) + (ca)b \otimes ((vu)w - w(uv) + (wu)v - v(uw))$$
$$+ (cb)a \otimes ((vu)w + w(vu) - (wv)u - v(uw)) + (bc)a \otimes ((vu)w + (vw)u - v(wu) - v(uw)) = 0.$$

From the right sides of the tensors, we obtain the identities (1), (2) and (4) which means that the operad $\mathcal{A}lt_3$ is self-dual. □

We denote by $\mathcal{P}_2$ and $\mathcal{P}_2\langle X\rangle$ the variety of binary perm algebras and free binary perm algebra. Let us calculate the dual operad of binary perm algebra ${\mathcal{P}_2}^!$. As above, we first fix the multilinear basis of binary perm algebra of degree 3. That is

$$(bc)a = (ba)c - (ac)b + (ab)c,$$

$$(cb)a = (ca)b + (ac)b - (ab)c,$$

$$a(bc) = c(ab) - (ca)b + (ab)c,$$

$$a(cb) = -c(ab) + (ca)b + (ac)b,$$

$$b(ac) = -c(ab) + (ca)b + (ba)c,$$

$$c(ba) = -c(ab) + 2(ca)b + (ac)b - (ab)c$$

and

$$b(ca) = c(ab) - (ca)b + (ba)c - (ac)b + (ab)c.$$

Performing similar calculations as above, we obtain

$$\begin{aligned}
&[[a\otimes u, b\otimes v], c\otimes w] + [[b\otimes v, c\otimes w], a\otimes u] + [[c\otimes w, a\otimes u], b\otimes v] = \\
&(ab)c\otimes (uv)w - (ba)c\otimes (vu)w - c(ab)\otimes w(uv) + c(ba)\otimes w(vu) \\
&+ (bc)a\otimes (vw)u - (cb)a\otimes (wv)u - a(bc)\otimes u(vw) + a(cb)\otimes u(wv) \\
&+ (ca)b\otimes (wu)v - (ac)b\otimes (uw)v - b(ca)\otimes v(wu) + b(ac)\otimes v(uw) = \\
&(ab)c\otimes (uv)w - (ba)c\otimes (vu)w - c(ab)\otimes w(uv) + (-c(ab) + 2(ca)b + (ac)b - (ab)c)\otimes w(vu)
\end{aligned}$$

$$\begin{aligned}
&+ ((ba)c - (ac)b + (ab)c)\otimes (vw)u - ((ca)b + (ac)b - (ab)c)\otimes (wv)u \\
&-(c(ab)-(ca)b+(ab)c)\otimes u(vw)+(-c(ab)+(ca)b+(ac)b)\otimes u(wv)+(ca)b\otimes (wu)v-(ac)b\otimes (uw)v \\
&- (c(ab) - (ca)b + (ba)c - (ac)b + (ab)c)\otimes v(wu) + (-c(ab) + (ca)b + (ba)c)\otimes v(uw) = \\
&(ab)c\otimes ((uv)w - w(vu) + (vw)u + (wv)u - u(vw) - v(wu)) \\
&+ (ba)c\otimes (-(vu)w + (vw)u - v(wu) + v(uw)) \\
&+ c(ab)\otimes (-w(uv) - w(vu) - u(vw) - u(wv) - v(wu) - v(uw)) \\
&+ (ca)b\otimes (2w(vu) - (wv)u + u(vw) + u(wv) + (wu)v + v(wu) + v(uw)) \\
&(ac)b\otimes (w(vu) - (vw)u - (wv)u + u(wv) - (uw)v + v(wu) + v(uw)) = 0.
\end{aligned}$$

From all calculations, we obtain the following result:

Theorem 5. *The following identities define an operad which corresponds to* ${\mathcal{P}_2}^!$*:*

$$(v, w, u) = (v, u, w),$$

$$(u,w,v)+(w,v,u)+(v,w,u)=0$$

and

$$w(uv)+w(vu)+u(vw)+u(wv)+v(wu)+v(uw)=0,$$

where (v,w,u) *stands for associator.*

Theorem 6. *The operad* ${\mathcal{P}_2}^!$ *is not Koszul.*

Proof. Calculating the dimension of the operad ${\mathcal{P}_2}^!$ by means of the package [2], we get the following result:

n	1	2	3	4	5
$\dim({\mathcal{P}_2}^!(n))$	1	2	7	26	67

According to the obtained table and [5], the first few terms of the Hilbert series of the operads ${\mathcal{P}_2}^!$ and $\mathcal{P}_2$ are

$$H(t)=-t+t^2-5t^3/6+6t^4/24-5t^5/120+O(t^6)$$

and

$$H^!(t)=-t+t^2-7t^3/6+26t^4/24-67t^5/120+O(t^6)$$

Thus,

$$H(H^!(t))=t+31t^5/60+O(t^6)\neq t.$$

By [4], the operad ${\mathcal{P}_2}^!$ is not Koszul. □

Proposition 7. *The polarization of* ${\mathcal{P}_2}^!$ *algebra gives*

$$[[a,b],c]+[[b,c],a]+[[c,a],b]=0,$$

$$\{\{a,b\},c\}+\{\{b,c\},a\}+\{\{c,a\},b\}=0,$$

$$[\{a,b\},c]+[\{b,c\},a]+[\{c,a\},b]=0$$

and

$$\{[b,c],a\}=\{\{a,b\},c\}+\{[c,a],b\}+[\{a,b\},c]+1/3[[b,c],a]-1/3[[c,a],b].$$

Proof. It can be stated by straightforward calculations. □

The next natural operad that we have to consider is an assosymmetric operad with identity $x^3=0$.

Definition 8. An algebra is called a 3-nil assosymmetric if it satisfies the following identities:

$$(x,y,z)=(x,z,y),$$

$$(x,y,z)=(y,x,z)$$

and

$$x(yz)+x(zy)+y(xz)+y(zx)+z(xy)+z(yx)=0.$$

In other words, this is an assosymmetric algebra with identity generated by a 1-dimensional invariant basis vector
$$\sum_{\sigma \in S_3} x_{\sigma(1)}(x_{\sigma(2)}x_{\sigma(3)}).$$

Remark 9. *A 3-nil assosymmetric algebra does not satisfy the identity (4) under anti-commutator.*

Let us calculate $S \otimes U$, where S is a 3-nil assosymmetric algebra.

$$\begin{aligned}
&[[a\otimes u, b\otimes v], c\otimes w] + [[b\otimes v, c\otimes w], a\otimes u] + [[c\otimes w, a\otimes u], b\otimes v] = \\
&(ab)c\otimes(uv)w - (ba)c\otimes(vu)w - c(ab)\otimes w(uv) + c(ba)\otimes w(vu) \\
&+(-c(ba) - 5c(ab) + 4(ca)b - (ba)c - (ac)b - (ab)c)\otimes(vw)u \\
&-(c(ba)-c(ab)+(ca)b)\otimes(wv)u - (c(ab)-(ca)b+(ab)c)\otimes u(vw) + (c(ab)-(ca)b+(ac)b)\otimes u(wv) \\
&+(ca)b\otimes(wu)v - (ac)b\otimes(uw)v - (-c(ba) - 4c(ab) + 3(ca)b - (ba)c - (ac)b - (ab)c)\otimes v(wu) \\
&+(c(ab) - (ca)b + (ba)c)\otimes v(uw) = \\
&(ab)c\otimes((uv)w - (vw)u - u(vw) + v(wu)) + (ba)c\otimes(-(vu)w - (vw)u + v(wu) + v(uw)) \\
&+c(ab)\otimes(-w(uv) - 5(vw)u + (wv)u - u(vw) + u(wv) - 4v(wu) + v(uw)) \\
&+c(ba)\otimes(w(vu) - (vw)u - (wv)u + v(wu)) \\
&+(ca)b\otimes(4(vw)u - (wv)u + u(vw) - u(wv) + (wu)v - 3v(wu) - v(uw)) \\
&+(ac)b\otimes(-(vw)u + u(wv) - (uw)v + v(wu)).
\end{aligned}$$

The above calculations give the following result:

Theorem 10. *The dual operad of 3-nil assosymmetric operad is an alternative operad with the additional identity:*

$$(uv)w - (vu)w - (uw)v + (wu)v + (vw)u - (wv)u = 0.$$

So, this is an alternative algebra with the additional identity generated by a 1-dimensional invariant basis vector
$$\sum_{\sigma \in S_3} (-1)^{\sigma}(x_{\sigma(1)}x_{\sigma(2)})x_{\sigma(3)}.$$

Theorem 11. *The operad governed by the variety of 3-nil assosymmetric algebras is not Koszul.*

Proof. Calculating the dimension of these operads by means of the package [2], the first few terms of the Hilbert series of these operads are

$$H(t) = -t + t^2 - t^3 + 13t^4/24 - 15t^5/120 + O(t^6)$$

and

$$H^!(t) = -t + t^2 - t^3 + 13t^4/24 - 9t^5/120 + O(t^6).$$

Thus,

$$H(H^!(t)) = t + 19t^5/20 + O(t^6) \neq t.$$

By [4], such operad is not Koszul. □

Let us define an assosymmetric algebra with the additional identity which is generated by the 1-dimensional invariant basis vector

$$\sum_{\sigma \in S_3} (-1)^\sigma (x_{\sigma(1)} x_{\sigma(2)}) x_{\sigma(3)}.$$

Such algebra also satisfies another identity

$$\sum_{\sigma \in S_3} (-1)^\sigma x_{\sigma(1)} (x_{\sigma(2)} x_{\sigma(3)}).$$

Theorem 12. *An assosymmetric operad with identity* $\sum_{\sigma \in S_3} (-1)^\sigma (x_{\sigma(1)} x_{\sigma(2)}) x_{\sigma(3)}$ *is self-dual.*

Proof. As before

$$\begin{gathered}
[[a\otimes u, b\otimes v], c\otimes w] + [[b\otimes v, c\otimes w], a\otimes u] + [[c\otimes w, a\otimes u], b\otimes v] = \\
(ab)c\otimes(uv)w - (ba)c\otimes(vu)w - c(ab)\otimes w(uv) + c(ba)\otimes w(vu) \\
+ (c(ba) - c(ab) + (ba)c + (ac)b - (ab)c)\otimes(vw)u \\
-(c(ba)-c(ab)+(ca)b)\otimes(wv)u-(c(ab)-(ca)b+(ab)c)\otimes u(vw)+(c(ab)-(ca)b+(ac)b)\otimes u(wv) \\
+ (ca)b\otimes(wu)v - (ac)b\otimes(uw)v \\
- (c(ba) - (ca)b + (ba)c + (ac)b - (ab)c)\otimes v(wu) + (c(ab) - (ca)b + (ba)c)\otimes v(uw) = \\
(ab)c\otimes((uv)w - (vw)u - u(vw) + v(wu)) + (ba)c\otimes(-(vu)w + (vw)u - v(wu) + v(uw)) \\
+ c(ab)\otimes(-w(uv) - (vw)u + (wv)u - u(vw) + u(wv) + v(uw)) \\
+ c(ba)\otimes(w(vu) + (vw)u - (wv)u - v(wu)) + (ac)b\otimes((vw)u + u(wv) - (uw)v - v(wu)) \\
+ (ca)b)\otimes(-(wv)u + u(vw) - u(wv) + (wu)v + v(wu) - v(uw)).
\end{gathered}$$

The right parts of the tensors are equal to 0 if and only if the given operad is self-dual. □

Theorem 13. *The operad governed by the variety of assosymmetric algebras with identity*

$$\sum_{\sigma \in S_3} (-1)^\sigma (x_{\sigma(1)} x_{\sigma(2)}) x_{\sigma(3)}$$

is not Koszul.

Proof. Calculating the dimension of this operad by means of the package [2], the first few terms of the Hilbert series of this operad is

$$H(t) = H^!(t) = -t + t^2 - t^3 + 14t^4/24 - 12t^5/120 + O(t^6)$$

Thus,

$$H(H^!(t)) = t + 7t^5/10 + O(t^6) \neq t.$$

As before, such operad is not Koszul. □

The last remaining algebra is assosymmetric algebra which admits the identity

$$\{\{x, y\}, z\} + \{\{y, z\}, x\} + \{\{z, x\}, y\} = 0$$

under anti-commutator. For such an operad, let us calculate its Koszul dual operad:

$$\begin{gathered}
[[a \otimes u, b \otimes v], c \otimes w] + [[b \otimes v, c \otimes w], a \otimes u] + [[c \otimes w, a \otimes u], b \otimes v] = \\
(ab)c \otimes (uv)w - (ba)c \otimes (vu)w - c(ab) \otimes w(uv) + c(ba) \otimes w(vu) \\
+ (-c(ba) - 2c(ab) + (ca)b - (ba)c - (ac)b - (ab)c) \otimes (vw)u - (c(ba) - c(ab) + (ca)b) \otimes (wv)u \\
- (c(ab) - (ca)b + (ab)c) \otimes u(vw) + (c(ab) - (ca)b + (ac)b) \otimes u(wv) \\
+ (ca)b \otimes (wu)v - (ac)b \otimes (uw)v \\
- (-c(ba) - c(ab) - (ba)c - (ac)b - (ab)c) \otimes v(wu) + (c(ab) - (ca)b + (ba)c) \otimes v(uw) = \\
(ab)c \otimes ((uv)w - (vw)u - u(vw) + v(wu)) + (ba)c \otimes (-(vu)w - (vw)u + v(wu) + v(uw)) \\
c(ab) \otimes (-w(uv) - 2(vw)u + (wv)u - u(vw) + u(wv) + v(wu) + v(uw)) \\
+ c(ba) \otimes (w(vu) - (vw)u - (wv)u + v(wu)) + (ac)b \otimes (-(vw)u + u(wv) - (uw)v + v(wu)) \\
+ (ca)b \otimes ((vw)u - (wv)u + u(vw) - u(wv) + (wu)v - v(uw)).
\end{gathered}$$

We obtain an alternative operad with the identity

$$(vw)u - (wv)u + (wu)v - u(wv) - v(uw) + u(vw) = 0.$$

Let us check Koszulness condition for the last considered operad:

Theorem 14. *An assosymmetric operad with identity* $\{\{x, y\}, z\} + \{\{y, z\}, x\} + \{\{z, x\}, y\} = 0$ *is not Koszul.*

Proof. Calculating the dimension of these operads by means of the package [2], the first few terms of the Hilbert series of these operads are

$$H(t) = -t + t^2 - t^3 + 12t^4/24 - 15t^5/120 + O(t^6)$$

and

$$H^!(t) = -t + t^2 - t^3 + 12t^4/24 - 9t^5/120 + O(t^6)$$

Thus,

$$H(H^!(t)) = t + 6t^5/5 + O(t^6) \neq t.$$

So, such operad is not Koszul. □

3 Some identities under commutator

For $\mathcal{A}\langle X\rangle$, we define commutator algebra $\mathcal{A}^{(-)}\langle X\rangle$ which is obtained from $\mathcal{A}\langle X\rangle$ under the operation

$$[x, y] = xy - yx.$$

Analogically, we define anti-commutator algebra $\mathcal{A}^{(+)}\langle X\rangle$ under the operation $\{x, y\} = xy + yx$.

Theorem 15. *An algebra ${\mathcal{P}_2}^{!\,(-)}\langle X\rangle$ satisfies the following identities:*

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0,$$

$$\frac{1}{3}([[[[a, b], d], c], e] + [[[[a, b], d], e], c] + [[[[a, d], b], c], e] + [[[[a, d], b], e], c]) \\ + [[[[a, c], b], e], d] + [[[[a, c], d], e], b] + [[[[a, e], b], c], d] + [[[[a, e], d], c], b] = 0,$$

and

$$\frac{1}{2}([[[[a, b], c], d], e] + [[[[a, b], d], c], e] + [[[[a, c], b], d], e] \\ + [[[[a, c], d], b], e] + [[[[a, d], b], c], e] + [[[[a, d], c], b], e]) = \\ [[[[a, b], c], e], d] + [[[[a, b], d], e], c] + [[[[a, c], b], e], d]+ \\ [[[[a, c], d], e], b] + [[[[a, d], b], e], c] + [[[[a, d], c], e], b].$$

Proof. The Jacobi identity follows from the fact that every pre-Lie algebra under commutator is a Lie. The identities of degree 5 can be obtained using means of the package [2]. Since both identities are written as a sum of basis monomials of the free Lie algebra, they are independent of anti-commutative and Jacobi identities. □

Theorem 16. *An algebra $Alt_3^{(-)}\langle X\rangle$ satisfies the following identities:*

$$[[a, c], [b, d]] = [[[a, b], c], d] + [[[b, c], d], a] + [[[c, d], a], b] + [[[d, a], b], c]$$

and

$$[[[[a, b], d], e], c] + [[[[a, b], e], d], c] + [[[[a, c], d], e], b] + [[[[a, c], e], d], b] \\ - [[[[a, d], b], c], e] - [[[[a, d], c], b], e] - [[[[a, e], b], c], d] - [[[[a, e], c], b], d] = 0.$$

Proof. The first identity corresponds to the Malcev identity. The identity of degree 5 can be obtained using means of the package [2]. □

4 Some identities under anti-commutator

Theorem 17. *An algebra* $Alt_3^{(+)}\langle X\rangle$ *satisfies the following identities:*

$$\{a,\{b,c\}\}+\{\{a,c\},b\}+\{\{a,b\},c\}=0, \tag{6}$$

$$\{\{a,d\},\{b,c\}\}=-\{\{a,c\},\{b,d\}\}+\{\{a,\{c,d\}\},b\}+\{a,\{b,\{c,d\}\}\}, \tag{7}$$

$$\begin{aligned}\{\{\{a,d\},c\},b\}+\{\{\{a,c\},d\},b\}+\{\{\{a,d\},b\},c\}+\{\{\{a,c\},b\},d\}\\ +\{\{\{a,b\},d\},c\}+\{\{\{a,b\},c\},d\}=0,\end{aligned} \tag{8}$$

$$\begin{aligned}&-\{\{\{\{a,e\},c\},d\},b\}+\{\{\{\{a,c\},d\},e\},b\}+\{\{\{\{a,d\},e\},b\},c\}+\{\{\{\{a,c\},e\},b\},d\}\\ &+\{\{\{\{a,e\},b\},c\},d\}-\{\{\{\{a,d\},b\},c\},e\}+2\{\{\{\{a,c\},b\},e\},d\}-\{\{\{\{a,b\},d\},e\},c\}\\ &-2\{\{\{\{a,b\},d\},c\},e\}-\{\{\{\{a,b\},c\},d\},e\}=0,\end{aligned} \tag{9}$$

$$\begin{aligned}&-\{\{\{\{a,d\},c\},e\},b\}-\{\{\{\{a,c\},d\},e\},b\}-\{\{\{\{a,d\},e\},b\},c\}-\{\{\{\{a,c\},e\},b\},d\}\\ &-\{\{\{\{a,e\},b\},d\},c\}-\{\{\{\{a,d\},b\},e\},c\}-\{\{\{\{a,e\},b\},c\},d\}+\{\{\{\{a,d\},b\},c\},e\}\\ &-\{\{\{\{a,c\},b\},e\},d\}+\{\{\{\{a,c\},b\},d\},e\}+\{\{\{\{a,b\},d\},e\},c\}+2\{\{\{\{a,b\},d\},c\},e\}\\ &+\{\{\{\{a,b\},c\},e\},d\}+2\{\{\{\{a,b\},c\},d\},e\}=0.\end{aligned} \tag{10}$$

Proof. The identity (6) is taken from (5). Other identities can be obtained using means of the package [2].

Proposition 18. *The identities (7), (8), (9) and (10) are consequence of commutative identity and (6).*

We use the identity (6) in two different ways on monomial $\{\{\{a,b\},c\},d\}$ as follows:

$$\begin{aligned}&\{\{\{a,b\},c\},d\}=^{(6)}-\{\{\{a,c\},b\},d\}-\{\{a,\{b,c\}\},d\}=^{(6)}\{\{\{a,c\},d\},b\}+\{\{a,c\},\{b,d\}\}\\ &+\{a,\{\{b,c\},d\}+\{\{a,d\},\{b,c\}\}=^{(6)}-\{\{\{a,d\},c\},b\}-\{\{a,\{c,d\}\},b\}+\{\{a,c\},\{b,d\}\}\\ &+\{\{a,d\},\{b,c\}\}-\{a,\{\{b,d\},c\}\}-\{a,\{b,\{c,d\}\}\},\\ &\{\{\{a,b\},c\},d\}=^{(6)}-\{\{\{a,b\},d\},c\}-\{\{a,b\},\{c,d\}\}=^{(6)}\{\{\{a,b\},d\},c\}+\{\{a,\{c,d\}\},b\}\\ &+\{a,\{b,\{c,d\}\}\}=^{(6)}-\{\{\{a,d\},c\},b\}-\{\{a,d\},\{b,c\}\}-\{\{a,c\},\{b,d\}\}-\{a,\{\{b,d\},c\}\}\\ &+\{\{a,\{c,d\}\},b\}+\{a,\{b,\{c,d\}\}\}.\end{aligned}$$

By equating and collecting similar monomials, we get

$$\{\{a,d\},\{b,c\}\}+\{\{a,c\},\{b,d\}\}-\{\{a,\{c,d\}\},b\}-\{a,\{b,\{c,d\}\}\}=0,$$

which correspond to (7).

To find the next identity, we use the previously obtained identities:

$$\{a,\{b,\{c,d\}\}\} =^{(6)} -\{a,\{\{b,d\},c\}\} - \{a,\{\{b,c\},d\}\} =^{(6)} \{\{a,c\},\{b,d\}\} + \{\{a,\{b,d\}\},c\}$$
$$+ \{\{a,d\},\{b,c\}\} + \{\{a,\{b,c\}\},d\} =^{(6)} -\{\{\{a,c\},d\},b\} - \{\{\{a,c\},b\},d\} - \{\{\{a,d\},b\},c\}$$
$$- \{\{\{a,b\},d\},c\} - \{\{\{a,d\},c\},b\} - \{\{\{a,d\},b\},c\} - \{\{\{a,c\},b\},d\} - \{\{\{a,b\},c\},d\},$$

$$\{a,\{b,\{c,d\}\}\} =^{(7)} \{\{a,\{c,d\}\},b\} - \{\{a,b\},\{c,d\}\} = \{\{\{a,d\},c\},b\}$$
$$+ \{\{\{a,c\},d\},b\} + \{\{\{a,b\},d\},c\} + \{\{\{a,b\},c\},d\}.$$

By equalizing them, we obtain the identity (8). In the same way, we obtain the remained identities. More explicitly, there identity (9) can be prove as follows:

$$\{\{\{a,d\},c\},\{b,e\}\} =^{(6)} -\{\{\{\{a,d\},c\},e\},b\} - \{\{\{\{a,d\},c\},b\},e\} =^{(8)} -\{\{\{\{a,d\},c\},e\},b\}$$
$$+ \{\{\{\{a,c\},d\},b\},e\} + \{\{\{\{a,d\},b\},c\},e\} + \{\{\{\{a,c\},b\},d\},e\} + \{\{\{\{a,b\},d\},c\},e\}$$
$$+ \{\{\{\{a,b\},c\},d\},e\},$$

$$\{\{\{a,d\},c\},\{b,e\}\} =^{(8)} -\{\{\{a,c\},d\},\{b,e\}\} - \{\{\{a,d\},\{b,e\}\},c\} - \{\{\{a,c\},\{b,e\}\},d\}$$
$$- \{\{\{a,\{b,e\}\},d\},c\} - \{\{\{a,\{b,e\}\},c\},d\} =^{(6)} \{\{\{\{a,c\},d\},e\},b\} + \{\{\{\{a,c\},d\},b\},e\}$$
$$+ \{\{\{\{a,d\},e\},b\},c\} + \{\{\{\{a,d\},e\},b\},c\} + \{\{\{\{a,e\},b\},c\},d\} + \{\{\{\{a,b\},e\},c\},d\}$$
$$+ \{\{\{\{a,e\},b\},d\},c\} + \{\{\{\{a,b\},e\},d\},c\} + \{\{\{\{a,c\},e\},b\},d\} + \{\{\{\{a,c\},b\},e\},d\}$$
$$=^{(8)} \{\{\{\{a,c\},d\},e\},b\} + \{\{\{\{a,c\},d\},b\},e\} + \{\{\{\{a,d\},e\},b\},c\} + \{\{\{\{a,d\},e\},b\},c\}$$
$$+ \{\{\{\{a,e\},b\},c\},d\} + \{\{\{\{a,b\},e\},c\},d\} + \{\{\{\{a,e\},b\},d\},c\} - \{\{\{\{a,b\},d\},e\},c\}$$
$$- \{\{\{\{a,b\},e\},c\},d\} - \{\{\{\{a,b\},d\},c\},e\} - \{\{\{\{a,b\},c\},e\},d\} - \{\{\{\{a,b\},c\},d\},e\}$$
$$+ \{\{\{\{a,c\},e\},b\},d\} + \{\{\{\{a,c\},b\},e\},d\}.$$

As before, we equalize them. Reducing the same monomials, we obtain (9).

For the last identity, we use previous identities to the monomial $\{\{\{a,e\},c\},\{b,d\}\}$ in two different ways as follows:

$$\{\{\{a,e\},c\},\{b,d\}\} =^{(6)} -\{\{\{\{a,e\},c\},d\},b\} - \{\{\{\{a,e\},c\},b\},d\} =^{(8)} -\{\{\{\{a,e\},c\},d\},b\}$$
$$+ \{\{\{\{a,c\},e\},b\},d\} + \{\{\{\{a,e\},b\},d\},c\} + \{\{\{\{a,c\},b\},e\},d\} + \{\{\{\{a,b\},e\},c\},d\}$$
$$+ \{\{\{\{a,b\},c\},e\},d\},$$

$$\{\{\{a,e\},c\},\{b,d\}\} =^{(8)} -\{\{\{a,c\},e\},\{b,d\}\} - \{\{\{a,e\},\{b,c\}\},d\} - \{\{\{a,c\},\{b,d\}\},e\}$$
$$- \{\{\{a,\{b,d\}\},e\},c\} - \{\{\{a,\{b,d\}\},c\},e\} =^{(8)} \{\{\{\{a,c\},e\},b\},d\} + \{\{\{\{a,c\},e\},d\},b\}$$
$$+ \{\{\{\{a,e\},d\},b\},c\} + \{\{\{\{a,e\},b\},d\},c\} + \{\{\{\{a,c\},d\},b\},e\} + \{\{\{\{a,c\},b\},d\},e\}$$
$$+ \{\{\{\{a,d\},b\},e\},c\} + \{\{\{\{a,b\},d\},e\},c\} + \{\{\{\{a,d\},b\},c\},e\} + \{\{\{\{a,b\},d\},c\},e\}$$
$$=^{(6),(8)} -\{\{\{\{a,c\},d\},e\},b\} - \{\{\{\{a,c\},e\},b\},d\} - \{\{\{\{a,c\},d\},b\},e\} - \{\{\{\{a,c\},b\},e\},d\}$$
$$- \{\{\{\{a,c\},b\},d\},e\} + \{\{\{\{a,c\},e\},b\},d\} - \{\{\{\{a,d\},e\},b\},c\} - \{\{\{\{a,e\},b\},d\},c\}$$
$$- \{\{\{\{a,d\},b\},e\},c\} - \{\{\{\{a,b\},d\},e\},c\} + \{\{\{\{a,e\},b\},d\},c\} + \{\{\{\{a,c\},d\},b\},e\}$$
$$+ \{\{\{\{a,c\},b\},d\},e\} + \{\{\{\{a,d\},b\},e\},c\} + \{\{\{\{a,b\},d\},e\},c\} + \{\{\{\{a,b\},e\},c\},d\}$$
$$+ \{\{\{\{a,b\},d\},c\},e\} + \{\{\{\{a,b\},c\},e\},d\} + \{\{\{\{a,b\},c\},d\},e\}.$$

Finally, we obtain the needed result. □

Remark 19. *An algebra* ${\mathcal{P}_2}^{!\,(+)}\langle X\rangle$ *satisfies the following identity:*

$$\{a,\{b,c\}\} + \{\{a,c\},b\} + \{\{a,b\},c\} = 0,$$

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Дүйсенбай Е. Қ., Сартаев Б. К., Текебай А. А. 3-НӨЛДІК АЛЬТЕРНАТИВТІ, ПРЕ-ЛИ ЖӘНЕ АССИММЕТРИЯЛЫҚ ОПЕРАДАЛАР

Альтернативті алгебралар қатаң ассоциативтіктен ауытқыған, бірақ пайдалы болу үшін жеткілікті құрылымды сақтайтын жүйелерді зерттеу және модельдеу үшін өте маңызды. Шынында да, альтернативті алгебралар ассоциативті алгебраларды қатаң ассоциативтік шартты босаңсыту арқылы жалпылайды. Альтернативті алгебралар, әрине, ассоциативті емес бөліну алгебрасының негізгі мысалы болып табылатын октониондарды қамтиды. Октониондар Кейли-Диксон құрылымының бөлігі болып табылады және геометрияда, топологияда және теориялық физикада, әсіресе желі теориясында және ерекше Ли топтарында маңызды рөл атқарады. Альтернативті алгебраның шығу тегі бөліну алгебрасын тарихи зерттеуде жатыр және олардың қолданылуы әртүрлі математикалық және физикалық пәндерге, әсіресе ассоциативті емес алгебралық құрылымдарды түсінуге таралады. Бұл жұмыста біз $x^3 = 0$ қосымша сәйкестігімен еркін альтернативті алгебраны қарастырамыз. Мотивация үшін альтернативті операданың қос операсына жүгінеміз. Сондай-ақ, біз екілік перм алгебрасынан $x^3 = 0$ сәйкестігі бар пре-Ли алгебрасын аламыз. Соңында, $x^3 = 0$ сәйкестігі бар ассосимметриялық алгебраны қарастырамыз.

Түйін сөздер: альтернативті алгебра, пре-Ли алгебра, ассосимметриялық алгебра, көпмүшелік сәйкестіктер.

Дуйсенбай Е. К., Сартаев Б. К., Текебай А. А. 3-НУЛЕВАЯ АЛЬТЕРНАТИВНАЯ, ПРЕ-ЛИ И АССОСИММЕТРИЧЕСКАЯ ОПЕРАДЫ

Альтернативные алгебры имеют решающее значение для изучения и моделирования систем, которые отклоняются от строгой ассоциативности, но сохраняют достаточную структуру, чтобы быть полезными в алгебре. Действительно, альтернативные алгебры обобщают ассоциативные алгебры, ослабляя условие строгой ассоциативности. Альтернативные алгебры естественным образом включают октонионы, которые являются ключевым примером неассоциативной алгебры с делением. Октонионы являются частью конструкции Кэли-Диксона и играют важную роль в геометрии, топологии и теоретической физике, особенно в теории струн и исключительных группах Ли. Происхождение альтернативных алгебр лежит в историческом исследовании алгебр с делением, и их приложения распространяются на различные математические и физические дисциплины, особенно в понимании неассоциативных алгебраических структур. В этой статье мы рассматриваем свободную альтернативную алгебру с дополнительным тождеством $x^3 = 0$. Для мотивации мы ссылаемся на двойственную операду альтернативной операды. Также мы получаем пре-Ли алгебру с тождеством $x^3 = 0$ из бинарной перм алгебры. Наконец, мы рассматриваем асосимметрическую алгебру с тождеством $x^3 = 0$.

Ключевые слова: альтернативная алгебра, пре-Ли алгебра, ассосимметрическая алгебра, полиномиальные тождества.