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On algebras of binary isolating formulas for weakly circularly minimal theories of convexity rank 2

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Abstract. This paper is devoted to the study of weakly circularly minimal circularly ordered structures. The simplest example of a circular order is a linear order with endpoints, in which the largest element is identified with the smallest. Another example is the order that arises when going around a circle. A circularly ordered structure is called weakly circularly minimal if any of its definable subsets is a finite union of convex sets and points. A theory is called weakly circularly minimal if all its models are weakly circularly minimal. Algebras of binary isolating formulas are described for $\aleph_0$-categorical 1-transitive non-primitive weakly circularly minimal theories of convexity rank 2 with a trivial definable closure having a monotonic-to-right function to the definable completion of a structure and non-having a non-trivial equivalence relation partitioning the universe of a structure into finitely many convex classes.

Keywords. algebra of binary formulas, $\aleph_0$-categorical theory, weak circular minimality, circularly ordered structure, convexity rank.

1 Preliminaries

Algebras of binary formulas are a tool for describing relationships between elements of the sets of realizations of a one-type at the binary level with respect to the superposition of binary definable sets. A *binary isolating formula* is a formula of the form $\varphi(x, y)$ such that for some parameter a the formula $\varphi(a, y)$ isolates a complete type in $S(\{a\})$. The concepts and notations related to these algebras can be found in the papers [1, 2]. In recent years, algebras of binary formulas have been studied intensively and have been continued in the works [3]–[11].

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Let L be a countable first-order language. Throughout we consider L-structures and assume that L contains a ternary relational symbol K, interpreted as a circular order in these structures (unless otherwise stated).

Let $M = \langle M, \leq \rangle$ be a linearly ordered set. If we connect two endpoints of M (possibly, $-\infty$ and $+\infty$), then we obtain a circular order. More formally, the *circular order* is described by a ternary relation K satisfying the following conditions:

(co1) $\forall x \forall y \forall z (K(x, y, z) \to K(y, z, x))$;

(co2) $\forall x \forall y \forall z (K(x, y, z) \wedge K(y, x, z) \Leftrightarrow x = y \vee y = z \vee z = x)$;

(co3) $\forall x \forall y \forall z (K(x, y, z) \to \forall t [K(x, y, t) \vee K(t, y, z)])$;

(co4) $\forall x \forall y \forall z (K(x, y, z) \vee K(y, x, z))$.

The following observation relates linear and circular orders.

Fact 1. [12] *(i) If $\langle M, \leq \rangle$ is a linear ordering and K is the ternary relation derived from $\leq$ by the rule $K(x, y, z) :\Leftrightarrow (x \leq y \leq z) \vee (z \leq x \leq y) \vee (y \leq z \leq x)$, then K is a circular order relation on M.*

(ii) If $\langle N, K \rangle$ is a circular ordering and $a \in N$, then the relation $\leq_a$ defined on $M := N \setminus \{a\}$ by the rule $y \leq_a z :\Leftrightarrow K(a, y, z)$ is a linear order.

Thus, any linearly ordered structure is circularly ordered, since the relation of circular order is $\emptyset$-definable in an arbitrary linearly ordered structure. However, the opposite is not true. The following example shows that there are circularly ordered structures not being linearly ordered (in the sense that a linear ordering relation is not $\emptyset$-definable in an arbitrary circularly ordered structure).

Example 2. [13, 14] Let $\mathbb{Q}_2^* := \langle \mathbb{Q}_2, K, L \rangle$ be a circularly ordered structure, where $L = \{\sigma_0^2, \sigma_1^2\}$, for which the following conditions hold:

(i) its domain $\mathbb{Q}_2$ is a countable dense subset of the unit circle, no two points making the central angle π;

(ii) for distinct $a, b \in \mathbb{Q}_2$

$$(a, b) \in \sigma_0 \Leftrightarrow 0 < \arg(a/b) < \pi,$$

$$(a, b) \in \sigma_1 \Leftrightarrow \pi < \arg(a/b) < 2\pi,$$

where $\arg(a/b)$ means the value of the central angle between a and b clockwise.

Indeed, one can check that the linear order relation is not $\emptyset$-definable in this structure.

The notion of *weak circular minimality* was studied initially in [15]. Let $A \subseteq M$, where M is a circularly ordered structure. The set A is called *convex* if for any $a, b \in A$ the following property is satisfied: for any $c \in M$ with $K(a, c, b)$, $c \in A$ holds, or for any $c \in M$ with $K(b, c, a)$, $c \in A$ holds. A *weakly circularly minimal structure* is a circularly ordered structure $M = \langle M, K, \ldots \rangle$ such that any definable (with parameters) subset of M is a union of finitely many convex sets in M. The study of weakly circularly minimal structures was continued in the papers [16]–[21].

Let M be an $\aleph_0$-categorical weakly circularly minimal structure, $G := \mathrm{Aut}(M)$. Following the standard group theory terminology, the group G is called *k-transitive* if for any pairwise distinct $a_1, a_2, \ldots, a_k \in M$ and pairwise distinct $b_1, b_2, \ldots, b_k \in M$ there exists $g \in G$ such that $g(a_1) = b_1, g(a_2) = b_2, \ldots, g(a_k) = b_k$. A *congruence* on M is an arbitrary G-invariant equivalence relation on M. The group G is called *primitive* if G is 1-transitive and there are no non-trivial proper congruences on M.

(1) $K_0(x, y, z) := K(x, y, z) \wedge y \neq x \wedge y \neq z \wedge x \neq z$.

(2) $K(u_1, \ldots, u_n)$ denotes a formula saying that all subtuples of the tuple $\langle u_1, \ldots, u_n\rangle$ having the length 3 (in ascending order) satisfy K; similar notations are used for K_0.

(3) Let A, B, C be disjoint convex subsets of a circularly ordered structure M. We write $K(A, B, C)$ if for any $a, b, c \in M$ with $a \in A$, $b \in B$, $c \in C$ we have $K(a, b, c)$. We extend naturally that notation using, for instance, the notation $K_0(A, d, B, C)$ if $d \notin A \cup B \cup C$ and $K_0(A, d, B) \wedge K_0(d, B, C)$ holds.

Further we need the notion of the definable completion of a circularly ordered structure, introduced in [15]. Its linear analog was introduced in [22]. A *cut* $C(x)$ in a circularly ordered structure M is a maximal consistent set of formulas of the form $K(a, x, b)$, where $a, b \in M$. A cut is said to be *algebraic* if there exists $c \in M$ that realizes it. Otherwise, such a cut is said to be *non-algebraic*. Let $C(x)$ be a non-algebraic cut. If there is some $a \in M$ such that either for all $b \in M$ the formula $K(a, x, b) \in C(x)$, or for all $b \in M$ the formula $K(b, x, a) \in C(x)$, then $C(x)$ is said to be *rational*. Otherwise, such a cut is said to be *irrational*. A *definable cut* in M is a cut $C(x)$ with the following property: there exist $a, b \in M$ such that $K(a, x, b) \in C(x)$ and the set $\{c \in M \mid K(a, c, b)$ and $K(a, x, c) \in C(x)\}$ is definable. The *definable completion* $\overline{M}$ of a structure M consists of M together with all definable cuts in M that are irrational (essentially $\overline{M}$ consists of endpoints of definable subsets of the structure M).

[15] Let $F(x, y)$ be an L-formula such that $F(M, b)$ is convex infinite co-infinite for each $b \in M$. Let $F^{\ell}(y)$ be the formula saying y is a left endpoint of $F(M, y)$:

$$\exists z_1 \exists z_2 [K_0(z_1, y, z_2) \wedge \forall t_1 (K(z_1, t_1, y) \wedge t_1 \neq y \to \neg F(t_1, y)) \wedge$$

$$\forall t_2 (K(y, t_2, z_2) \wedge t_2 \neq y \to F(t_2, y))].$$

We say that $F(x, y)$ is *convex-to-right* if

$$M \models \forall y \forall x [F(x, y) \to F^{l}(y) \wedge \forall z (K(y, z, x) \to F(z, y))].$$

If $F_1(x, y), F_2(x, y)$ are arbitrary convex-to-right formulas we say F_2 is *bigger than* F_1 if there is $a \in M$ with $F_1(M, a) \subset F_2(M, a)$. If M is 1-transitive and this holds for some a, it holds for all a. This gives a total ordering on the (finite) set of all convex-to-right formulas $F(x, y)$ (viewed up to equivalence modulo $Th(M)$).

Consider $F(M, a)$ for arbitrary $a \in M$. In general, $F(M, a)$ has no right endpoint in M. For example, if $dcl(\{a\}) = \{a\}$ holds for some $a \in M$ then for any convex-to-right formula $F(x, y)$ and any $a \in M$ the formula $F(M, a)$ has no right endpoint in M. We write $f(y) :=$ rend $F(M, y)$, assuming that $f(y)$ is the right endpoint of the set $F(M, y)$ that lies in general in the definable completion $\overline{M}$ of M. Then f is a function mapping M in $\overline{M}$.

Let $F(x, y)$ be a convex-to-right formula. We say that $F(x, y)$ is *equivalence-generating* if for any $a, b \in M$ such that $M \models F(b, a)$ the following holds:

$$M \models \forall x(K(b, x, a) \wedge x \neq a \rightarrow [F(x, a) \leftrightarrow F(x, b)]).$$

Lemma 3. [20] *Let M be an $\aleph_0$-categorical 1-transitive weakly circularly minimal structure, $F(x, y)$ be a convex-to-right formula that is equivalence-generating. Then $E(x, y) := F(x, y) \vee F(y, x)$ is an equivalence relation partitioning M into infinite convex classes.*

Let $E(x, y)$ be an $\emptyset$–definable equivalence relation partitioning M into infinite convex classes. Suppose that y lies in $\overline{M}$ (non-obligatory in M). Then

$$E^*(x, y) := \exists y_1 \exists y_2 [y_1 \neq y_2 \wedge \forall t(K(y_1, t, y_2) \rightarrow E(t, x)) \wedge K_0(y_1, y, y_2)].$$

Let M, N be circularly ordered structures. The 2-*reduct* of M is a circularly ordered structure with the same universe of M and consisting of predicates for each $\emptyset$-definable relation on M of arity ≤ 2 as well as of the ternary predicate K for the circular order, but does not have other predicates of arities more than two. We say that the structure M is *isomorphic* to N *up to binarity* or *binarily isomorphic to* N if the 2-reduct of M is isomorphic to the 2-reduct of N.

Let f be a unary function from M to $\overline{M}$. We say that f is *monotonic-to-right (left) on* M if it preserves (reverses) the relation K_0, i.e. for any $a, b, c \in M$ such that $K_0(a, b, c)$, we have $K_0(f(a), f(b), f(c))$ $(K_0(f(c), f(b), f(a)))$.

The following definition can be used in a circular ordered structure as well.

Definition 4. [23], [24] Let T be a weakly o-minimal theory, M be a sufficiently saturated model of T, $A \subseteq M$. *The rank of convexity of the set* A $(RC(A))$ is defined as follows:

1) $RC(A) = -1$ if $A = \emptyset$.

2) $RC(A) = 0$ if A is finite and non-empty.

3) $RC(A) \geq 1$ if A is infinite.

4) $RC(A) \geq \alpha + 1$ if there exist a parametrically definable equivalence relation $E(x, y)$ and an infinite sequence of elements $b_i \in A, i \in \omega$, such that:

- For every $i, j \in \omega$ whenever $i \neq j$ we have $M \models \neg E(b_i, b_j)$;
- For every $i \in \omega$, $RC(E(x, b_i)) \geq \alpha$ and $E(M, b_i)$ is a convex subset of A.

5) $RC(A) \geq \delta$ if $RC(A) \geq \alpha$ for all $\alpha < \delta$, where δ is a limit ordinal.

If $RC(A) = \alpha$ for some α, we say that $RC(A)$ is defined. Otherwise (i.e. if $RC(A)) \geq \alpha$ for all α), we put $RC(A) = \infty$.

The rank of convexity of a formula $\phi(x, \bar{a})$, where $\bar{a} \in M$, is defined as the rank of convexity of the set $\phi(M, \bar{a})$, i.e. $RC(\phi(x, \bar{a})) := RC(\phi(M, \bar{a}))$.

The rank of convexity of a 1-type p is defined as the rank of convexity of the set $p(M)$, i.e. $RC(p) := RC(p(M))$.

In particular, a theory has convexity rank 1 if there is no definable (with parameters) equivalence relations with infinitely many infinite convex classes.

The following theorem characterizes up to binarity $\aleph_0$–categorical 1-transitive non-primitive weakly circularly minimal structures M of convexity rank greater than 1 having both a trivial definable closure and a convex-to-right formula $R(x, y)$ such that $r(y) := R(M, y)$ is monotonic-to-right on M:

Theorem 5. [16] *Let M be an $\aleph_0$–categorical 1-transitive non-primitive weakly circularly minimal structure of convexity rank greater than 1, $dcl(\{a\}) = \{a\}$ for some $a \in M$. Suppose that there exists a convex-to-right formula $R(x, y)$ such that $r(y) := R(M, y)$ is monotonic-to-right on M. Then M is isomorphic up to binarity to*

$$M'_{s,m,k} := \langle M, K^3, E_1^2, E_2^2, \ldots, E_s^2, E_{s+1}^2, R^2 \rangle,$$

where M is a circularly ordered structure, M is densely ordered, $s \geq 1$; E_{s+1} is an equivalence relation partitioning M into m infinite convex classes without endpoints; E_i for every $1 \leq i \leq s$ is an equivalence relation partitioning every E_{i+1}-class into infinitely many infinite convex E_i-subclasses without endpoints so that the induced order on E_i-subclasses is dense without endpoints; $R(M, a)$ has no right endpoint in M and $r^k(a) = a$ for all $a \in M$ and some $k \geq 2$, where $r^k(y) := r(r^{k-1}(y))$; for every $1 \leq i \leq s + 1$ and any $a \in M$

$$M'_{s,m,k} \models \neg E_i^*(a, r(a)) \wedge \forall y (E_i(y, a) \rightarrow \exists u [E_i^*(u, r(a)) \wedge E_i^*(u, r(y))]),$$

$m = 1$ or k divides m.

In [7] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories with a primitive automorphism group. In [8] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank 1 with a 1-transitive non-primitive automorphism group and a non-trivial definable closure. In [9]–[10] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank greater than 1 with a 1-transitive non-primitive automorphism group and a non-trivial definable closure. In [11] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank

1 with a 1-transitive non-primitive automorphism group and a trivial definable closure. Here we describe algebras of binary isolating formulas for $\aleph_0$-categorical 1-transitive non-primitive weakly circularly minimal theories of convexity rank 2 with a trivial definable closure having a monotonic-to-right function to the definable completion of a structure and non-having a non-trivial equivalence relation partitioning the universe of a structure into finitely many convex classes.

2 Results

Definition 6. [2] Let $p \in S_1(\emptyset)$ be non-algebraic. The algebra $\mathcal{P}_{\nu(p)}$ is said to be *deterministic* if $u_1 \cdot u_2$ is a singleton for any labels $u_1, u_2 \in \rho_{\nu(p)}$.

Generalizing the last definition, we say that the algebra $\mathcal{P}_{\nu(p)}$ is *m-deterministic* if the product $u_1 \cdot u_2$ consists of at most m elements for any labels $u_1, u_2 \in \rho_{\nu(p)}$. We also say that an m-deterministic algebra $\mathcal{P}_{\nu(p)}$ is *strictly m-deterministic* if it is not $(m-1)$-deterministic. Obviously, strict 1-determinacy of an algebra is equivalent its determinacy.

Example 7. Consider the structure $M'_{1,1,2} := \langle M, K^3, E_1^2, R^2 \rangle$ from Theorem 5 with the condition that the function $r(y) := R(M, y)$ is monotonic-to-right on M.

We assert that $Th(M'_{1,1,2})$ has seven binary isolating formulas:

$$\theta_0(x,y) := x = y,$$

$$\theta_1(x,y) := K_0(x,y,r(x)) \wedge E_1(x,y),$$

$$\theta_2(x,y) := K_0(x,y,r(x)) \wedge \neg E_1(x,y) \wedge \neg E_1^*(y,r(x)),$$

$$\theta_3(x,y) := K_0(x,y,r(x)) \wedge \neg E_1(x,y) \wedge E_1^*(y,r(x)),$$

$$\theta_4(x,y) := K_0(r(x),y,x) \wedge \neg E_1(x,y) \wedge E_1^*(y,r(x)),$$

$$\theta_5(x,y) := K_0(r(x),y,x) \wedge \neg E_1(x,y) \wedge \neg E_1^*(y,r(x)),$$

$$\theta_6(x,y) := K_0(r(x),y,x) \wedge E_1(x,y),$$

and the following holds for any $a \in M$:

$$K_0(\theta_0(a,M), \theta_1(a,M), \theta_2(a,M), \theta_3(a,M), \theta_4(a,M), \theta_5(a,M), \theta_6(a,M)).$$

Define labels for these formulas as follows:

$$\text{label } k \text{ for } \theta_k(x,y), \text{ where } 0 \le k \le 6.$$

It is easy to check that for the algebra $\mathfrak{P}_{M'_{1,1,2}}$ the Cayley table has the following form:

$\cdot$	0	1	2	3	4	5	6
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$	$\{6\}$
1	$\{1\}$	$\{1\}$	$\{2\}$	$\{3,4\}$	$\{4\}$	$\{5\}$	$\{0,1,6\}$
2	$\{2\}$	$\{2\}$	$\{2,3,4,5\}$	$\{5\}$	$\{5\}$	$\{0,1,2,5,6\}$	$\{2\}$
3	$\{3\}$	$\{3,4\}$	$\{5\}$	$\{6\}$	$\{0,1,6\}$	$\{2\}$	$\{3\}$
4	$\{4\}$	$\{4\}$	$\{5\}$	$\{0,1,6\}$	$\{1\}$	$\{2\}$	$\{3,4\}$
5	$\{5\}$	$\{5\}$	$\{0,1,2,5,6\}$	$\{2\}$	$\{2\}$	$\{2,3,4,5\}$	$\{5\}$
6	$\{6\}$	$\{0,1,6\}$	$\{2\}$	$\{3\}$	$\{3,4\}$	$\{5\}$	$\{6\}$

By the Cayley table the algebra $\mathfrak{P}_{M'_{1,1,2}}$ is commutative and strictly 5-deterministic.

Theorem 8. *The algebra $\mathfrak{P}_{M'_{1,1,k}}$ of binary isolating formulas with monotonic-to-right function r has $3k+1$ labels, is commutative and strictly 5-deterministic for every $k \geq 2$.*

Proof of Theorem 8. We assert that the algebra $\mathfrak{P}_{M'_{1,1,k}}$ has $3k+1$ binary isolating formulas:

$$\theta_0(x,y) := x = y,$$

$$\theta_1(x,y) := K_0(x,y,r(x)) \wedge E_1(x,y),$$

$$\theta_2(x,y) := K_0(x,y,r(x)) \wedge \neg E_1(x,y) \wedge \neg E_1^*(y,r(x)),$$

$$\theta_3(x,y) := K_0(x,y,r(x)) \wedge E_1^*(y,r(x)),$$

$$\theta_{3l-2}(x,y) := K_0(r^{l-1}(x),y,r^l(x)) \wedge E_1^*(y,r^{l-1}(x)), \text{ where } 2 \leq l \leq k-1,$$

$$\theta_{3l-1}(x,y) := K_0(r^{l-1}(x),y,r^l(x)) \wedge \neg E_1^*(y,r^{l-1}(x)) \wedge \neg E_1^*(y,r^l(x)), \text{ where } 2 \leq l \leq k-1,$$

$$\theta_{3l}(x,y) := K_0(r^{l-1}(x),y,r^l(x)) \wedge E_1^*(y,r^l(x)), \text{ where } 2 \leq l \leq k-1,$$

$$\theta_{3k-2}(x,y) := K_0(r^{k-1}(x),y,x) \wedge E_1^*(y,r^{k-1}(x)),$$

$$\theta_{3k-1}(x,y) := K_0(r^{k-1}(x),y,x) \wedge \neg E_1^*(y,r^{k-1}(x)) \wedge \neg E_1(y,x),$$

$$\theta_{3k}(x,y) := K_0(r^{k-1}(x),y,x) \wedge E_1(y,x).$$

Thus, we have $1+3+3(k-2)+3 = 3k+1$ binary isolating formulas. Moreover, we have defined the formulas so that for any $a \in M$ the following holds:

$$K_0(\theta_0(a,M), \theta_1(a,M), \theta_2(a,M), \ldots, \theta_{3k-1}(a,M), \theta_{3k}(a,M)).$$

Prove now that the algebra $\mathfrak{P}_{M'_{1,1,k}}$ is commutative and strictly 5-deterministic for every $k \geq 2$.

Firstly, obviously that $0 \cdot l = l \cdot 0 = \{l\}$ for any $0 \leq l \leq 3k$. Suppose further that $l_1 \neq 0$ and $l_2 \neq 0$.

Consider the following formula

$$\exists t[\theta_{l_1}(x,t) \wedge \theta_{l_2}(t,y)].$$

Case 1: $l_1 = 3m_1 - 2$ for some $1 \leq m_1 \leq k-1$.

We have: $K_0(r^{m_1-1}(x), t, r^{m_1}(x))$ and $E_1^*(t, r^{m_1-1}(x))$.

Let $l_2 = 3m_2-2$ for some $1 \leq m_2 \leq k-1$, i.e. $K_0(r^{m_2-1}(t), y, r^{m_2}(t))$ and $E_1^*(y, r^{m_2-1}(t))$. Then we obtain the following:

$$K_0(r^{m_1+m_2-2}(x), y, r^{m_1+m_2-1}(x)) \text{ and } E_1^*(y, r^{m_1+m_2-2}(x)).$$

Suppose firstly that $(3m_1-2)+(3m_2-2) < 3k+1$. We assert that in this case $m_1+m_2-1 \leq k$. Then we have $l_1 \cdot l_2 = \{3(m_1+m_2-1)-2\}$.

Obviously, $(3m_1-2)+(3m_2-2) \neq 3k+1$.

Suppose now that $(3m_1-2)+(3m_2-2) > 3k+1$. Let $s = (m_1+m_2-1)[\text{mod k}]$. Obviously, $0 \leq s \leq k-1$. If $s = 0$, we have $K_0(r^{k-1}(x), y, x)$ and $E_1^*(y, r^{k-1}(x))$, i.e. $l_1 \cdot l_2 = \{3k-2\}$. If $1 \leq s \leq k-1$ then we have $K_0(r^{s-1}(x), y, r^s(x))$ and $E_1^*(y, r^{s-1}(x))$, i.e. $l_1 \cdot l_2 = \{3s-2\}$.

Let now $l_2 = 3m_2-1$ for some $1 \leq m_2 \leq k-1$. Then we have the following: $K_0(r^{m_2-1}(t)$, y, $r^{m_2}(t))$, $\neg E_1^*(y, r^{m_2-1}(t))$ and $\neg E_1^*(y, r^{m_2}(t))$. Whence we obtain:

$$K_0(r^{m_1+m_2-2}(x), y, r^{m_1+m_2-1}(x)), \neg E_1^*(y, r^{m_1+m_2-2}(x)) \text{ and } \neg E_1^*(y, r^{m_1+m_2-1}(x)).$$

Suppose firstly $(3m_1-2)+(3m_2-1) < 3k+1$. We assert that $m_1+m_2-1 \leq k$. In this case we have $l_1 \cdot l_2 = \{3(m_1+m_2-1)-1\}$.

Obviously, also $(3m_1-2)+(3m_2-1) \neq 3k+1$.

Suppose now $(3m_1-2)+(3m_2-1) > 3k+1$. Let $s = (m_1+m_2-1)[\text{mod k}]$. Obviously, $0 \leq s \leq k-1$. If $s = 0$, we have $K_0(r^{k-1}(x), y, x)$, $\neg E_1^*(y, r^{k-1}(x))$ and $\neg E_1(y, x)$, i.e. $l_1 \cdot l_2 = \{3k-1\}$. If $1 \leq s \leq k-1$ then we have $K_0(r^{s-1}(x), y, r^s(x))$, $\neg E_1^*(y, r^{s-1}(x))$ and $\neg E_1^*(y, r^s(x))$, i.e. $l_1 \cdot l_2 = \{3s-1\}$.

Consider the product $l_2 \cdot l_1$. We have: $K_0(r^{m_2-1}(x), t, r^{m_2}(x))$, $\neg E_1^*(t, r^{m_2-1}(x))$, $\neg E_1^*(t$, $r^{m_2}(x))$, $K_0(r^{m_1-1}(t), y, r^{m_1}(t))$ and $E_1^*(y, r^{m_1-1}(t))$. Whence we obtain:

$$K_0(r^{m_1+m_2-2}(x), y, r^{m_1+m_2-1}(x)), \neg E_1^*(y, r^{m_1+m_2-2}(x)) \text{ and } \neg E_1^*(y, r^{m_1+m_2-1}(x)).$$

If $(3m_1-2)+(3m_2-1) < 3k+1$ then $l_2 \cdot l_1 = \{3(m_1+m_2-1)-1\}$. If $(3m_1-2)+(3m_2-1) > 3k+1$ then in case $s = 0$ we obtain $l_2 \cdot l_1 = \{3k-1\}$, and in case $1 \leq s \leq k-1$ we obtain $l_2 \cdot l_1 = \{3s-1\}$.

Let now $l_2 = 3m_2$ for some $1 \leq m_2 \leq k-1$. Then we have: $K_0(r^{m_2-1}(t), y, r^{m_2}(t))$ and $E_1^*(y, r^{m_2}(t))$. Whence we obtain:

$$E_1^*(y, r^{m_1+m_2-1}(x)), \text{ and either } K_0(r^{m_1+m_2-2}(x), y, r^{m_1+m_2-1}(x)) \text{ or}$$

$$K_0(r^{m_1+m_2-1}(x), y, r^{m_1+m_2}(x)).$$

Suppose firstly that $(3m_1-2)+3m_2 < 3k+1$. Then we assert that $m_1+m_2-1 < k$. Whence we obtain: $l_1 \cdot l_2 = \{3(m_1+m_2-1), 3(m_1+m_2-1)+1\}$.

Suppose now that $(3m_1-2)+3m_2 = 3k+1$. This case is possible since $(3m_1-2)+3m_2 = 3(m_1+m_2-1)+1$. We also have: $(3m_1-2)+3m_2 = 3k+1$ iff $m_1+m_2-1 = k$. Thus, we obtain: $E_1(y,x)$ and either $K_0(r^{k-1}(x), y, x)$ or $K_0(x, y, r(x))$, i.e. $l_1 \cdot l_2 = \{3k, 0, 1\}$.

Let now $(3m_1-2)+3m_2 > 3k+1$. Consider $s = (m_1+m_2-1)[\text{mod k}]$. We prove that $0 < s < k-1$. Indeed, $(3m_1-2)+3m_2 = 3(m_1+m_2-1)+1 > 3k+1$ iff $m_1+m_2-1 > k$. Since $m_1 \le k-1$ and $m_2 \le k-1$, $m_1+m_2-1 \le (k-1)+(k-1) = 1 = 2k-3$. Thus, $k < m_1+m_2-1 \le 2k-3$, whence $0 < s < k-1$. We have:

$$E_1^*(y, r^s(x)) \text{ and either } K_0(r^{s-1}(x), y, r^s(x)) \text{ or } K_0(r^s(x), y, r^{s+1}(x)).$$

Whence we obtain: $l_1 \cdot l_2 = \{3s, 3s+1\}$.

Consider the product $l_2 \cdot l_1$. We have: $K_0(r^{m_2-1}(x), t, r^{m_2}(x))$, $E_1^*(t, r^{m_2}(x))$, $K_0(r^{m_1-1}(t), y, r^{m_1}(t))$, and $E_1^*(y, r^{m_1-1}(t))$. Whence we obtain the following:

$$E_1^*(y, r^{m_1+m_2-1}(x)), \text{ and either } K_0(r^{m_1+m_2-2}(x), y, r^{m_1+m_2-1}(x)) \text{ or}$$

$$K_0(r^{m_1+m_2-1}(x), y, r^{m_1+m_2}(x)).$$

If $3m_2+(3m_1-1) < 3k+1$ then $l_2 \cdot l_1 = \{3(m_1+m_2-1), 3(m_1+m_2-1)+1\}$. If $3m_2+(3m_1-1) = 3k+1$ then $l_2 \cdot l_1 = \{3k, 0, 1\}$. If $3m_2+(3m_1-1) > 3k+1$ then $l_2 \cdot l_1 = \{3s, 3s+1\}$.

Let now $l_2 = 3k-2$, i.e. $K_0(r^{k-1}(t), y, t)$ and $E_1^*(y, r^{k-1}(t))$. Whence we obtain:

$$K_0(r^{m_1+k-2}(x), y, r^{m_1-1}(x)) \text{ and } E_1^*(y, r^{m_1+k-2}(x)),$$

i.e. $K_0(r^{m_1-2}(x), y, r^{m_1-1}(x))$ and $E_1^*(y, r^{m_1-2}(x))$. Consequently, $l_1 \cdot l_2 = \{3(m_1-1)-2\}$.

Consider the product $l_2 \cdot l_1$. We have: $K_0(r^{k-1}(x), t, x)$, $E_1^*(t, r^{k-1}(x))$, $K_0(r^{m_1-1}(t), y, r^{m_1}(t))$, and $E_1^*(y, r^{m_1-1}(t))$. Whence we obtain:

$$K_0(r^{m_1-2}(x), y, r^{m_1-1}(x)) \text{ and } E_1^*(y, r^{m_1-2}(x)),$$

i.e. $l_2 \cdot l_1 = \{3(m_1-1)-2\}$.

Let now $l_2 = 3k-1$, i.e. $K_0(r^{k-1}(t), y, t)$, $\neg E_1^*(y, r^{k-1}(t))$ and $\neg E_1^*(y, t)$. Whence we obtain:

$$K_0(r^{m_1-2}(x), y, r^{m_1-1}(x)), \neg E_1^*(y, r^{m_1-2}(x)) \text{ and } \neg E_1^*(y, r^{m_1-1}(x)).$$

Thus, $l_1 \cdot l_2 = \{3(m_1-1)-1\}$.

Consider the product $l_2 \cdot l_1$. Then we have: $K_0(r^{k-1}(x), t, x)$, $\neg E_1^*(t, r^{k-1}(x))$, $\neg E_1^*(x, t)$, $K_0(r^{m_1-1}(t), y, r^{m_1}(t))$, and $E_1^*(y, r^{m_1-1}(t))$. Whence we obtain:

$$K_0(r^{m_1-2}(x), y, r^{m_1-1}(x)), \neg E_1^*(y, r^{m_1-2}(x)) \text{ and } \neg E_1^*(y, r^{m_1-1}(x)),$$

i.e. $l_2 \cdot l_1 = \{3(m_1 - 1) - 1\}$.

Let now $l_2 = 3k$, i.e. $K_0(r^{k-1}(t), y, t)$, and $E_1^*(y, t)$. Whence we obtain:

$$E_1^*(y, r^{m_1-1}(x)) \text{ and either } K_0(r^{m_1-2}(x), y, r^{m_1-1}(x)) \text{ or } K_0(r^{m_1-1}(x), y, r^{m_1}(x)),$$

i.e. $l_1 \cdot l_2 = \{3(m_1 - 1), 3m_1 - 2\}$.

Consider the product $l_2 \cdot l_1$. Then we have: $K_0(r^{k-1}(x), t, x)$, $E_1^*(x, t)$, $K_0(r^{m_1-1}(t), y, r^{m_1}(t))$, and $E_1^*(y, r^{m_1-1}(t))$. Whence we obtain:

$$E_1^*(y, r^{m_1-1}(x)) \text{ and either } K_0(r^{m_1-2}(x), y, r^{m_1-1}(x)) \text{ or } K_0(r^{m_1-1}(x), y, r^{m_1}(x)),$$

i.e. $l_2 \cdot l_1 = \{3(m_1 - 1), 3m_1 - 2\}$.

Case 2. $l_1 = 3m_1 - 1$ for some $1 \le m_1 \le k - 1$.

We have the following: $K_0(r^{m_1-1}(x), t, r^{m_1}(x))$, $\neg E_1^*(t, r^{m_1-1}(x))$ and $\neg E_1^*(t, r^{m_1}(x))$.

Let $l_2 = 3m_2 - 1$ for some $1 \le m_2 \le k - 1$, i.e. $K_0(r^{m_2-1}(t), y, r^{m_2}(t))$, $\neg E_1^*(y, r^{m_2-1}(t))$ and $\neg E_1^*(y, r^{m_2}(t))$. Whence we obtain:

$$\text{either } K_0(r^{m_1+m_2-2}(x), y, r^{m_1+m_2-1}(x)) \text{ or } K_0(r^{m_1+m_2-1}(x), y, r^{m_1+m_2}(x)).$$

Suppose firstly that $(3m_1 - 1) + (3m_2 - 1) < 3k + 1$. It can be checked that $(3m_1 - 1) + (3m_2 - 1) < 3k + 1$ iff $m_1 + m_2 - 1 < k$. Then

$$l_1 \cdot l_2 = \{3(m_1 + m_2 - 1) - 1, 3(m_1 + m_2 - 1), 3(m_1 + m_2 - 1) + 1, 3(m_1 + m_2 - 1) + 2\}.$$

Let now $(3m_1-1)+(3m_2-1) = 3k+1$. This case is possible, and $(3m_1-1)+(3m_2-1) = 3k + 1$ iff $m_1 + m_2 - 1 = k$. Then we have: either $K_0(r^{k-1}(x), y, x)$ or $K_0(x, y, r(x))$. Consequently, $l_1 \cdot l_2 = \{3k - 1, 3k, 0, 1, 2\}$.

Let now $(3m_1 - 1) + (3m_2 - 1) > 3k + 1$. Clearly, $(3m_1 - 1) + (3m_2 - 1) > 3k + 1$ iff $m_1+m_2-1 > k$. Let $s = (m_1+m_2-1)[\text{mod k}]$. Since $k < m_1+m_2-1 \le k-1+k-1-1 = 2k-3$, we have $0 < s \le k-3$. Thus, we obtain: either $K_0(r^{s-1}(x), y, r^s(x))$ or $K_0(r^s(x), y, r^{s+1}(x)))$, whence $l_1 \cdot l_2 = \{3s - 1, 3s, 3s + 1, 3s + 2\}$.

Let $l_2 = 3m_2$ for some $1 \le m_2 \le k - 1$, i.e. $K_0(r^{m_2-1}(t), y, r^{m_2}(t))$ and $E_1^*(y, r^{m_2}(t))$. Whence we obtain:

$$K_0(r^{m_1+m_2-1}(x), y, r^{m_1+m_2}(x)), \neg E_1^*(y, r^{m_1+m_2-1}(x)) \text{ and } \neg E_1^*(y, r^{m_1+m_2}(x)).$$

Suppose firstly that $(3m_1 - 1) + 3m_2 < 3k + 1$. It can be checked that $(3m_1 - 1) + 3m_2 < 3k + 1$ iff $m_1 + m_2 - 1 < k$. In this case $l_1 \cdot l_2 = \{3(m_1 + m_2) - 1\}$.

The case $(3m_1-1)+3m_2 = 3k+1$ is impossible. Suppose that $(3m_1-1)+3m_2 > 3k+1$. It can be checked that $(3m_1-1)+3m_2 > 3k+1$ iff $m_1+m_2-1 \ge k$. Let $s = (m_1+m_2-1)[\text{mod k}]$. Since $k \le m_1 + m_2 - 1 \le k - 1 + k - 1 - 1 = 2k - 3$, we have $0 \le s \le k - 3$. Thus, we obtain:

$$K_0(r^s(x), y, r^{s+1}(x)), \neg E_1^*(y, r^s(x)) \text{ and } \neg E_1^*(y, r^{s+1}(x)),$$

whence $l_1 \cdot l_2 = \{3(s+1)-1\}$.

Consider the product $l_2 \cdot l_1$. We have: $K_0(r^{m_2-1}(x), t, r^{m_2}(x))$, $E_1^*(t, r^{m_2}(x))$, $K_0(r^{m_1-1}(t), y, r^{m_1}(t))$, $\neg E_1^*(y, r^{m_1-1}(t))$ and $\neg E_1^*(y, r^{m_1}(t))$. Whence we obtain:

$$K_0(r^{m_1+m_2-1}(x), y, r^{m_1+m_2}(x)), \neg E_1^*(y, r^{m_1+m_2-1}(x)) \text{ and } \neg E_1^*(y, r^{m_1+m_2}(x)).$$

If $(3m_1-1)+3m_2 < 3k+1$ then $l_2 \cdot l_1 = \{3(m_1+m_2)-1\}$. If $(3m_1-1)+3m_2 > 3k+1$ then $l_2 \cdot l_1 = \{3(s+1)-1\}$, where $s = (m_1+m_2-1)[\text{mod k}]$.

Let now $l_2 = 3k-2$, i.e. $K_0(r^{k-1}(t), y, t)$ and $E_1^*(y, r^{k-1}(t))$. Whence we obtain:

$$K_0(r^{m_1-2}(x), y, r^{m_1-1}(x)), \neg E_1^*(y, r^{m_1-2}(x)) \text{ and } \neg E_1^*(y, r^{m_1-1}(x)).$$

Consequently, $l_1 \cdot l_2 = \{3(m_1-1)-1\}$.

Consider the product $l_2 \cdot l_1$. We have: $K_0(r^{m_1-2}(x), y, r^{m_1-1}(x))$, $\neg E_1^*(y, r^{m_1-2}(x))$ and $\neg E_1^*(y, r^{m_1-1}(x))$, whence $l_2 \cdot l_1 = \{3(m_1-1)-1\}$.

Let now $l_2 = 3k-1$, i.e. $K_0(r^{k-1}(t), y, t)$, $\neg E_1^*(y, r^{k-1}(t))$ and $\neg E_1^*(y, t)$. Whence we obtain:

$$\text{either } K_0(r^{m_1-2}(x), y, r^{m_1-1}(x)) \text{ or } K_0(r^{m_1-1}(x), y, r^{m_1}(x)).$$

Clearly, $(3m_1-1)+3(k-1) > 3k+1$. Let $s = 3(m_1-2)$, whence $4 \le s \le k-5$. Then $l_1 \cdot l_2 = \{3(m_1-1)-1, 3(m_1-1), 3m_2-2, 3m_2-1\}$.

Consider the product $l_2 \cdot l_1$. Then we have: $K_0(r^{k-1}(x), t, x)$, $\neg E_1^*(t, r^{k-1}(x))$, $\neg E_1^*(x, t)$, $K_0(r^{m_1-1}(t), y, r^{m_1}(t))$, $\neg E_1^*(y, r^{m_1-1}(t))$, and $\neg E_1^*(y, r^{m_1}(t))$. Whence we obtain:

$$\text{either } K_0(r^{m_1-2}(x), y, r^{m_1-1}(x)) \text{ or } K_0(r^{m_1-1}(x), y, r^{m_1}(x)),$$

i.e. $l_2 \cdot l_1 = \{3(m_1-1)-1, 3(m_1-1), 3m_2-2, 3m_2-1\}$.

Let now $l_2 = 3k$, i.e. $K_0(r^{k-1}(t), y, t)$, and $E_1^*(y, t)$. Whence we obtain:

$$K_0(r^{m_1-2}(x), y, r^{m_1-1}(x)), \neg E_1^*(y, r^{m_1-1}(x)) \text{ and } \neg E_1^*(y, r^{m_1}(x)),$$

i.e. $l_1 \cdot l_2 = \{3m_1-1\}$.

Consider the product $l_2 \cdot l_1$. Then we have: $K_0(r^{k-1}(x), t, x)$, $E_1^*(x, t)$, $K_0(r^{m_1-1}(t), y, r^{m_1}(t))$, $\neg E_1^*(y, r^{m_1-1}(x))$ and $\neg E_1^*(y, r^{m_1}(x))$. Whence we obtain:

$$K_0(r^{m_1-1}(x), y, r^{m_1}(x)), \neg E_1^*(y, r^{m_1-1}(x)) \text{ and } \neg E_1^*(y, r^{m_1}(x)),$$

i.e. $l_2 \cdot l_1 = \{3m_1-1\}$.

Case 3. $l_1 = 3m_1$ for some $1 \le m_1 \le k-1$.

We have: $K_0(r^{m_1-1}(x), t, r^{m_1}(x))$ and $E_1^*(t, r^{m_1}(x))$.

Let $l_2 = 3m_2$ for some $1 \le m_2 \le k-1$, i.e. $K_0(r^{m_2-1}(t), y, r^{m_2}(t))$ and $E_1^*(y, r^{m_2}(t))$. Whence we obtain the following:

$$K_0(r^{m_1+m_2-1}(x), y, r^{m_1+m_2}(x)) \text{ and } E_1^*(y, r^{m_1+m_2}(x)).$$

Suppose firstly that $3m_1+3m_2<3k+1$. It can be checked that $3m_1+3m_2<3k+1$ iff $m_1+m_2\le k$. If $m_1+m_2=k$ then $l_1\cdot l_2=\{3k\}$. If $m_1+m_2<k$ then $l_1\cdot l_2=\{3(m_1+m_2)\}$.

The case $3m_1+3m_2=3k+1$ is impossible. Suppose that $3m_1+3m_2>3k+1$. It can be checked that $3m_1+3m_2>3k+1$ iff $m_1+m_2>k$ iff $m_1+m_2-1\ge k$. Let $s=(m_1+m_2-1)[\text{mod k}]$. Since $k\le m_1+m_2-1\le k-1+k-1-1=2k-3$, we have $0\le s\le k-3$. Thus, we obtain:

$$K_0(r^s(x),y,r^{s+1}(x)) \text{ and } E_1^*(y,r^{s+1}(x)),$$

whence $l_1\cdot l_2=\{3s\}$.

Let now $l_2=3k-2$, i.e. $K_0(r^{k-1}(t),y,t)$ and $E_1^*(y,r^{k-1}(t))$. Whence we obtain:

$$E_1^*(y,r^{m_1-1}(x)) \text{ and either } K_0(r^{m_1-2}(x),y,r^{m_1-1}(x)) \text{ or } K_0(r^{m_1-1}(x),y,r^{m_1}(x)).$$

Consequently, $l_1\cdot l_2=\{3(m_1-1),3m_1-2\}$.

Consider the product $l_2\cdot l_1$. We have: $K_0(r^{k-1}(x),t,x)$, $E_1^*(t,r^{k-1}(x))$, $K_0(r^{m_1-1}(t), y, r^{m_1}(t))$, and $E_1^*(y,r^{m_1}(t))$. Whence we obtain:

$$E_1^*(y,r^{m_1-1}(x)) \text{ and either } K_0(r^{m_1-2}(x),y,r^{m_1-1}(x)) \text{ or } K_0(r^{m_1-1}(x),y,r^{m_1}(x)),$$

i.e. $l_2\cdot l_1=\{3(m_1-1),3m_1-2\}$.

Let now $l_2=3k-1$, i.e. $K_0(r^{k-1}(t),y,t)$, $\neg E_1^*(y,r^{k-1}(t))$ and $\neg E_1(y,t)$. Whence we obtain:

$$K_0(r^{m_1-1}(x),y,r^{m_1}(x)), \neg E_1^*(y,r^{m_1-1}(t)), \text{ and } \neg E_1^*(y,r^{m_1}(t)),$$

i.e. $l_1\cdot l_2=\{3m_1-1\}$.

Consider the product $l_2\cdot l_1$. We have the following: $K_0(r^{k-1}(x),t,x)$, $\neg E_1^*(t,r^{k-1}(x))$, $\neg E_1^*(x,t)$, $K_0(r^{m_1-1}(t),y, r^{m_1}(t))$ and $E_1^*(y,r^{m_1}(t))$. Whence we obtain:

$$K_0(r^{m_1-1}(x),y,r^{m_1}(x)), \neg E_1^*(y,r^{m_1-1}(t)), \text{ and } \neg E_1^*(y,r^{m_1}(t)),$$

i.e. $l_2\cdot l_1=\{3m_1-1\}$.

Let now $l_2=3k$, i.e. $K_0(r^{k-1}(t),y,t)$, and $E_1(y,t)$. Whence we obtain:

$$K_0(r^{m_1-1}(x),y,r^{m_1}(x)) \text{ and } E_1(y,r^{m_1}(x)),$$

i.e. $l_1\cdot l_2=\{3m_1\}$.

Consider the product $l_2\cdot l_1$. Then we have: $K_0(r^{k-1}(x),t,x)$, $E_1(x,t)$, $K_0(r^{m_1-1}(t),y, r^{m_1}(t))$ and $E_1^*(y,r^{m_1}(x))$. Whence we obtain: $K_0(r^{m_1-1}(x),y, r^{m_1}(x))$ and $E_1^*(y,r^{m_1}(x))$, i.e. $l_2\cdot l_1=\{3m_1\}$.

Case 4. $l_1=3k-2$.

We have: $K_0(r^{k-1}(x),t,x)$ and $E_1^*(t,r^{k-1}(x))$.

Let $l_2 = 3k-2$, i.e. $K_0(r^{k-1}(t), y, t)$ and $E_1^*(y, r^{k-1}(t))$. Whence we obtain the following: $K_0(r^{k-2}(x), y, r^{k-1}(x))$ and $E_1^*(y, r^{k-1}(x))$, i.e. $l_1 \cdot l_2 = \{3(k-1)-2\}$.

Let now $l_2 = 3k-1$, i.e. $K_0(r^{k-1}(t), y, t)$, $\neg E_1^*(y, r^{k-1}(t))$ and $\neg E_1(y, t)$. Whence we obtain:

$$K_0(r^{k-2}(x), y, r^{k-1}(x)), \neg E_1^*(y, r^{k-2}(x)), \text{ and } \neg E_1(y, r^{k-1}(x)),$$

i.e. $l_1 \cdot l_2 = \{3(k-1)-1\}$.

Consider the product $l_2 \cdot l_1$. We have the following: $K_0(r^{k-1}(x), t, x)$, $\neg E_1^*(t, r^{k-1}(x))$, $\neg E_1^*(x, t)$, $K_0(r^{m_1-1}(t), y, r^{m_1}(t))$ and $E_1^*(y, r^{m_1}(t))$. Whence we obtain:

$$\neg E_1^*(y, r^{k-2}(x)), K_0(r^{k-2}(x), y, r^{k-1}(x)) \text{ and } \neg E_1(y, r^{k-1}(x)),$$

i.e. $l_2 \cdot l_1 = \{3(k-1)-1\}$.

Let now $l_2 = 3k$, i.e. $K_0(r^{k-1}(t), y, t)$, and $E_1(y, t)$. Whence we obtain:

$$E_1^*(y, r^{k-1}(x)) \text{ and either } K_0(r^{k-2}(x), y, r^{k-1}(x)) \text{ or } K_0(r^{k-1}(x), y, x),$$

i.e. $l_1 \cdot l_2 = \{3k-2, 3(k-1)\}$.

Consider the product $l_2 \cdot l_1$. Then we have: $K_0(r^{k-1}(x), t, x)$, $E_1(x, t)$, $K_0(r^{k-1}(t), y, t)$ and $E_1^*(y, r^{k-1}(t))$. Whence we obtain:

$$E_1^*(y, r^{k-1}(x)) \text{ and either } K_0(r^{k-2}(x), y, r^{k-1}(x)) \text{ or } K_0(r^{k-1}(x), y, x),$$

i.e. $l_2 \cdot l_1 = \{3k-2, 3(k-1)\}$.

Case 5. $l_1 = 3k-1$.

We have: $K_0(r^{k-1}(x), t, x)$, $\neg E_1^*(t, r^{k-1}(x))$ and $\neg E_1(t, x)$.

Let now $l_2 = 3k-1$, i.e. $K_0(r^{k-1}(t), y, t)$, $\neg E_1^*(y, r^{k-1}(t))$ and $\neg E_1(y, t)$. Whence we obtain:

$$K_0(r^{k-2}(x), y, x), \neg E_1^*(y, r^{k-2}(x)), \text{ and } \neg E_1(y, x),$$

i.e. $l_1 \cdot l_2 = \{3k-4, 3k-3, 3k-2, 3k-1\}$.

Let now $l_2 = 3k$, i.e. $K_0(r^{k-1}(t), y, t)$, and $E_1(y, t)$. Whence we obtain:

$$K_0(r^{k-1}(x), t, x), \neg E_1^*(t, r^{k-1}(x)) \text{ and } \neg E_1(t, x),$$

i.e. $l_1 \cdot l_2 = \{3k-1\}$.

Consider the product $l_2 \cdot l_1$. Then we have: $K_0(r^{k-1}(x), t, x)$, $E_1(x, t)$, $K_0(r^{k-1}(t), y, t)$, $\neg E_1^*(y, r^{k-1}(t))$ and $\neg E_1(y, t)$. Whence we obtain:

$$K_0(r^{k-1}(x), t, x), \neg E_1^*(t, r^{k-1}(x)) \text{ and } \neg E_1(t, x),$$

i.e. $l_2 \cdot l_1 = \{3k-1\}$.

Case 6. $l_1 = 3k$.

We have: $K_0(r^{k-1}(x), t, x)$ and $E_1(t, x)$.

Let $l_2 = 3k$, i.e. $K_0(r^{k-1}(t), y, t)$ and $E_1(y, t)$. Whence we obtain: $K_0(r^{k-1}(x), t, x)$ and $E_1(y, x)$, i.e. $l_1 \cdot l_2 = \{3k\}$.

Thus, we established that the algebra $\mathfrak{P}_{M'_{1,1,k}}$ is commutative and strictly 5-deterministic for every $k \geq 2$.

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Кулпешов Б.Ш., Судоплатов С.В. ДӨҢЕСТІК РАНГІСІ 2 ӘЛСІЗ ЦИКЛДІК МИНИМАЛДЫ ТЕОРИЯЛАР ҮШІН БИНАРЛЫҚ ОҚШАУЛАУ ФОРМУЛАЛАРЫ АЛГЕБРАЛАРЫНДА

Бұл жұмыс циклдік реттелген әлсіз циклді минималды құрылымдарды зерттеуге арналған. Циклдік тәртіптің ең қарапайым мысалы — соңғы нүктелері бар сызықтық тәртіп, онда ең үлкен элемент ең кішімен сәйкестендіріледі. Тағы бір мысал, шеңбер бойымен жүру кезінде пайда болатын тәртіп. Циклдік реттелген құрылым, егер оның формулалық ішкі жиындарының кез келгені дөңес жиындар мен нүктелердің ақырлы бірлестігі болса, оны әлсіз циклдік минималды деп атайды. Теория әлсіз циклдік минималды деп аталады, егер оның барлық модельдері әлсіз циклдік минималды болса. Біз құрылымның анықталатын аяқталуына оң-монотонды функцияға ие және құрылымның негізгі жиынын дөңес класстарына шектеулі санына бөлетін тривиалды емес эквиваленттік қатынас бар тривиалды анықталатын түйықталуына ие дөңестік рангісі 2 санаулы категориялық 1-отпелі примитивтік емес әлсіз циклдік минималды теориялары ушін бинарлық оқшаулау формулаларынын алгебрасын сипаттаймыз.

Түйін сөздер: бинарлық формулалар алгебрасы, $\aleph_0$-категориялық теория, әлсіз циклдік минималдылық, циклдік реттелген құрылым, дөңестік рангісі.

Кулпешов Б.Ш., Судоплатов С.В. ОБ АЛГЕБРАХ БИНАРНЫХ ИЗОЛИРУЮЩИХ ФОРМУЛ ДЛЯ СЛАБО ЦИКЛИЧЕСКИ МИНИМАЛЬНЫХ ТЕОРИЙ РАНГА ВЫПУКЛОСТИ 2

Данная работа посвящена исследованию слабо циклически минимальных циклически упорядоченных структур. Простейший пример циклического порядка — это линейный порядок с концевыми точками, в котором наибольший элемент отождествили с наименьшим. Другой пример — это порядок, возникающий при обходе окружности. Циклически упорядоченная структура называется слабо циклически минимальной, если любое ее формульное подмножество является конечным объединением выпуклых множеств и точек. Теория называется слабо циклически минимальной, если все ее модели являются слабо циклически минимальными. Описываются алгебры бинарных изолирующих формул для счетно категоричных 1-транзитивных непримитивных слабо циклически минимальных теорий ранга выпуклости 2 с тривиальным определимым замыканием, имеющих монотонную вправо функцию в определимое пополнение структуры и не имеющих нетривиального отношения эквивалентности, разбивающего основное множество структуры на конечное число выпуклых классов.

Ключевые слова. алгебра бинарных формул, $\aleph_0$-категоричная теория, слабая циклическая минимальность, циклически упорядоченная структура, ранг выпуклости.