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Quasistationary nonlinear problem of thermal conduction with spherical symmetry in the region with a free boundary

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Abstract. The article discusses a boundary value single-phase problem with a free boundary for a quasi-stationary nonlinear inhomogeneous heat conduction equation with a spherically symmetric thermal field. The domain of the problem degenerates at the initial moment of time. To find a free boundary, the problem is supplemented by Stefan's condition. It is assumed that thermal conductivity and latent heat of melting depend on temperature. The study of the boundary value problem was carried out by transforming it into an equivalent system of integral equations, for which a proof of solvability and uniqueness of the solution was obtained. An iterative algorithm for the numerical solution of a system of integral equations has been developed. The results of a computational experiment on the analytical solution of a boundary value problem have been presented.

Keywords. nonlinear heat equation, spherical symmetry, boundary condition, Stefan's problem, integral equations, electrical contacts.

1 Introduction

We consider the following single-phase spherically symmetric Stephan problem for the quasi-stationary equation of thermal conduction

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\lambda(r,t,\theta)\frac{\partial\theta}{\partial r}\right)+\frac{q(r,t)}{r^4}=0,\quad \theta=\theta(r,t) \tag{1}$$

specified in the domain

$$\Omega=\left\{(r,t)|0<b<r<\alpha(t)<\infty,\ 0<t<t_a\right\},$$

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under the boundary conditions:

$$\alpha(0) = b, \tag{2}$$

$$-\lambda(r,t;\theta(r,t)) \left.\frac{\partial\theta}{\partial r}\right|_{r=b} = P(t) \tag{3}$$

$$\theta(\alpha(t),t) = \theta_\alpha(t), \tag{4}$$

$$-\lambda(r,t;\theta(r,t)) \left.\frac{\partial\theta}{\partial r}\right|_{r=\alpha(t)} = L(t;\theta_\alpha(t))\frac{d\alpha(t)}{dt}, \tag{5}$$

The domain Ω of Problem (1)–(5) is bounded by a free boundary $\alpha(t)$. Due to Condition (2), the domain Ω degenerates at the initial moment (turns into a point). Equation (5) is called Stefan's condition. It defines the law of change (motion) of the boundary $\alpha(t)$.

When analyzing problem (1)–(5), we assume that the following requirements for parameters and boundary values are met:

$$\lambda(r,t;z) \in C\left\{[b,\infty)\times[0,t_a]\times(-\infty,\infty)\right\}, \quad \lambda(r,t;z) \geq \lambda_{\min} > 0, \tag{6}$$

and $\lambda(r,t;z)$ satisfies the Lipschitz condition on the variable z

$$|\lambda(r,t;z_2) - \lambda(r,t;z_1)| \leq k_\lambda \left|z_2 - z_1\right|, \tag{7}$$

$$q(r,t) \in C\left\{[b,\infty)\times[0,t_a]\right\}, \quad q(r,t) \geq 0, \tag{8}$$

$$L(t;z) \in C\left\{[0,t_a]\times(-\infty,\infty\right\}, \quad L(t;z) \geq L_{\min} > 0, \tag{9}$$

$$P(t) \in C[0,t_a], \quad P(t) \geq 0, \tag{10}$$

$$\theta_\alpha(t) \in C[0,t_a]. \tag{11}$$

Problem (1)–(5) describes, in particular, the quasi-stationary heating of the electrical contacts of circuit breakers with the formation of a liquid phase in mathematical models of the opening processes in electric current circuits. In this case, the functions λ, q, θ_α and L define the specific thermal conductivity, the density of heat sources, the temperature at the mobile boundary of the liquid phase, and the latent heat of melting of the contact material, respectively.

The quasi–stationarity of the problem (1)–(5) assumes that the change in the internal energy of the contacts occurs only as a result of conductive heat transfer. This assumption is natural under the condition of a significant value of the thermal Fourier number Fo [1], which reflects the ratio of the density of thermal energy transferred as a result of the thermal conductivity of the coolant material to the density of thermal energy used to increase its internal energy. In particular, this fact occurs for short-term processes of opening metal contacts, the penetration depth of the liquid phase in which is small, and the magnitude of the flowing electric current is significant ($\geq 100A$).

This mathematical model of the dynamics of the thermal field in electrical contact is based on the use of the Holm sphere [2]. Such an approach leads to equations of type (1), for which the isothermal surfaces are concentric hemispheres with a center coinciding with the center of the arc spot on the flat surface of the electrode, and heat flows are directed along the radii of these hemispheres.

A significant number of scientific papers, both linear and non-linear, are devoted to such problems, in particular [3, 4, 5]. The issues of the existence and uniqueness of the solution of the Stephan problem in single-phase and multiphase formulation in linear cases are considered in monographs [6, 7, 8].

2 Transformation of the boundary value problem (1)–(5) to a system of integral equations

Suppose that there are a pair of functions $\alpha(t)$ and $\theta(r,t)$, which are the solution of the problem (1)–(5). In this case, equation (1) is an identity. Multiply it by r^2 and integrate it by the interval (b,r). As a result, taking into account the boundary condition (3), we obtain the equation

$$\lambda(r,t;\theta(r,t))\frac{\partial\theta(r,t)}{\partial r} = -\frac{b^2P(t)}{r^2} - \frac{1}{r^2}\int\limits_b^{\alpha(t)}\frac{q(\xi,t)}{\xi^2}d\xi. \tag{12}$$

Let us pass in equality (12) to the limit at $r\to\alpha(t)$

$$\left[\lambda(r,t;\theta(r,t))\frac{\partial\theta(r,t)}{\partial r}\right]_{r=\alpha(t)} = -\frac{1}{\alpha^2(t)}\left[b^2P(t) + \int\limits_b^{\alpha(t)}\frac{q(\xi,t)}{\xi^2}d\xi\right]. \tag{13}$$

Then, in view of the boundary condition (4) and Stefan's condition (5), we obtain an ordinary differential equation with respect to $\alpha(t)$

$$\frac{d\alpha(t)}{dt} = \frac{1}{L(t;\theta_\alpha(t))\alpha^2(t)}\left[b^2P(t) + \int\limits_b^{\alpha(t)}\frac{q(\xi,t)}{\xi^2}d\xi\right]. \tag{14}$$

Integrating (14) over the interval $(0,t)$, $t\le t_a$, we transform it to the integral equation

$$\alpha(t) = b + \int\limits_0^t\left[b^2P(\tau) + \int\limits_b^{\alpha(\tau)}\frac{q(\xi,\tau)}{\xi^2}d\xi\right]\frac{d\tau}{L(\tau;\theta_\alpha(\tau))\alpha^2(\tau)}, \tag{15}$$

which is more convenient for analyzing the problem.

Further, dividing equation (12) by $\lambda(r, t; \theta(r, t))$, which in view of (6) is strictly positive, we find the derivative

$$\frac{\partial\theta(r,t)}{\partial r} = -\frac{1}{\lambda(r,t;\theta(r,t))r^2}\left[b^2P(t) + \int\limits_b^r \frac{q(\xi,t)}{\xi^2}d\xi\right] \tag{16}$$

We integrate the left and right parts of (16) over the interval $(r, \alpha(t))$ and apply Condition (4). As a result, we obtain the following integral equation for $\theta(r, t)$:

$$\theta(r,t) = \theta_\alpha(t) + \int\limits_0^{\alpha(t)}\left[b^2P(t) + \int\limits_b^\eta \frac{q(\xi,t)}{\xi^2}d\xi\right]\frac{d\eta}{\lambda(\eta,t;\theta(\eta,t))\eta^2}. \tag{17}$$

Thus, the boundary value problem (1)–(5) turns out to be reduced to a system of integral equations (15) and (17) with unknown functions $\alpha(t)$ and $\theta(r, t)$ [9, 10, 11]. The differential equation (14) and its integral equivalent (15) are independent of the function $\theta(r, t)$. After finding the moving boundary $\alpha(t)$, the temperature $\theta(r, t)$ inside the liquid phase will be determined from equation (17).

In the case of functions $q(r, t) \equiv q_0(t)$ and $\lambda(r, t; \theta(r, t)) \equiv \lambda_0(t)$, which depend only on time, the differential equation (14) and the integral equations (15) and (17) are simplified and take the form respectively

$$\frac{d\alpha(t)}{dt} = \frac{1}{L(t;\theta_\alpha(t))\alpha^2(t)}\left[b^2P(t) + q_0(t)\left(\frac{1}{b} - \frac{1}{\alpha(t)}\right)\right], \tag{18}$$

$$\alpha(t) = b + \int\limits_0^t \left[b^2P(\tau) + q_0(\tau)\left(\frac{1}{b} - \frac{1}{\alpha(\tau)}\right)\right]\frac{d\tau}{L(\tau;\theta_\alpha(\tau))\alpha^2(\tau)}, \tag{19}$$

$$\theta(r,t) = \theta_\alpha(t) + \frac{1}{\lambda_0(t)}\left(\frac{1}{r} - \frac{1}{\alpha(t)}\right)\left\{b^2P(t) + \frac{q_0(t)}{2}\left[\left(\frac{1}{b} + \frac{1}{r}\right) + \left(\frac{1}{b} - \frac{1}{\alpha(t)}\right)\right]\right\}. \tag{20}$$

3 Solvability of integral equations

3.1 Equation (15)

Let (C_α, ρ_α) be a metric space in which C_α is a set of continuous functions $\alpha(t) \geq b$, $t \in [0, t_a]$, and ρ_α is a metric

$$\rho_\alpha(\alpha_1, \alpha_2) = \max_{0\leq t\leq t_a} |\alpha_2(t) - \alpha_1(t)|. \tag{21}$$

We consider the operator

$$A\alpha(t) = b + \int\limits_0^t \left[b^2 P(t) + \int\limits_b^{\alpha(\tau)} \frac{q(\xi,\tau)}{\xi^2} d\xi \right] \frac{d\tau}{L(\tau;\theta_\alpha(\tau))\alpha^2(\tau)}. \tag{22}$$

The integral in the right-hand side of (22) is defined for everyone $t \in [0, t_a]$ and everyone $\alpha(t) \in C_\alpha$ by virtue of the above properties of the integrative functions (8)–(10).

We show that the operator $A\alpha(t)$ is bounded on the set C_α. Conditions (8)–(10) imply $A\alpha(t) \geq b$. Further, by virtue of the same conditions, we have

$$\int\limits_b^{\alpha(\tau)} \frac{q(\xi,\tau)}{\xi^2} d\xi \leq \int\limits_b^{\alpha(\tau)} \frac{q_{\max}}{\xi^2} d\xi \leq q_{\max} \int\limits_b^{\infty} \frac{d\xi}{\xi^2} = \frac{q_{\max}}{b}, \tag{23}$$

$$A\alpha(t) \leq b + \int\limits_0^t \left[b^2 P_{\max} + \frac{q_{\max}}{b} \right] \frac{d\tau}{L_{\min} b^2} \leq b + \frac{t_a}{L_{\min}} \left[P_{\max} + \frac{q_{\max}}{b^3} \right]. \tag{24}$$

Let

$$M_\alpha = b + \frac{t_a}{L_{\min}} \left[P_{\max} + \frac{q_{\max}}{b^3} \right],$$

then we obtain $b \leq A\alpha(t) \leq b + M_\alpha$ and $A\alpha(t) \in C_\alpha$.

Thus, the operator $A\alpha(t)$ maps C_α to the set $C'_\alpha \subset C_\alpha$, where

$$C'_\alpha = \{\alpha(t) : b \leq \alpha(t) \leq b + M_\alpha\},$$

and as a consequence, $A : C'_\alpha \to C'_\alpha$.

Since the space (C_α, ρ_α) is complete and the set C'_α is closed, the space C'_α is also complete [12].

We show that the operator $A\alpha(t)$ is contractive in the space (C'_α, ρ_α).

Let $\alpha_1(t)$ and $\alpha_2(t) \in C'_\alpha$ be arbitrary. We consider the difference

$$A\alpha_2(t) - A\alpha_1(t) = \int\limits_0^t \left[b^2 P(\tau) + \int\limits_b^{\alpha_2(\tau)} \frac{q(\xi,\tau)}{\xi^2} d\xi \right] \frac{d\tau}{L(\tau;\theta_\alpha)\alpha_2^2(\tau)} -$$

$$- \int\limits_0^t \left[b^2 P(\tau) + \int\limits_b^{\alpha_1(\tau)} \frac{q(\xi,\tau)}{\xi^2} d\xi \right] \frac{d\tau}{L(\tau;\theta_\alpha)\alpha_1^2(\tau)}.$$

Regrouping the expression of the right side, we obtain

$$A\alpha_2(t) - A\alpha_1(t) = \int\limits_0^t \left[\frac{1}{\alpha_2^2(t)} - \frac{1}{\alpha_1^2(t)}\right]\left[b^2 P(\tau) + \int\limits_b^{\alpha_2(\tau)} \frac{q(\xi,\tau)}{\xi^2} d\xi\right] \frac{d\tau}{L(\tau;\theta_\alpha(\tau))} +$$

$$+ \int\limits_0^t \frac{d\tau}{L(\tau;\theta_\alpha(\tau))\alpha_1^2(\tau)} \int\limits_{\alpha_1(\tau)}^{\alpha_2(\tau)} \frac{q(\xi,\tau)}{\xi^2} d\xi.$$

Since

$$\left|\frac{1}{\alpha_2^2(t)} - \frac{1}{\alpha_1^2(t)}\right| = \frac{[\alpha_2(\tau)+\alpha_1(\tau)]\,|\alpha_2(\tau)-\alpha_1(\tau)|}{\alpha_1^2(\tau)\alpha_2^2(\tau)} \le$$

$$\le \frac{2(b+M_\alpha}{b^4}\,|\alpha_2(\tau)-\alpha_1(\tau)| \le \frac{2(b+M_\alpha}{b^4}\rho_\alpha(\alpha_1,\alpha_2),$$

$$\left|\int\limits_b^{\alpha_2(\tau)} \frac{q(\xi,\tau)}{\xi^2} d\xi\right| \le \left|\int\limits_b^{\alpha_2(\tau)} \frac{q_{\max}}{\xi^2} d\xi\right| \le q_{\max}\int\limits_b^{\infty}\frac{d\xi}{\xi^2} = \frac{q_{\max}}{b},$$

$$\left|\int\limits_{\alpha_1(\tau)}^{\alpha_2(\tau)} \frac{q(\xi,\tau)}{\xi^2} d\xi\right| \le \frac{q_{\max}}{b^2}\left|\int\limits_{\alpha_1(\tau)}^{\alpha_2(\tau)} d\xi\right| = \frac{q_{\max}}{b^2}\,|\alpha_2(\tau)-\alpha_1(\tau)| \le \frac{q_{\max}}{b^2}\rho_\alpha(\alpha_1,\alpha_2),$$

that

$$|A\alpha_2(t) - A\alpha_1(t)| \le \int\limits_0^t \frac{2(b+M_\alpha}{b^4 L_{\min}}\left[b^2 P_{\max} + \frac{q_{\max}}{b}\right]|\alpha_2(\tau)-\alpha_1(\tau)|\,d\tau +$$

$$+ \int\limits_0^t \frac{q_{\max}}{b^4 L}\,|\alpha_2(\tau)-\alpha_1(\tau)|\,d\tau$$

or

$$|A\alpha_2(t) - A\alpha_1(t)| \le K_\alpha \int\limits_0^t |\alpha_2(\tau)-\alpha_1(\tau)|\,d\tau,$$

where

$$K_\alpha = \frac{1}{bL_{\min}}\left(\frac{2P_{\max}(b+M_\alpha)}{b} + \frac{2(b+M_\alpha)q_{\max}}{b^4} + \frac{q_{\max}}{b^3}\right).$$

Further

$$\left|A^2\alpha_2(t)-A^2\alpha_1(t)\right| \le K_\alpha \int\limits_0^t \left|A\alpha_2(\tau)-A\alpha_1(\tau)\right| d\tau \le$$

$$\le K_\alpha^2 \int\limits_0^t d\tau \int\limits_0^\tau \left|\alpha_2(\tau_1)-\alpha_1(\tau_1)\right| d\tau_1 = K_\alpha^2 \int\limits_0^\tau (t-\tau)\left|\alpha_2(\tau)-\alpha_1(\tau)\right| d\tau.$$

Continuing further

$$\left|A^n\alpha_2(t)-A^n\alpha_1(t)\right| \le K_\alpha^n \int\limits_0^t \frac{(t-\tau)^{n-1}}{(n-1)!}\left|\alpha_2(\tau)-\alpha_1(\tau)\right| d\tau,$$

$$\rho\left(A^n\alpha_1, A^n\alpha_2\right) = \max_{0\le t\le t_a}\left|A^n\alpha_2(t)-A^n\alpha_1(t)\right| \le$$

$$\le K_\alpha^n \max_{0\le t\le t_a} \int\limits_0^t \frac{(t-\tau)^{n-1}}{(n-1)!}\left|\alpha_2(\tau)-\alpha_1(\tau)\right| d\tau \le \frac{K_\alpha^n t_a^n}{n!}\rho_\alpha(\alpha_1,\alpha_2).$$

There exists some N_α such that the equality $k_\alpha = \frac{K_\alpha^n t_a^n}{n!}\rho_\alpha(\alpha_1,\alpha_2)$ holds. Then

$$\rho\left(A^{N_\alpha}\alpha_1, A^{N_\alpha}\alpha_2\right) \le k_\alpha \rho_\alpha(\alpha_1,\alpha_2),$$

that is, the operator $B_\alpha\alpha(t) = A^{N_\alpha}\alpha(t)$ is contractive in the space (C'_α, ρ_α) and therefore there exists $\alpha^\star(t) \in C'_\alpha$ such that $B_\alpha\alpha^\star = \alpha^\star$. Then, following [12], we obtain $A\alpha^\star(t) = \alpha^\star(t)$, and $\alpha^\star(t)$ is a unique fixed point of the operator $A\alpha(t)$.

Thus, equation (19) is solvable and has the only solution that can be found using the iterative Piccard procedure [9]

$$\alpha_{n+1}(t) = A\alpha_n(t), \quad n = 1, 2, ..., \tag{25}$$

where $\alpha_1(t)$ arbitrary function of the set C'_α.

3.2 Equation (17)

We introduce a metric space (C_θ, ρ_θ) in which C_θ is the set of all continuous functions $\theta(r,t)$, where $(r,t) \in D_\theta = \{[b,\infty) \times [0,t_a]\}$, and $\theta(r,t) \ge \theta_{\min}$, for $\theta_{\min} = \min\limits_{0\le t\le t_a} \theta_\alpha(t)$,

$$\rho_\theta(\theta_1,\theta_2) = \max_{(r,t)\in D_\theta}\left|\theta_2(r,t)-\theta_1(r,t)\right|. \tag{26}$$

We consider

$$\Theta\theta(r,t) = \theta_\alpha(t) + \int\limits_r^{\alpha(t)} \left[b^2P(t) + \int\limits_b^\eta \frac{q(xi,t)}{\xi^2}d\xi\right] \frac{d\eta}{\lambda(\eta,t;\theta(\eta,t))\eta^2}. \tag{27}$$

The integral in the right-hand side of (27) is defined by virtue of the continuity and boundedness of integrand for all $t \in [0, t_a]$, $r \in [b, \alpha(t)]$, $\alpha(t) \in C'_\alpha$.

We show that the operator $\Theta\theta(r,t)$ is bounded on the set C_θ,

$$\theta_{\min} \le \Theta\theta(r,t) \le \theta_{\max} + M_\theta, \tag{28}$$

where

$$\theta_{\max} = \max_{0\le t\le t_a} \theta_\alpha(t), \quad M_\theta = \left[b^2 P_{\max} + \frac{q_{\max}}{b}\right] \frac{1}{b\lambda_{\min}}. \tag{29}$$

Assuming that conditions (6)–(11) hold, we obtain $\Theta\theta(r,t) \ge \theta_\alpha(t)$. Further, by virtue of the same conditions (6)–(11), we have

$$\int_b^\eta \frac{q(\xi,\tau)}{\xi^2} d\xi \le \int_b^\eta \frac{q_{\max}}{\xi^2} d\xi \le q_{\max} \int_b^\infty \frac{d\xi}{\xi^2} = \frac{q_{\max}}{b},$$

$$\Theta\theta(r,t) \le \theta_{\max} + \int_b^{\alpha(t)} \left[b^2 P_{\max} + \frac{q_{\max}}{b}\right] \frac{d\eta}{\lambda_{\min}\eta^2} \le$$

$$\le \theta_{\max} + \int_b^{\infty} \left[b^2 P_{\max} + \frac{q_{\max}}{b}\right] \frac{d\eta}{\lambda_{\min}\eta^2} \le \theta_{\max} + \frac{M_\theta}{b},$$

Therefore, the operator $\Theta\theta(r,t)$ maps the set C_θ in $C'_\theta \subset C_\theta$, where

$$C'_\theta = \{\theta(r,t) | \theta_{\min} \le \theta(r,t) \le \theta_{\max} + M_\theta\},$$

and therefore $\Theta : C'_\theta \to C_\theta$.

Since the space (C_θ, ρ_θ) is complete and the set C'_θ is closed, the space (C'_θ, ρ_θ) is also complete.

We show that the operator $\Theta\theta(r,t)$ is contractive in the space C'_θ. Let $\theta_1(r,t)$ and $\theta_2(t,t) \in C'_\theta$ be arbitrarily as well as $\alpha(t) \in C'_\alpha$. We consider the difference

$$\Theta\theta_2(r,t) - \Theta\theta_1(r,t) = \int_r^{\alpha(t)} \left[b^2 P(t) + \int_b^\eta \frac{q(\xi,t)}{\xi^2} d\xi\right] \frac{d\eta}{\lambda(\eta,t;\theta_2(\eta,t)\eta^2} -$$

$$- \int_r^{\alpha(t)} \left[b^2 P(t) + \int_b^\eta \frac{q(\xi,t)}{\xi^2} d\xi\right] \frac{d\eta}{\lambda(\eta,t;\theta_1(\eta,t)\eta^2} =$$

$$= \int\limits_r^{\alpha(t)} \left[b^2 P(t) + \int\limits_b^{\eta} \frac{q(\xi,t)}{\xi^2} d\xi \right] \left[\frac{1}{\lambda(\eta,t;\theta_2(\eta,t)} - \frac{1}{\lambda(\eta,t;\theta_1(\eta,t)} \right] \frac{d\eta}{\eta^2}.$$

In view of (7) we obtain

$$\left| \frac{1}{\lambda(\eta,t;\theta_2(\eta,t)} - \frac{1}{\lambda(\eta,t;\theta_1(\eta,t)} \right| \le \frac{|\lambda(\eta,t;\theta_1(\eta,t) - \lambda(\eta,t;\theta_1(\eta,t)|}{\lambda_{\min}^2} \le$$

$$\le \frac{k_\lambda \left| \theta_2(\eta,t) - \theta_1(\eta,t) \right|}{\lambda_{\min}^2}, \qquad t \in [0,t_a], \quad \eta \in [b,\infty),$$

that

$$|\Theta\theta_2(r,t) - \Theta\theta_1(r,t)| \le \int\limits_r^{\alpha(t)} \left[b^2 P_{\max} + \frac{q_{\max}}{b} \right] \frac{k_\lambda \left| \theta_2(\eta,t) - \theta_2(\eta,t) \right|}{\lambda_{\min}^2} \frac{d\eta}{\eta^2} \le$$

$$\le K_\theta \int\limits_r^{\alpha(t)} |\theta_2(\eta,t) - \theta_2(\eta,t)| \frac{d\eta}{\eta^2} \le K_\theta \rho_\theta(\theta_1,\theta_2) \int\limits_r^{\infty} \frac{d\eta}{\eta^2} = \frac{K_\theta}{r} \rho_\theta(\theta_1,\theta_2).$$

where

$$K_\theta = \left[b^2 P_{\max} + \frac{q_{\max}}{b} \right] \frac{k_\lambda}{\lambda_{\min}^2}.$$

For the second degree of the operator $\Theta\theta(r,t)$, we find

$$\left| \Theta^2\theta_2(r,t) - \Theta^2\theta_1(r,t) \right| \le K_\theta \int\limits_r^{\alpha(t)} |\Theta\theta_2(\eta,t) - \Theta\theta_2(\eta,t)| \frac{d\eta}{\eta^2} \le$$

$$\le K_\theta \int\limits_r^{\alpha(t)} \frac{K_\theta}{\eta} \rho_\theta(\theta_1,\theta_2) \frac{d\eta}{\eta^2} \le K_\theta^2 \rho_\theta(\theta_1,\theta_2) \int\limits_r^{\infty} \frac{d\eta}{\eta^3} = \frac{K_\theta^2}{2r^2} \rho_\theta(\theta_1,\theta_2).$$

Continuing further, for the free skates, $n = 1, 2, ...$, we obtain

$$|\Theta^n\theta_2(r,t) - \Theta^n\theta_1(r,t)| \le K_\theta^n \rho_\theta(\theta_1,\theta_2) \frac{1}{n! r^n} \le \frac{K_\theta^n}{n! b^n} \rho_\theta(\theta_1,\theta_2).$$

There is N_θ such that

$$\frac{K_\theta^{N_\theta}}{N_\theta! b^{N_\theta}} \rho_\theta(\theta_1,\theta_2) < 1.$$

.

In this case, the operator $B_\theta\theta(r,t) = \Theta^{N_\theta}\theta(r,t)$ is contractive and there exists $\theta^\star(r,t)$ such that $B_\theta\theta^\star(r,t) = \theta^\star(r,t)$ and, therefore, $\Theta\theta^\star(r,t) = \theta^\star(r,t)$.

Thus, the sequence

$$\theta_{n+1}(r,t) = \Theta\theta_n(r,t), \quad n = 1,2,..., \tag{30}$$

where $\theta_1(r,t) \in C'_\theta$ and $\alpha(t) \in C'_\alpha$ are arbitrarily chosen functions, is fundamental in space (C'_θ, ρ_θ) and $\theta^*(r,t)$ is its limit point.

4 Algorithm for numerical solution of Problem (1)–(5)

1) On the time interval $[0, t_a]$ we introduce a uniform grid with $N_t + 1$ nodes $t_i = \frac{t_a}{N_t} i$, $i = 0, 1, 2, ..., N_t$.

2) We find the values $\alpha_i = \alpha(t_i)$, $i = 0, 1, ..., N_t$, by solving the integral equation (15) (or by solving the differential equation (14)).

3) For each moment t_i on the segment $[b, \alpha_i]$, $i = 1, 2, ..., N_t$, we introduce a grid with nodes

$$r_{ij} = b + \frac{\alpha_i - b}{N_r} j, \quad j = 0, 1, 2, ..., N_r.$$

4) We calculate the values $\theta_{i,j} = \theta(r_{ij}, t_i)$, $j = 0, 1, ..., N_r$ by repeating the solution of the integral equation (17) for each t_i.

The resulting grid is r condensed by the variable in the vicinity of the initial moment $t = 0$.

5 Computational experiment

In order to check the convergence and stability of the algorithm for solving the system of integral equations, a computational experiment was carried out on the exact solution of the next test problem:

$$\theta_{\text{test}}(r,t) = b + \alpha_0 t - r, \quad b < r < b + \alpha_0 t, \tag{31}$$

$$\alpha_{\text{test}}(t) = b + \alpha_0 t, \quad t \in [0, t_a]. \tag{32}$$

Based on (31)–(32) and taking into account the boundary conditions (2)–(5), we find

$$\theta_\alpha(t) = 0,$$

$$\lambda(r,t;\theta) = \lambda_c + \theta(r,t),$$

$$P(t) = \lambda_c + \alpha_0 t,$$

$$L(t, \theta_\alpha) = \frac{\lambda_c + \theta_\alpha(t)}{\alpha_0},$$

$$q(r,t) = 2r^3(\lambda_c + b + \alpha_0 t) - 3r^4.$$

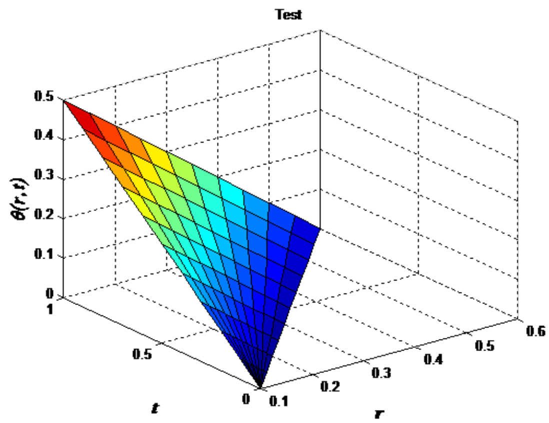

Fig 1. Temperature $\theta_{\text{test}}(r,t)$

The calculation is performed with the values of the parameters: $b = 0.1$, $\alpha_0 = 0.5$, $t_a = 1$, $\lambda_c = 1$, $N_t = 11$, $N_r = 11$, $\alpha_1(t) \equiv b$, $\theta_1(r,t) \equiv \theta_\alpha(t)$, $\varepsilon_\alpha = 0.0001$, $\varepsilon_\theta = 0.001$. The results of these calculations are given in Tables 1 and 2.

Table 1: Convergence of the computational process for $\alpha(t)$.

Iteration, n	$\rho_\alpha(\alpha_n(t), \alpha_{n-1}(t))$	$\rho_\alpha(\alpha_n(t), \alpha_{\text{test}}(t))$
1	0.625	0.125
2	0.14594	0.020938
3	0.023594	0.0026563
4	0.0029298	0.00027348
5	0.00028738	$2.3896 \cdot 10^{-5}$
6	$2.5725 \cdot 10^{-5}$	$1.8289 \cdot 10^{-6}$

Table 2: Convergence of the computational process for $\theta(r,t)$

Iteration, n	$\rho_\theta(\theta_n(r,t), \theta_{n-1}(r,t))$	$\rho_\theta(\theta_n(r,t), \theta_{\text{test}}(r,t))$
1	2.2247	0.31666
2	0.34689	0.030686
3	0.033256	0.0025988
4	0.0027876	0.00018998
5	0.00019953	$9.7715 \cdot 10^{-6}$

The calculation of the test task was carried out in the MATLAB computing environment.

6 Conclusion

For the thermal problem formulated in the work, the existence and uniqueness of the solution are proved, and an algorithm for numerically finding its solution is proposed.

If in the condition of problem (1)–(5) $P(t) \equiv 0$ then the uniqueness of the solution of equation (15) implies that $\alpha(t) \equiv b$ and problem (1)–(5) is unsolvable in the sense that a liquid phase is not formed in the electrode.

The computational experiment shows that the numerical solution differs from the exact solution by an amount that is an order of magnitude less than the specified accuracy of the iterative process. You can use this result to obtain the necessary calculation accuracy.

It should also be noted that the quasi-stationary model can be used to indirectly estimate the heating depth of the molten electrode zone and the maximum achievable temperature. Its values are the upper faces of sets C'_α and C'_θ.

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Шпади Ю.Р. ЕРКІН ШЕКАРАСЫ БАР АЙМАҚТА СФЕРАЛЫҚ СИММЕТРИЯМЕН ЖЫЛУ ӨТКІЗГІШТІКТІҢ КВАЗИ-СТАЦИОНАРЛЫҚ СЫЗЫҚТЫ ЕМЕС МӘСЕЛЕСІ

Мақалада сфералық симметриялы жылу өрісі бар квази-стационарлық сызықты емес біртекті емес жылу өткізгіштік теңдеуі үшін бос шекарасы бар шекаралық мәннің бір фазалы мәселесі қарастырылады. Мәселенің домені уақыттың бастапқы сәтінде деградацияға ұшырайды. Еркін шекараны табу үшін мәселе Стефанның жағдайымен толықтырылады. Балқудың жылу өткізгіштігі мен жасырын жылуы температураға байланысты болады деп болжанады. Шекаралық шама мәселесін зерттеу оны интегралдық теңдеулердің эквивалентті жүйесіне айналдыру арқылы жүзеге асырылды, ол үшін шешімнің шешімділігі мен бірегейлігінің дәлелі алынды. Интегралдық теңдеулер жүйесін сандық шешудің итеративті алгоритмі жасалды. Шекаралық есептің аналитикалық шешімі бойынша есептеу экспериментінің нәтижелері ұсынылды.

Түйін сөздер: сызықты емес жылу теңдеуі, сфералық симметрия, шекаралық шарт, Стефан есебі, интегралдық теңдеулер, электрлік контактілер.

Шпади Ю.Р. КВАЗИСТАЦИОНАРНАЯ НЕЛИНЕЙНАЯ ЗАДАЧА ТЕПЛОПРОВОДНОСТИ СО СФЕРИЧЕСКОЙ СИММЕТРИЕЙ В ОБЛАСТИ СО СВОБОДНОЙ ГРАНИЦЕЙ

В статье рассматривается краевая однофазная задача со свободной границей для квазистационарного нелинейного неоднородного уравнения теплопроводности со сферически симметричным тепловым полем. В начальный момент времени область задачи вырождена. Чтобы найти свободную границу, задача дополняется условием Стефана. Предполагается, что теплопроводность и скрытая теплота плавления зависят от температуры. Исследование краевой задачи проводилось путем преобразования ее в эквивалентную систему интегральных уравнений, для которой было получено доказательство разрешимости и единственности решения. Разработан итерационный алгоритм численного решения системы интегральных уравнений. Представлены результаты вычислительного эксперимента по аналитическому решению краевой задачи.

Ключевые слова: нелинейное уравнение теплопроводности, сферическая симметрия, граничное условие, задача Стефана, интегральные уравнения, электрические контакты.