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Globally unsolvability of the pseudo-parabolic inequalities in annulus

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Abstract. In the present paper, we consider the pseudo-parabolic inequality with a singular potential function in an annular domain. The problem is studied with the Neumann-type and Dirichlet-type boundary conditions on the upper boundary of annulus. The non-existence of global weak solutions were established for both cases based on the test function method.

Keywords. pseudo-parabolic equation, critical exponent, existence and nonexistence of global solution.

1 Introduction

In [8], Fujita studied the following parabolic problem

$$\begin{cases} u_t - \Delta u = u^p, \ \ (t,x) \in (0,T) \times \mathbb{R}^N, \\ u\left(0,x\right) = u_0\left(x\right) \geq 0, \ \ x \in \mathbb{R}^N, \end{cases} \tag{1}$$

and proved the following results:

(a) If $1 < p < p_c$, then the problem (1) admits no global positive solutions;

(b) If $p > p_c$, then for sufficiently small initial data, the problem (1) admits positive global solutions.

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The exponent $p_c = 1 + \frac{2}{N}$ is called the Fujita critical exponent, which distinguishes between the existence and nonexistence of global in time solutions of (1).

We have to mention that, when $p = p_c$, this problem was considered by Hayakawa in [10] for $N = 1, 2$ and Kobayashi et. al. in [12] for arbitrary N. For any nontrivial nonnegative initial data, it was proven that there is no nonnegative global solution.

Suppose that $\Omega(\subset \mathbb{R}^N)$ is any domain, bounded or unbounded. In [2], Bandle and Levine considered the initial boundary value problem

$$\begin{cases} u_t - \Delta u = u^p, \ \ (t,x) \in (0,T) \times \Omega, \\ u(0,x) = u_0(x), \ \ x \in \Omega \\ u(t,x) = 0, \ \ (t,x) \in (0,T) \times \partial\Omega. \end{cases} \tag{2}$$

They proved that for the exterior problem they also have $p_c = 1 + \frac{2}{N}$.

In some of the references, the nonlinearity u^p is replaced by $t^k|x|^\sigma u^p$ (see [1, 2]). The critical exponent is changed to reflect the dependence on k and σ, respectively, in the following form

$$p_c = 1 + \frac{2 + 2k + \sigma}{N}.$$

Later, Qi in [15] studied the equation

$$\begin{cases} u_t = \Delta u^m + |x|^\sigma t^s u^p, \ \ t > 0, \ x \in \mathbb{R}^N \\ u(0,x) = u_0(x) \geq 0, \ \text{for } x \in \mathbb{R}^N, \end{cases}$$

and had shown that the critical exponent for this problem is equal to

$$p_c = (m-1)(s-1) + \frac{2 + 2s + \sigma}{N}.$$

Other known results on critical exponents have been studied by a large number of authors, as shown in [3, 5, 6, 7, 11, 13, 14, 17] and the references therein. The author apologizes to the authors of the referenced and (unintentionally) unreferenced articles for any omissions.

Recently, Jleli et al. in [9] have studied the nonexistence of global weak solutions to systems of semilinear parabolic inequalities

$$\begin{cases} u_t - \Delta u \geq (|x|-1)^{-\rho}|v|^q, \ \ (t,x) \in (0,T) \times A, \\ v_t - \Delta v \geq (|x|-1)^{-\rho}|u|^p, \ \ (t,x) \in (0,T) \times A \end{cases} \tag{3}$$

and single inequality of the form

$$u_t - \Delta u \geq (|x|-1)^{-\rho}|u|^p, \;\; (t,x) \in (0,T) \times A, \tag{4}$$

where $A = \{x \in \mathbb{R}^N : 1 < |x| \leq 2\}$, $N \geq 2$, $\rho > 0$ and $p, q > 1$.

Problem (3) considered under two types of inhomogeneous boundary conditions:

• the Neumann-type

$$\frac{\partial u}{\partial \nu}(t,x) \geq f(x), \;\; \frac{\partial v}{\partial \nu}(t,x) \geq g(x) \;\; \text{on} \quad (t,x) \in (0,T) \times \partial B_2 = \Gamma, \tag{5}$$

• the Dirichlet-type

$$u(t,x) \geq f(x), \;\; v(t,x) \geq g(x) \;\; \text{on} \quad (t,x) \in (0,T) \times \partial B_2 = \Gamma, \tag{6}$$

where $\partial B_2 = \{x \in \mathbb{R}^N : |x| = 2\}$, ν is the outward unit normal vector on ∂B_2, relative to A and $f, g \in L^1(\partial B_2)$ are nontrivial functions.

Mainly, they have proved the following theorem.

Theorem 1. *Let $N \geq 2$ and $p, q > 1$.*

(I). *Let $f, g \in L^1(\partial B_2)$. If*

$$\min\left\{\int_{\partial B_2} f(x)d\sigma, \int_{\partial B_2} g(x)d\sigma\right\} > 0$$

and

$$\rho > 1 + \min\left\{\frac{p(q+1)}{p+1}, \frac{q(p+1)}{q+1}\right\},$$

the system (3) with either Neumann-type (5) or Dirichlet-type (6) boundary conditions, does not admit global in time weak solutions.

(II). *If*

$$0 < \rho < 1 + \min\left\{\frac{p(q+1)}{p+1}, \frac{q(p+1)}{q+1}\right\},$$

the system (3) with either Neumann-type (5) or Dirichlet-type (6) boundary conditions, admits global in time weak solutions for some f and g.

It is obvious that Theorem 1 also provides for a single inequality (4) (if $f = g$ and $p = q$).

Initially, it was unclear whether the critical value $\rho = p + 1$ was part of the non-existence scenario. However, recent work by the authors in [4] established that $\rho = p + 1$ indeed falls under the non-existence case.

This paper is devoted to studying the nonexistence of global weak solutions for the following pseudo-parabolic inequality

$$u_t - k\Delta u_t - \Delta u \geq (|x|-1)^{-\rho}|u|^p, \ \ (t,x) \in (0,T) \times A = \Pi_T, \tag{7}$$

where $A = \{x \in \mathbb{R}^N : 1 < |x| \leq 2\}$, $N \geq 2$, $\rho > 0$ and $k \geq 0$, $p > 1$.

The problem (7) is supplemented by inhomogeneous boundary conditions

• the Neumann-type

$$\frac{\partial u}{\partial \nu}(t,x) \geq f(x) \ \ \text{on} \quad (t,x) \in (0,T) \times \partial B_2 = \Gamma, \tag{8}$$

• the Dirichlet-type

$$u(t,x) \geq f(x) \ \ \text{on} \quad (t,x) \in (0,T) \times \partial B_2 = \Gamma, \tag{9}$$

where $\partial B_2 = \{x \in \mathbb{R}^N : |x| = 2\}$, ν is the outward unit normal vector on ∂B_2, relative to A and $f \in L^1(\partial B_2)$.

From the results cited above, we deduce that the size of the dimension plays a crucial role in determining whether or not a blowup occurs. Our purpose here is to show an interesting fact that the dimension of space does not affect the critical exponent in an annual domain.

2 Main results

In this section, we derive the main results of this work.

Definition 2 (Weak solution)**.** We say that $u \in L^p_{\text{loc}}(\Pi_T)$ is a local weak solution to (7)–(8), if the following inequality

$$\begin{aligned}\int_{\Pi_T} (|x|-1)^{-\rho}|u|^p \varphi dx dt + \int_\Gamma f\varphi d\sigma dt + k\int_\Gamma f\varphi_t d\sigma dt \leq \\ \leq -\int_{\Pi_T} u\varphi_t dx dt - k\int_{\Pi_T} u\Delta\varphi_t dx dt - \int_{\Pi_T} u\Delta\varphi dx dt,\end{aligned} \tag{10}$$

holds for all $\varphi \in C^2_c(\Pi_T), \varphi \geq 0, \varphi(T,\cdot) = 0$ with $\dfrac{\partial \varphi}{\partial \nu}|_\Gamma = 0$ and the notation $d\sigma$ is the surface measure on $\partial \Pi_T$.

Definition 3 (Weak solution)**.** We say that $u \in L^p_{\text{loc}}(\Pi_T)$ is a local weak solution to (7)–(9), if

$$\begin{aligned}\int_{\Pi_T} (|x|-1)^{-\rho}|u|^p \psi dx dt - \int_\Gamma f\frac{\partial \psi}{\partial \nu} d\sigma dt - k\int_\Gamma f\frac{\partial \psi_t}{\partial \nu} d\sigma dt \leq \\ \leq -\int_{\Pi_T} u\psi_t dx dt - k\int_{\Pi_T} u\Delta\psi_t dx dt - \int_{\Pi_T} u\Delta\psi dx dt,\end{aligned} \tag{11}$$

holds true for any $\psi \in C_c^2(\Pi_T), \psi \geq 0, \psi(T, \cdot) = 0$ and $\psi|_\Gamma = 0$, $\dfrac{\partial \psi}{\partial \nu}|_\Gamma \leq 0$.

If $T = +\infty$, then u is called a global in time weak solution.

Theorem 4. *Let $N \geq 2$ and $p > 1$. Suppose that $f \in L^1(\partial B_2)$. If*

$$\int_{\partial B_2} f(x) d\sigma > 0 \quad \textit{and} \quad \rho \geq p + 1,$$

the problem (7) *with either Neumann-type* (8) *or Dirichlet-type* (9) *boundary conditions, does not admit global in time weak solution.*

2.1. Test functions

This section will cover some test functions and their properties. Additionally, we will establish some useful estimates related to the test functions.

Now, we introduce the test function in the form

$$\varphi(t, x) = \varphi_1(t)\varphi_2(x) = \left(1 - \frac{t}{T}\right)^\lambda \xi_R(x), \;\; t \in (0, T), \, x \in A, \tag{12}$$

where $\lambda > \frac{p+1}{p-1}$ and the coefficients T, R are sufficiently large.

Let ξ_R be a family of smooth functions in A satisfying

$$0 \leq \xi_R \leq 1, \;\; \mathrm{supp}(\xi_R) \subset\subset 1 + \frac{1}{2R} < |x| \leq 2, \, \xi_R = 1 \;\text{ in }\; 1 + \frac{1}{R} < |x| \leq 2, \tag{13}$$

with

$$|\nabla \xi_R| \leq CR \text{ and } |\Delta \xi_R| \leq CR^2. \tag{14}$$

As an example we can give functions of the form (see [9])

$$\xi_R(x) = \Phi(R(|x| - 1)), \;\; x \in A,$$

where $\Phi : \mathbb{R}_+ \to [0, 1]$ is an increasing smooth function

$$\Phi(s) = \begin{cases} 0, \;\text{if}\;\; 0 \leq s \leq \dfrac{1}{2}, \\ 1, \;\text{if}\;\; s \geq 1, \end{cases}$$

which satisfies the conditions (13) and (14), respectively.

Lemma 5. *Let $\rho > 0, p > 1$. For sufficiently large T, R we have*

$$\mathcal{I}_1 = \int_{\Pi_T} (|x| - 1)^{\frac{\rho}{p-1}} \varphi^{-\frac{1}{p-1}} |\varphi_t|^{\frac{p}{p-1}} dx dt \leq C T^{1 - \frac{p}{p-1}}, \tag{15}$$

$$\mathcal{I}_2 = \int_{\Pi_T} (|x|-1)^{\frac{\rho}{p-1}} \varphi^{-\frac{1}{p-1}} |\Delta\varphi_t|^{\frac{p}{p-1}} dxdt \leq CT^{1-\frac{p}{p-1}} R^{\frac{p-\rho+1}{p-1}} \tag{16}$$

and

$$\mathcal{I}_3 = \int_{\Pi_T} (|x|-1)^{\frac{\rho}{p-1}} \varphi^{-\frac{1}{p-1}} |\Delta\varphi|^{\frac{p}{p-1}} dxdt \leq CTR^{\frac{p-\rho+1}{p-1}}. \tag{17}$$

Proof of Lemma 5. According to the test function (12), we have

$$\mathcal{I}_1 = \left(\int_0^T \varphi_1^{-\frac{1}{p-1}} |\varphi_1'|^{\frac{p}{p-1}} dt\right) \left(\int_A (|x|-1)^{\frac{\rho}{p-1}} \varphi_2^{-\frac{1}{p-1}} |\varphi_2|^{\frac{p}{p-1}} dx\right).$$

Furthermore, it can be deduced from basic calculations and the properties of the test functions (12)–(13) that

$$\begin{aligned}\int_0^T \varphi_1^{-\frac{1}{p-1}} |\varphi_1'|^{\frac{p}{p-1}} dt = T^{-\frac{p}{p-1}} \int_0^T \left(1-\frac{t}{T}\right)^{-\frac{\lambda}{p-1}} \left|\lambda\left(1-\frac{t}{T}\right)^{\lambda-1}\right|^{\frac{p}{p-1}} dt = \\ = CT^{1-\frac{p}{p-1}}\end{aligned} \tag{18}$$

and

$$\begin{aligned}\int_A (|x|-1)^{\frac{\rho}{p-1}} \varphi_2^{-\frac{1}{p-1}} |\varphi_2|^{\frac{p}{p-1}} dx &= \int\limits_{1+\frac{1}{2R}<|x|\leq 2} (|x|-1)^{\frac{\rho}{p-1}} \xi_R(x) dx \leq \\ &\leq \int\limits_{1+\frac{1}{2R}<|x|\leq 2} (|x|-1)^{\frac{\rho}{p-1}} dx \\ &\leq C.\end{aligned}$$

We get (15) by combining the aforementioned integrals.
Similarly, the integral $\mathcal{I}_2$ can be rewritten in the following form

$$\mathcal{I}_2 = \left(\int_0^T \varphi_1^{-\frac{1}{p-1}} |\varphi_1'|^{\frac{p}{p-1}} dt\right) \left(\int_A (|x|-1)^{\frac{\rho}{p-1}} \varphi_2^{-\frac{1}{p-1}} |\Delta\varphi_2|^{\frac{p}{p-1}} dx\right).$$

At this point, noting (14), we obtain

$$
\begin{aligned}
\int_A (|x|-1)^{\frac{\rho}{p-1}} \varphi_2^{-\frac{1}{p-1}} |\Delta\varphi_2|^{\frac{p}{p-1}} dx &= \int\limits_{1+\frac{1}{2R}}^{1+\frac{1}{R}} (|x|-1)^{\frac{\rho}{p-1}} \xi_R^{-\frac{1}{p-1}} |\Delta\xi_R|^{\frac{p}{p-1}} dx \\
&\le CR^{\frac{2p}{p-1}} \int\limits_{1+\frac{1}{2R}}^{1+\frac{1}{R}} (|x|-1)^{\frac{\rho}{p-1}} \xi_R^{-\frac{1}{p-1}} dx \\
&\le CR^{\frac{2p}{p-1}} \int\limits_{1+\frac{1}{2R}}^{1+\frac{1}{R}} (|x|-1)^{\frac{\rho}{p-1}} dx \\
&\overset{|x|=r}{=} CR^{\frac{2p}{p-1}} \int\limits_{1+\frac{1}{2R}}^{1+\frac{1}{R}} (r-1)^{\frac{\rho}{p-1}} r^{N-1} dr \\
&\overset{r-1=s}{\le} CR^{\frac{2p}{p-1}} \int\limits_{\frac{1}{2R}}^{\frac{1}{R}} s^{\frac{\rho}{p-1}} ds \\
&\le CR^{\frac{p-\rho+1}{p-1}}.
\end{aligned}
\tag{19}
$$

The combination of (18) with the last inequality gives (16).
In a similar manner, the integral $\mathcal{I}_3$ can be expressed as follows

$$
\mathcal{I}_3 = \left(\int_0^T \varphi_1^{-\frac{1}{p-1}} |\varphi_1|^{\frac{p}{p-1}} dt \right) \left(\int_A (|x|-1)^{\frac{\rho}{p-1}} \varphi_2^{-\frac{1}{p-1}} |\Delta\varphi_2|^{\frac{p}{p-1}} dx \right).
$$

It is also easy to verify by a direct calculation that

$$
\begin{aligned}
\int_0^T \varphi_1^{-\frac{1}{p-1}} |\varphi_1|^{\frac{p}{p-1}} dt &= \int_0^T \left(1-\frac{t}{T}\right)^{\lambda} dt \\
&= CT.
\end{aligned}
\tag{20}
$$

Hence, from (19) and (20), we get

$$
\mathcal{I}_3 \le CTR^{\frac{p-\rho+1}{p-1}},
$$

which completes the proof. □

Lemma 6. *Suppose that $f \in L^1(\partial B_2)$, then there holds*

$$\int_\Gamma f\varphi d\sigma \leq CT \int_{\partial B_2} f(x)d\sigma$$

and

$$\int_\Gamma f\varphi_t d\sigma \leq C \int_{\partial B_2} f(x)d\sigma.$$

Proof of Lemma 6. In view of (12)–(13), it yields that

$$\begin{aligned}\int_\Gamma f\varphi d\sigma = \int_\Gamma f\varphi_1(t)\varphi_2(x)d\sigma dt &= \left(\int_0^T \left(1-\frac{t}{T}\right)^\lambda dt\right)\left(\int_{\partial B_2} f(x)\xi_R(x)d\sigma\right)\\ &\leq CT\left(\int_{\partial B_2} f(x)d\sigma\right)\end{aligned}$$

and

$$\begin{aligned}\int_\Gamma f\varphi_t d\sigma = \int_\Gamma f\varphi_1'(t)\varphi_2(x)d\sigma dt &= \left(\int_0^T \frac{\lambda}{T}\left(1-\frac{t}{T}\right)^{\lambda-1} dt\right)\left(\int_{\partial B_2} f(x)\xi_R(x)d\sigma\right)\\ &\leq C\int_{\partial B_2} f(x)d\sigma,\end{aligned}$$

which give the desired results. □

At this stage, we introduce another test function for T, R sufficiently large coefficients as

$$\psi(t,x) = \psi_1(t)\psi_2(x) = \left(1-\frac{t}{T}\right)^\lambda H(x)\xi(x),\ \ \lambda > \frac{1}{p-1},\ \ t\in(0,T),\ x\in A, \tag{21}$$

where $\xi(x)$ be a family of smooth functions in A satisfying

$$0 \leq \xi(x) \leq 1,\ \ \mathrm{supp}(\xi) \subset\subset 1+\frac{1}{R} < |x| \leq 2,\ \xi = 1 \ \text{ in } \ 1+\frac{1}{\sqrt{R}} < |x| \leq 2. \tag{22}$$

In addition, the function $H(x)$ defined in A by

$$H(x) = \begin{cases} \ln 2 - \ln|x|, & \text{if } \ N=2,\\ 2^{N-2}|x|^{2-N}, & \text{if } \ N\geq 3.\end{cases} \tag{23}$$

In addition, $H(x)$ is nonnegative, harmonic and satisfies the condition $H|_{\partial B_2} = 0$.

Lemma 7. *Let the test function $\xi(x)$ be introduced in the form*

$$\xi(x) = \Psi^s\left(\frac{\ln(R(|x|-1))}{\ln R}\right), \ s > \frac{2p}{p-1}, \tag{24}$$

where $\Psi : (-\infty, 1] \to [0, 1]$ be a smooth function function such that

$$\Psi(z) = \begin{cases} 0 & \text{if} \quad -\infty < z \le 0, \\ \nearrow & \text{if} \quad 0 < z < \frac{1}{2}, \\ 1 & \text{if} \quad \frac{1}{2} \le z \le 1. \end{cases}$$

Consequently,

$$\Psi\left(\frac{\ln(R(|x|-1))}{\ln R}\right) = \begin{cases} 0 & \text{if} \quad 1 \le |x| \le 1 + \frac{1}{R}, \\ \nearrow & \text{if} \quad 1 + \frac{1}{R} < |x| < 1 + \frac{1}{\sqrt{R}}, \\ 1 & \text{if} \quad 1 + \frac{1}{\sqrt{R}} \le |x| \le 2. \end{cases}$$

Assume that

$$|\Psi'(z)| \le C, \ \ |\Psi''(z)| \le C, \tag{25}$$

then, the following estimate holds true

$$\begin{aligned} |\Delta\xi(x)| \le & \frac{C}{(|x|-1)^2 \ln^2 R}\Psi^{s-2}\left(\frac{\ln(R(|x|-1))}{\ln R}\right) \\ & + \frac{C}{(|x|-1)^2 \ln R}\Psi^{s-1}\left(\frac{\ln(R(|x|-1))}{\ln R}\right). \end{aligned} \tag{26}$$

Proof of Lemma 7. In because of the function $\Psi(x)$ is radial, we arrive at

$$\begin{aligned} \Delta\xi(r) &= \left(\frac{d^2}{dr^2} + \frac{N-1}{r}\frac{d}{dr}\right)\Psi^s\left(\frac{\ln(R(r-1))}{\ln R}\right) \\ &= s(s-1)\Psi^{s-2}\left(\frac{\ln(R(r-1))}{\ln R}\right)\left[\Psi'\left(\frac{\ln(R(r-1))}{\ln R}\right)\right]^2 \frac{1}{\ln^2 R(r-1)^2} \\ &+ s\Psi^{s-1}\left(\frac{\ln(R(r-1))}{\ln R}\right)\Psi''\left(\frac{\ln(R(r-1))}{\ln R}\right)\frac{1}{\ln^2 R(r-1)^2} \\ &+ s\Psi^{s-1}\left(\frac{\ln(R(r-1))}{\ln R}\right)\Psi'\left(\frac{\ln(R(r-1))}{\ln R}\right)\frac{N-2}{\ln R(r-1)^2}, \end{aligned}$$

where $r = |x| = (x_1^2 + x_2^2 + ... + x_n^2)^{\frac{1}{2}}$ and $C > 0$ is an arbitrary constant.

Moreover, from (25), we obtain

$$|\Delta\xi(r)| \le \frac{C}{\ln^2 R(r-1)^2}\Psi^{s-2}\left(\frac{\ln(R(r-1))}{\ln R}\right) + \frac{C}{\ln R(r-1)^2}\Psi^{s-1}\left(\frac{\ln(R(r-1))}{\ln R}\right),$$

which completes the proof. □

Lemma 8. *Let $\rho > 0$, $p > 1$ and T, R are sufficiently large, then we have*

$$\mathcal{J}_1 = \int\limits_{\Pi_T} (|x|-1)^{\frac{\rho}{p-1}} \psi^{-\frac{1}{p-1}} |\psi_t|^{\frac{p}{p-1}} dxdt \leq CT^{1-\frac{p}{p-1}}, \tag{27}$$

$$\begin{aligned}\mathcal{J}_2 &= \int\limits_{\Pi_T} (|x|-1)^{\frac{\rho}{p-1}} \psi^{-\frac{1}{p-1}} |\Delta\psi|^{\frac{p}{p-1}} dxdt \leq \\ &\leq CT \left[(\ln R)^{-\frac{2p}{p-1}} + (\ln R)^{-\frac{p}{p-1}} \right] R^{-\frac{1}{2}\left(\frac{\rho-p-1}{p-1}\right)}\end{aligned} \tag{28}$$

and

$$\begin{aligned}\mathcal{J}_3 &= \int\limits_{\Pi_T} (|x|-1)^{\frac{\rho}{p-1}} \psi^{-\frac{1}{p-1}} |\Delta\psi_t|^{\frac{p}{p-1}} dxdt \leq \\ &\leq CT^{1-\frac{p}{p-1}} \left[(\ln R)^{-\frac{2p}{p-1}} + (\ln R)^{-\frac{p}{p-1}} \right] R^{-\frac{1}{2}\left(\frac{\rho-p-1}{p-1}\right)}.\end{aligned} \tag{29}$$

Proof of Lemma 8. Taking into account (21), it obvious that

$$\mathcal{J}_1 = \left(\int\limits_0^T \psi_1^{-\frac{1}{p-1}} |\psi_1'|^{\frac{p}{p-1}} dt \right) \left(\int\limits_{1+\frac{1}{R}<|x|\leq 2} (|x|-1)^{\frac{\rho}{p-1}} \psi_2 dx \right).$$

Therefore, taking into account the property of the function $H(x)$ and $\xi(x)$

$$H(x) \leq C,\ \xi(x) = 1 \ \text{ on } \ 1 + \frac{1}{R} < |x| \leq 2, \tag{30}$$

it yields that

$$\begin{aligned}\int\limits_{1+\frac{1}{R}<|x|\leq 2} (|x|-1)^{\frac{\rho}{p-1}} \psi_2 dx &= \int\limits_{1+\frac{1}{R}<|x|\leq 2} (|x|-1)^{\frac{\rho}{p-1}} H(x)\xi(x) dx \\ &\leq C \int\limits_{1+\frac{1}{R}<|x|\leq 2} (|x|-1)^{\frac{\rho}{p-1}} dx \\ &\leq C.\end{aligned} \tag{31}$$

Moreover, combining (18) and the last estimate, we obtain the desired result.

Similarly, we have

$$\mathcal{J}_2 = \left(\int\limits_0^T \psi_1^{-\frac{1}{p-1}} |\psi_1'|^{\frac{p}{p-1}} dt \right) \left(\int\limits_A (|x|-1)^{\frac{\rho}{p-1}} \psi_2^{-\frac{1}{p-1}} |\Delta\psi_2|^{\frac{p}{p-1}} dx \right).$$

Moreover, by (23), we get

$$H^{-\frac{1}{p-1}}(x) \leq C, |H(x)| \leq C, |\nabla H(x)| \leq C, \;\; 1+\frac{1}{R} < x < 1+\frac{1}{\sqrt{R}}. \tag{32}$$

Therefore, taking into account that the function $H(x)$ is harmonic with (32) and the remaining (26), we arrive at

$$\begin{aligned} |\Delta\psi_2(x)| = |\Delta\left[H(x)\xi(x)\right]| &\leq H(x)|\Delta\left[\xi(x)\right]| + 2|\nabla H(x)||\nabla\xi(x)| \\ &\leq \frac{CH(x)}{(|x|-1)^2\ln^2 R}\Psi^{s-2}\left(\frac{\ln(R(|x|-1))}{\ln R}\right) \\ &+ \frac{CH(x)}{(|x|-1)^2\ln R}\Psi^{s-1}\left(\frac{\ln(R(|x|-1))}{\ln R}\right) \\ &+ \frac{C}{(|x|-1)\ln R}\Psi^{s-1}\left(\frac{\ln(R(|x|-1))}{\ln R}\right) \\ &\leq C\left(\frac{1}{(|x|-1)^2\ln^2 R}\xi^{\frac{s-2}{s}} + \frac{1}{(|x|-1)\ln R}\xi^{\frac{s-1}{s}}\right). \end{aligned}$$

In view of the following inequality

$$(a+b)^m \leq 2^{m-1}(a^m+b^m), \;\; a\geq 0,\, b\geq 0,\, m=\frac{p}{p-1},$$

we obtain

$$\int\limits_A (|x|-1)^{\frac{\rho}{p-1}}\psi_2^{-\frac{1}{p-1}}|\Delta\psi_2|^{\frac{p}{p-1}}dx = \int\limits_{1+\frac{1}{R}<|x|<1+\frac{1}{\sqrt{R}}} (|x|-1)^{\frac{\rho}{p-1}}\xi^{-\frac{1}{p-1}}|\Delta\xi|^{\frac{p}{p-1}}dx$$

and

$$\begin{aligned} &\int\limits_{1+\frac{1}{R}<|x|<1+\frac{1}{\sqrt{R}}} (|x|-1)^{\frac{\rho}{p-1}}\xi^{-\frac{1}{p-1}}|\Delta\xi|^{\frac{p}{p-1}}dx \leq \\ &\leq C\int\limits_{1+\frac{1}{R}<|x|<1+\frac{1}{\sqrt{R}}} (|x|-1)^{\frac{\rho}{p-1}}\xi^{-\frac{1}{p-1}}\left[\frac{1}{(|x|-1)^2\ln^2 R}\left|\xi^{\frac{s-2}{s}}\right|\right]^{\frac{p}{p-1}}dx+ \\ &+ C\int\limits_{1+\frac{1}{R}<|x|<1+\frac{1}{\sqrt{R}}} (|x|-1)^{\frac{\rho}{p-1}}\xi^{-\frac{1}{p-1}}\left[\frac{1}{(|x|-1)\ln R}\left|\xi^{\frac{s-1}{s}}\right|\right]^{\frac{p}{p-1}}dx. \end{aligned}$$

Then, noting that $s > \frac{2p}{p-1}$ and (22) we arrive at

$$\int\limits_{1+\frac{1}{R}<|x|<1+\frac{1}{\sqrt{R}}} (|x|-1)^{\frac{\rho}{p-1}} \xi^{-\frac{1}{p-1}} |\Delta\xi|^{\frac{p}{p-1}} dx$$

$$\leq C \int\limits_{1+\frac{1}{R}<|x|<1+\frac{1}{\sqrt{R}}} (|x|-1)^{\frac{\rho}{p-1}} \left[\frac{1}{(|x|-1)^2 \ln^2 R}\right]^{\frac{p}{p-1}} dx$$

$$+ C \int\limits_{1+\frac{1}{R}<|x|<1+\frac{1}{\sqrt{R}}} (|x|-1)^{\frac{\rho}{p-1}} \left[\frac{1}{(|x|-1) \ln R}\right]^{\frac{p}{p-1}} dx.$$

At this stage, we calculate the last integral in the following form

$$\begin{aligned}
&\int\limits_{1+\frac{1}{R}<|x|<1+\frac{1}{\sqrt{R}}} (|x|-1)^{\frac{\rho}{p-1}} \left[\frac{1}{(|x|-1)^2 \ln^2 R}\right]^{\frac{p}{p-1}} dx \\
&= \left[\frac{1}{\ln^2 R}\right]^{\frac{p}{p-1}} \int\limits_{1+\frac{1}{R}<|x|<1+\frac{1}{\sqrt{R}}} (|x|-1)^{\frac{\rho}{p-1}-\frac{2p}{p-1}} dx \\
&\overset{|x|=r}{=} \left[\frac{1}{\ln^2 R}\right]^{\frac{p}{p-1}} \int\limits_{1+\frac{1}{R}<r<1+\frac{1}{\sqrt{R}}} (r-1)^{\frac{\rho}{p-1}-\frac{2p}{p-1}} r^{N-1} dr \\
&\overset{r-1=s}{\leq} C \left[\frac{1}{\ln^2 R}\right]^{\frac{p}{p-1}} \int\limits_{\frac{1}{R}<s<\frac{1}{\sqrt{R}}} s^{\frac{\rho}{p-1}-\frac{2p}{p-1}} ds \\
&\leq (\ln R)^{-\frac{2p}{p-1}} R^{-\frac{1}{2}\left(\frac{\rho-p-1}{p-1}\right)}.
\end{aligned} \tag{33}$$

Similarly, one obtains

$$\int\limits_{1+\frac{1}{R}<|x|<1+\frac{1}{\sqrt{R}}} (|x|-1)^{\frac{\rho}{p-1}} \left[\frac{1}{(|x|-1)^2 \ln R}\right]^{\frac{p}{p-1}} dx \leq (\ln R)^{-\frac{p}{p-1}} R^{-\frac{1}{2}\left(\frac{\rho-p-1}{p-1}\right)}. \tag{34}$$

Combining (33), (34) and (20) we obtain (28).

Consequently, the integral $\mathcal{J}_3$ expressed by

$$\mathcal{J}_3 = \left(\int\limits_0^T \psi_1^{-\frac{1}{p-1}} |\psi_1'|^{\frac{p}{p-1}} dt\right) \left(\int\limits_{1+\frac{1}{R}<|x|\leq 2} (|x|-1)^{\frac{\rho}{p-1}} \psi_2^{-\frac{1}{p-1}} |\Delta\psi_2|^{\frac{p}{p-1}} dx\right).$$

Then, the combination of (18) with (33), (34) gives us

$$\mathcal{J}_3 = \left(\int_0^T \psi_1^{-\frac{1}{p-1}} |\psi_1|^{\frac{p}{p-1}} dt\right) \left(\int_{1+\frac{1}{R}<|x|\leq 2} (|x|-1)^{\frac{\rho}{p-1}} \psi_2^{-\frac{1}{p-1}} |\Delta\psi_2|^{\frac{p}{p-1}} dx\right) \leq$$
$$\leq CT^{1-\frac{p}{p-1}} \left[(\ln R)^{-\frac{2p}{p-1}} + (\ln R)^{-\frac{p}{p-1}}\right] R^{-\frac{1}{2}\left(\frac{\rho-p-1}{p-1}\right)},$$

for all $(t,x) \in \Pi_T$, which completes the proof. □

Lemma 9. *For the function $\psi(t,x)$ on Γ the following inequalities*

$$\frac{\partial \psi}{\partial \nu}(t,x) = -C(N)\psi_1(t) \leq 0$$

and

$$\frac{\partial \psi_t}{\partial \nu}(t,x) = -C(N)\frac{\partial}{\partial t}\psi_1(t) \leq 0$$

hold, where

$$C(N) = \begin{cases} 2^{-1}, & \text{if } N = 2, \\ 2^{-1}(N-2), & \text{if } N \geq 3. \end{cases}$$

Proof of Lemma 9. Using the property of the test function (22), we obtain

$$\begin{aligned} \nabla\psi_2(x) &= \nabla\left(H(x)\xi(x)\right) \\ &= H(x)\nabla\xi(x) + \xi(x)\nabla H(x) \\ &= \nabla H(x), \end{aligned}$$

here we have used $\nabla\xi(x) = 0$ and thanks to $\xi(x) = 1$ on $1 + \frac{1}{\sqrt{R}} < |x| \leq 2$.

Hence, using (23) for $x \in \partial B_2$, we arrive at

$$\frac{\partial \psi_2}{\partial \nu}(x) = -C(N),$$

where

$$C(N) = \begin{cases} 2^{-1}, & \text{if } N = 2, \\ 2^{-1}(N-2), & \text{if } N \geq 3. \end{cases}$$

Consequently, from (21) we conclude that

$$\frac{\partial \psi}{\partial \nu}(t,x) = -C(N)\psi_1(t) = \begin{cases} 2^{-1}\psi_1(t) \leq 0, & \text{if } N = 2, \\ 2^{-1}(N-2)\psi_1(t) \leq 0, & \text{if } N \geq 3 \end{cases}$$

and

$$\frac{\partial \psi_t}{\partial \nu}(t,x) = -C(N)\frac{\partial}{\partial t}\psi_1(t) = \begin{cases} 2^{-1}\frac{\partial}{\partial t}\psi_1(t) \leq 0, & \text{if } N = 2, \\ 2^{-1}(N-2)\frac{\partial}{\partial t}\psi_1(t) \leq 0, & \text{if } N \geq 3, \end{cases}$$

hold true for all $(t,x) \in \Gamma$. □

2.2. Proof of basic theorems

In this subsection, we prove the main theorems in detail.

Proof of Theorem 4. **(i) The case with the Neumann-type boundary condition** (8)**.**
• **Subcritical case** $\rho > p+1$. From Definition 2, we deduce that

$$\begin{aligned}&\int_{\Pi_T}(|x|-1)^{-\rho}|u|^p\varphi dxdt+\int_{\Gamma}f\varphi d\sigma dt+k\int_{\Gamma}f\varphi_t d\sigma dt\\&\leq\int_{\Pi_T}|u||\varphi_t|dxdt+k\int_{\Pi_T}|u||\Delta\varphi_t|dxdt+\int_{\Pi_T}|u||\Delta\varphi|dxdt.\end{aligned}\tag{35}$$

Therefore, by Hölder's inequality, we obtain

$$\int_{\Pi_T}|u||\varphi_t|dxdt\leq\left(\int_{\Pi_T}(|x|-1)^{-\rho}|u|^p\varphi dxdt\right)^{\frac{1}{p}}\Bigg(\underbrace{\int_{\Pi_T}(|x|-1)^{\frac{\rho}{p-1}}\varphi^{-\frac{1}{p-1}}|\varphi_t|^{\frac{p}{p-1}}dxdt}_{\mathcal{I}_1}\Bigg)^{\frac{p-1}{p}},$$

$$\int_{\Pi_T}|u||\Delta\varphi_t|dxdt\leq\left(\int_{\Pi_T}(|x|-1)^{-\rho}|u|^p\varphi dxdt\right)^{\frac{1}{p}}\Bigg(\underbrace{\int_{\Pi_T}(|x|-1)^{\frac{\rho}{p-1}}\varphi^{-\frac{1}{p-1}}|\Delta\varphi_t|^{\frac{p}{p-1}}dxdt}_{\mathcal{I}_2}\Bigg)^{\frac{p-1}{p}}$$

and

$$\int_{\Pi_T}|u||\Delta\varphi|dxdt\leq\left(\int_{\Pi_T}(|x|-1)^{-\rho}|u|^p\varphi dxdt\right)^{\frac{1}{p}}\Bigg(\underbrace{\int_{\Pi_T}(|x|-1)^{\frac{\rho}{p-1}}\varphi^{-\frac{1}{p-1}}|\Delta\varphi|^{\frac{p}{p-1}}dxdt}_{\mathcal{I}_3}\Bigg)^{\frac{p-1}{p}}.$$

On the other hand, using the ε-Young inequality with $\varepsilon=\dfrac{p}{3}$ in the last inequalities, we can rewrite (35) in the following form

$$\int_{\Gamma}f\varphi d\sigma dt+k\int_{\Gamma}f\varphi_t d\sigma dt\leq C(p)(\mathcal{I}_1+k\mathcal{I}_2+\mathcal{I}_3),\tag{36}$$

where $C(p)=\dfrac{p-1}{p}\left(\dfrac{p}{3}\right)^{-\frac{1}{p-1}}$.
From Lemma 5 and Lemma 6 we deduce that

$$\left(C+kCT^{-1}\right)\int_{\partial B_2}f(x)d\sigma\leq C(p)\left(T^{-\frac{p}{p-1}}+kT^{-\frac{p}{p-1}}R^{\frac{p-\rho+1}{p-1}}+R^{\frac{p-\rho+1}{p-1}}\right).$$

Finally, taking $R=T$ passing to the limit $T\to\infty$ taking account $\rho>p+1$, we get a contradiction with

$$\int_{\partial B_2}f(x)d\sigma>0,$$

which completes the proof.

• **Critical case** $\rho = p+1$. Using the same technique as the previous case choosing the test function as (21), we obtain the estimate instead of (36) in the following form

$$\int_\Gamma f\psi d\sigma dt + k\int_\Gamma f\psi_t d\sigma dt \le C(p)(\mathcal{J}_1 + k\mathcal{J}_2 + \mathcal{J}_3).$$

According to Lemma 6 and Lemma 8, we conclude that

$$\begin{aligned}&\left(C + kCT^{-1}\right)\int_{\partial B_2} f(x)d\sigma\\ &\le C(p)\left(T^{-\frac{p}{p-1}}R^{-\frac{\rho+p-1}{p-1}} + kT^{-\frac{p}{p-1}}(\ln R)^{-\frac{p}{p-1}}R^{-\frac{\rho-p-1}{p-1}} + (\ln R)^{-\frac{p}{p-1}}R^{-\frac{p-\rho+1}{p-1}}\right).\end{aligned}$$

Furthermore, choosing $R = T$ and passing to the limit $T \to \infty$ taking account $\rho = p+1$, we get a contradiction with

$$\int_{\partial B_2} f(x)d\sigma > 0,$$

which claims our theorem.

• **The case with the Dirichlet boundary condition** (9). Assume that $u \in L^p(\Pi_T)$ is a global weak solution to (7)-(9). Then, acting in the same way as in the above case, we get the next estimate instead of (36)

$$-\int_\Gamma f\frac{\partial\psi}{\partial\nu}d\sigma dt - k\int_\Gamma f\frac{\partial\psi_t}{\partial\nu}d\sigma dt \le C(p)(\mathcal{I}_1 + k\mathcal{I}_2 + \mathcal{I}_3), \tag{37}$$

where $C(p) = \dfrac{p-1}{p}\left(\dfrac{p}{3}\right)^{-\frac{1}{p-1}}$.

Hence, using (21) and Lemma 9 on the left-hand side of the last inequality, we obtain

$$\begin{aligned}\int_\Gamma f\frac{\partial\psi}{\partial\nu}d\sigma dt &= -C(N)\left(\int_0^T\left(1-\frac{t}{T}\right)^\lambda dt\right)\left(\int_{\partial B_2} f(x)d\sigma\right)\\ &= -C_1T\left(\int_{\partial B_2} f(x)d\sigma\right)\end{aligned} \tag{38}$$

and

$$\begin{aligned}\int_\Gamma f\frac{\partial\psi_t}{\partial\nu}d\sigma dt &= -C(N)\left(-\frac{\lambda}{T}\int_0^T\left(1-\frac{t}{T}\right)^{\lambda-1}dt\right)\left(\int_{\partial B_2} f(x)d\sigma\right)\\ &= C_2\left(\int_{\partial B_2} f(x)d\sigma\right).\end{aligned} \tag{39}$$

From the last estimates and Lemma 5 we can rewrite the inequality (37) as

$$(C_1 - kC_2T^{-1}) \int_{\partial B_2} f(x)d\sigma \leq C(p)\left(T^{-\frac{p}{p-1}} + kT^{-\frac{p}{p-1}}R^{\frac{p-\rho+1}{p-1}} + R^{\frac{p-\rho+1}{p-1}}\right).$$

Finally, taking $R = T$ passing to the limit as $T \to \infty$ noting $\rho > p+1$, we get a contradiction.
• **Critical case** $\rho = p+1$. We obtain the estimate in instead of (36) using the same method as in the previous case, and it has the following form

$$-\int_\Gamma f\frac{\partial\psi}{\partial\nu}d\sigma dt - k\int_\Gamma f\frac{\partial\psi_t}{\partial\nu}d\sigma dt \leq C(p)(\mathcal{J}_1 + k\mathcal{J}_2 + \mathcal{J}_3).$$

In view of (38), (39) and Lemma 8, we conclude that

$$\begin{aligned}&\left(C_1 + kC_2T^{-1}\right)\int_{\partial B_2} f(x)d\sigma \\ &\leq C(p)\left(T^{-\frac{p}{p-1}}R^{-\frac{\rho+p-1}{p-1}} + kT^{-\frac{p}{p-1}}\left(\ln R\right)^{-\frac{p}{p-1}}R^{-\frac{\rho-p-1}{p-1}} + \left(\ln R\right)^{-\frac{p}{p-1}}R^{-\frac{p-\rho+1}{p-1}}\right).\end{aligned}$$

Furthermore, choosing $R = T$ and passing to the limit $T \to \infty$ taking account $\rho = p+1$, we get a contradiction with

$$\int_{\partial B_2} f(x)d\sigma > 0,$$

which completes the proof. □

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Бөріханов М.Б., Төребек Б.Т. САҚИНАДАҒЫ ПСЕВДО-ПАРАБОЛАЛЫҚ ТЕҢСІЗДІКТЕРДІҢ ГЛОБАЛДЫ ШЕШІЛІМСІЗДІГІ

Бұл жұмыста сақинадағы сингулярлық потенциалдық функциясы бар псевдопараболалық теңсіздікті қарастырамыз. Есеп сақинаның жоғарғы шекарасында Нейман типті және Дирихле типті шекаралық шарттармен зерттеледі. Екі жағдайда да сынақ функциясы әдісі негізінде глобалды әлсіз шешімдердің жоқтығы анықталды.

Түйін сөздер: дифференциалдық оператор, Лаплас теңдеуі, шекаралық шарт, шешімнің бірегейлігі, оператордың меншікті мәндері, толық ортогональды жүйелер, оператор спектрі.

Бориханов М. Б., Торебек Б. Т. ГЛОБАЛЬНАЯ НЕРАЗРЕШИМОСТЬ ПСЕВДОПАРАБОЛИЧЕСКИХ НЕРАВЕНСТВ В КОЛЬЦЕ

В настоящей работе рассматривается псевдопараболическое неравенство с сингулярной потенциальной функцией в кольце. Задача изучается с граничными условиями типа Неймана и типа Дирихле на верхней границе кольца. Для обоих случаев на основе метода пробных функций установлено отсутствие глобальных слабых решений.

Ключевые слова: дифференциальный оператор, уравнение Лапласа, граничное условие, единственность решения, собственные значения оператора, полные ортогональные системы, спектр оператора.