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Solution to the periodic problem for the impulsive hyperbolic equation with discrete memory

Anar T. Assanova[1], Altynai Molybaikyzy[2]

[1]Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
[2]Al-Farabi Kazakh National University, Almaty, Kazakhstan

[1]assanova@math.kz; anartasan@gmail.com, [2]altynaimolybai@gmail.com

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Abstract. In this article, we consider the periodic problem for the impulsive hyperbolic equation with discrete memory. Impulsive hyperbolic equations with discrete memory arise as a mathematical model for describing physical processes in the neural networks, discontinuous dynamical systems, hybrid systems, and etc. Questions of the existence and construction of solutions to periodic problems for impulsive hyperbolic equations with discrete memory remain important issues in the theory of discontinuous partial differential equations. To find the solvability conditions of this problem we apply Dzhumabaev's parametrization method. The coefficient conditions for the existence and uniqueness of the periodic problem for the impulsive hyperbolic equation with discrete memory are established. We offer an algorithm for determining the approximate solution to this problem and show its convergence to the exact solution of the periodic problem for the impulsive hyperbolic equation with discrete memory.

Keywords. hyperbolic equation, impulse effects, periodic condition, discrete memory, partition of domain, problem with parameters, solvability conditions.

1 Introduction

On the domain $\Omega = [0,T]\times[0,\omega]$ we consider the periodic problem for the impulsive hyperbolic equation with discrete memory in the following form

$$\frac{\partial^2 u}{\partial t \partial x} = A(t,x)\frac{\partial u(t,x)}{\partial x} + A_0(t,x)\frac{\partial u(\gamma(t),x)}{\partial x} + B(t,x)\frac{\partial u(t,x)}{\partial t} + C(t,x)u(t,x) + f(t,x), \quad (1)$$

$$t \neq \theta_j, \qquad j = \overline{1, N-1},$$

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$$u(0,x) = u(T,x), \qquad x \in [0,\omega], \tag{2}$$

$$\lim_{t\to\theta_p+0} u(t,x) - \lim_{t\to\theta_p-0} u(t,x) = \varphi_p(x), \qquad x \in [0,\omega], \quad p = \overline{1, N-1}, \tag{3}$$

$$u(t,0) = \psi(t), \qquad t \in [0,T], \tag{4}$$

where $u(t,x)$ is unknown function, the functions $A(t,x)$, $B(t,x)$, $C(t,x)$, $A_0(t,x)$ and n vector function $f(t,x)$ are continuous on Ω;

$$\gamma(t) = \zeta_s \text{ if } t \in [\theta_{s-1}, \theta_s),\ s = \overline{1,N};$$

$$\theta_{s-1} < \zeta_s < \theta_s \text{ for all } s = 1,2,...,N;\ 0 = \theta_0 < \theta_1 < ... < \theta_{N-1} < \theta_N = T;$$

the functions $\varphi_p(x)$ are continuously differentiable on $[0,\omega]$, $p = \overline{1, N-1}$; the function $\psi(t)$ is continuously differentiable on $[0,T]$ and satisfies the compatibility condition: $\psi(0) = \psi(T)$.

We introduce the notation

$$\Omega_s = [\theta_{s-1}, \theta_s) \times [0,\omega],\ s = \overline{1,N},\ \text{i.e. } \Omega = \bigcup_{s=1}^{N} \Omega_s.$$

Let $PC(\Omega, \{\theta_j\}_{j=1}^{N-1}, \mathbb{R})$ be the space of piecewise continuous on Ω functions $u(t,x)$ with possible discontinuities on lines $t = \theta_j$, $j = \overline{1, N-1}$, and the norm

$$||u||_1 = \max_{s=\overline{1,N}} \sup_{(t,x)\in\Omega_s} |u(t,x)|.$$

A function $u(t,x) \in PC(\Omega, \{\theta_j\}_{j=1}^{N-1}, \mathbb{R})$ is a solution to problem (1)–(4) if:

(i) $u(t,x)$ has partial derivatives

$$\frac{\partial u(t,x)}{\partial x} \in PC(\Omega, \{\theta_j\}_{j=1}^{N-1}, \mathbb{R}),\ \frac{\partial u(t,x)}{\partial t} \in PC(\Omega, \{\theta_j\}_{j=1}^{N-1}, \mathbb{R});$$

(ii) the mixed partial derivative $\frac{\partial^2 u(t,x)}{\partial t \partial x}$ exists at each point $(t,x) \in \Omega$ with the possible exception of the points (θ_{s-1}, x), $s = \overline{1,N}$, for all $x \in [0,\omega]$, where the one-sided mixed partial derivatives exist;

(iii) hyperbolic equation (1) is satisfied for $u(t,x)$ on each subdomain $(\theta_{s-1}, \theta_s) \times [0,\omega]$, $s = \overline{1,N}$, and it holds for the right mixed partial derivative of $u(t,x)$ at the points (θ_{s-1}, x), $s = \overline{1,N}$, $\quad x \in [0,\omega]$;

(iv) periodic condition (2) and initial condition (4) are satisfied for $u(t,x)$ at the lines $t = 0$, $t = T$, and $x = 0$, respectively;

(v) the conditions of the impulse effects (3) are satisfied for $u(t,x)$ at the lines $t = \theta_p$, $p = \overline{1, N-1}$, $x \in [0,\omega]$.

Differential equations with discrete memory (or generalized piecewise constant argument) are more suitable for modeling and solving various application problems, including areas of neural networks, discontinuous dynamical systems, biological and medical models, etc. [1, 2, 3, 4, 5, 6, 7].

Questions of solvability and construction of solutions to boundary value problems for differential and hyperbolic equations with generalized piecewise constant argument on a finite interval were studied in [8, 9, 10, 11].

For impulsive partial differential equations with discrete memory, however, the questions of solvability of boundary value problems on a finite interval still remain open [12].

This issue can be resolved by developing constructive methods.

The non-local problem for a system of hyperbolic equations with impulse discrete memory were considered in [13]. Conditions for the existence and uniqueness solution to the non-local problem for a system of hyperbolic equations with impulse discrete memory were established in the term of special matrix composed by coefficient matrices and boundary matrices.

In the present paper, we propose a new approach for solving periodic problems for the impulsive hyperbolic equation with discrete memory (1)–(4) based on the introduction of new functions and on Dzhumabaev's parametrization method [14].

2 Introduction of new functions and algorithm of Dzhumabaev's parametrization method

First, we introduce new functions $v(t,x) = \frac{\partial u(t,x)}{\partial x}$, $w(t,x) = \frac{\partial u(t,x)}{\partial t}$.

We have a periodic problem for a family of impulsive differential equations with discrete memory in the next form

$$\frac{\partial v}{\partial t} = A(t,x)v(t,x) + A_0(t,x)v(\gamma(t),x) + f(t,x) + B(t,x)w(t,x) + C(t,x)u(t,x), \quad (5)$$

$$t \neq \theta_j, \quad j = \overline{1, N-1},$$

$$v(0,x) = v(T,x), \qquad x \in [0,\omega], \quad (6)$$

$$\lim_{t \to \theta_p + 0} v(t,x) - \lim_{t \to \theta_p - 0} v(t,x) = \dot{\varphi}_p(x), \qquad x \in [0,\omega], \quad p = \overline{1, N-1}, \quad (7)$$

$$u(t,x) = \psi(t) + \int\limits_0^x v(t,\xi)d\xi, \qquad w(t,x) = \dot{\psi}(t) + \int\limits_0^x \frac{\partial v(t,\xi)}{\partial t} d\xi, \quad (8)$$

$$(t,x) \in \Omega_s, \qquad s = \overline{1, N}.$$

A triple of functions $\{v(t,x), u(t,x), w(t,x)\}$ is a solution to the problem for the family of impulsive differential equations with discrete memory (5)–(8) if:

(i) the function $v(t,x) \in PC(\Omega, \{\theta_j\}_{j=1}^{N-1}, \mathbb{R})$ has partial derivative

$$\frac{\partial v(t,x)}{\partial t} \in PC(\Omega, \{\theta_j\}_{j=1}^{N-1}, \mathbb{R});$$

(ii) the family of the differential equations (5) is satisfied for $v(t,x)$, $u(t,x)$ and $w(t,x)$ on each subdomain $(\theta_{s-1}, \theta_s) \times [0,\omega]$, $s = \overline{1,N}$, and it holds for the right partial derivative of $v(t,x)$ by t at the points (θ_{s-1}, x), $s = \overline{1,N}$, $x \in [0,\omega]$;

(iii) the periodic condition (6) is satisfied for $v(t,x)$ at the lines $t = 0$ and $t = T$.

(iv) the functions $u(t,x)$ and $w(t,x)$ are connected with $v(t,x)$ and $\frac{\partial v(t,x)}{\partial t}$ by the integral equations (8).

(v) Denote by $\Delta_N(\omega)$ a partition of the domain Ω by lines $t = \theta_s$:

$$\Omega_s = [\theta_{s-1}, \theta_s) \times [0,\omega], \quad s = \overline{1,N}.$$

Let $C(\Omega, \Delta_N(\omega), \mathbb{R}^N)$ be the space of functions systems

$$v([t],x) = (v_1(t,x), v_2(t,x), \ldots, v_N(t,x))',$$

where $v_s : \Omega_s \to \mathbb{R}$ are continuous and have finite left-hand side limits $\lim\limits_{t \to \theta_s - 0} v_s(t,x)$ for all $s = \overline{1,N}$, and $x \in [0,\omega]$ with the norm

$$\|v([\cdot],x)\|_2 = \max_{s=\overline{1,N}} \sup_{t \in [\theta_{s-1}, \widetilde{\theta}_s)} ||v_s(t,x)||.$$

We denote by $v_s(t,x)$ the restriction of a function $v(t,x)$ on the s-th subdomain $\widetilde{\Omega}_s$, i.e.

$$v_s(t,x) = v(t,x) \text{ for } (t,x) \in \Omega_s, \ s = \overline{1,N}.$$

Then the function system $v([t],x) = (v_1(t,x), v_2(t,x), \ldots, v_N(t,x))$ belongs to the space $C(\Omega, \Delta_N(\omega), \mathbb{R}^N)$, and its elements $v_s(t,x)$, $s = \overline{1,N}$, satisfy the following family of differential equations with discrete memory

$$\frac{\partial v_s}{\partial t} = A(t,x)v_s(t,x) + A_0(t,x)v_s(\zeta_s,x) + f(t,x) + B(t,x)w(t,x) + C(t,x)u(t,x), \tag{9}$$

$$(t,x) \in \Omega_s, \qquad s = \overline{1,N},$$

$$v_1(0,x) = \lim_{t \to T-0} v_N(t,x), \qquad x \in [0,\omega], \tag{10}$$

$$v_{p+1}(\theta_p,x) - \lim_{t \to \theta_p - 0} v_p(t,x) = \dot{\varphi}_p(x), \qquad x \in [0,\omega], \qquad p = \overline{1,N-1}, \tag{11}$$

$$u(t,x) = \psi(t) + \int_0^x v_s(t,\xi)d\xi, \;\; w(t,x) = \dot{\psi}(t) + \int_0^x \frac{\partial v_s(t,\xi)}{\partial t}d\xi, \quad (t,x) \in \Omega_s, \;\; s = \overline{1,N}. \quad (12)$$

In (9) we take into account that $\gamma(t) = \zeta_s$ for all $t \in [\theta_{s-1}, \theta_s)$, $s = \overline{1,N}$.

We introduce functional parameters in the next form: $\mu_s(x) = v_s(\zeta_s, x)$ for all $s = \overline{1,N}$ and $x \in [0,\omega]$.

Making the substitution $\widetilde{v}_s(t,x) = v_s(t,x) - \mu_s(x)$, $(t,x) \in \Omega_s$, $s = \overline{1,N}$, we obtain a problem with parameters for the family of differential equations in the following form

$$\frac{\partial \widetilde{v}_s}{\partial t} = A(t,x)\widetilde{v}_s(t,x) + [A(t,x) + A_0(t,x)]\mu_s(x) + f(t,x) + \\ + B(t,x)w(t,x) + C(t,x)u(t,x), \quad (t,x) \in \Omega_s, \;\; s = \overline{1,N}, \quad (13)$$

the initial conditions are

$$\widetilde{v}_s(\zeta_s, x) = 0, \qquad x \in [0,\omega], \qquad s = \overline{1,N}, \quad (14)$$

the periodic condition is

$$\widetilde{v}_1(0,x) + \mu_1(x) = \lim_{t \to T-0} \widetilde{v}_N(t,x) + \mu_N(x), \qquad x \in [0,\omega], \quad (15)$$

the conditions with impulse effects

$$\widetilde{v}_{p+1}(\theta_p, x) + \mu_{p+1}(x) - \lim_{t \to \theta_p - 0} \widetilde{v}_p(t,x) - \mu_p(x) = \dot{\varphi}_p(x), \quad x \in [0,\omega], \quad p = \overline{1,N-1}, \quad (16)$$

and the integral equations

$$u(t,x) = \psi(t) + \int_0^x [\widetilde{v}_s(t,\xi) + \mu_s(\xi)]d\xi, \quad w(t,x) = \dot{\psi}(t) + \int_0^x \frac{\partial \widetilde{v}_s(t,\xi)}{\partial t}d\xi, \\ (t,x) \in \Omega_s, \;\; s = \overline{1,N}. \quad (17)$$

A solution to the problem with parameters (13)–(17) is called a quadruple

$$\{\widetilde{v}([t],x), \mu(x), u(t,x), w(t,x)\},$$

with elements $\{\widetilde{v}_s(t,x), \mu_s(x), u(t,x), w(t,x)\}$, where the functions $\widetilde{v}_s(t,x) \in C(\Omega, \Delta_N(\omega), \mathbb{R})$ have the derivative $\frac{\partial \widetilde{v}_s(t,x)}{\partial t} \in C(\Omega, \Delta_N(\omega), \mathbb{R})$, the functional parameters $\mu_s(x) \in C([0,\omega], \mathbb{R})$, $s = \overline{1,N}$, the functions $u(t,x), w(t,x) \in C(\Omega, \Delta_N(\omega), \mathbb{R})$, and satisfies to the family of the differential equations (13) for all $(t,x) \in \Omega_s$, $s = \overline{1,N}$, the initial conditions (14), the boundary condition (15), the conditions with impulse effects (16) for all $x \in [0,\omega]$. The functions $u(t,x)$

and $w(t,x)$ are connected with $\widetilde{v}_s(t,x)$ and $\frac{\partial \widetilde{v}_s(t,x)}{\partial t}$ by the integral equations (17) for all $(t,x) \in \Omega_s,\ s = \overline{1,N}$.

At fixed $\mu_s(x)$, $w(t,x)$, $u(t,x)$ the problem (13)–(14) is a family of Cauchy problems for differential equations.

Let $\alpha(t,x) = \int\limits_{\zeta_s}^{t} A(\tau,x)d\tau, \quad (t,x) \in \Omega_s, \quad s = \overline{1,N}.$

A solution of the family Cauchy problems (13)–(14) is unique and has the next form

$$v\widetilde{v}_s(t,x) = e^{\alpha(t,x)} \int\limits_{\zeta_s}^{t} e^{-\alpha(\tau,x)}[A(\tau,x) + A_0(\tau,x)]\mu_s(x)d\tau + e^{\alpha(t,x)} \int\limits_{\zeta_s}^{t} e^{-\alpha(\tau,x)} f(\tau,x)d\tau +$$
$$+ e^{\alpha(t,x)} \int\limits_{\zeta_s}^{t} e^{-\alpha(\tau,x)} \Big[B(\tau,x)w(\tau,x) + C(\tau,x)u(\tau,x)\Big] d\tau, \quad (t,x) \in \Omega_s,\ s = \overline{1,N}. \quad (18)$$

We introduce the following notations:

$$D_s(t,x) = e^{\alpha(t,x)} \int\limits_{\zeta_s}^{t} e^{-\alpha(\tau,x)}[A(\tau,x) + A_0(\tau,x)]d\tau,$$
$$H_s(t,x,w,u) = e^{\alpha(t,x)} \int\limits_{\zeta_s}^{t} e^{-\alpha(\tau,x)} \Big[B(\tau,x)w(\tau,x) + C(\tau,x)u(\tau,x)\Big] d\tau,$$
$$F_s(t,x) = e^{\alpha(t,x)} \int\limits_{\zeta_s}^{t} e^{-\alpha(\tau,x)} f(\tau,x)d\tau, \quad (t,x) \in \Omega_r,\ r = \overline{1,N}.$$

From the integral representation (18) we find

$$\widetilde{v}_1(0,x),\ \lim_{t \to T-0} \widetilde{v}_N(t,x),\ \widetilde{v}_{p+1}(\theta_p,x),\ \lim_{t \to \theta_p - 0} \widetilde{v}_p(t,x), \quad p = \overline{1,N-1}.$$

Substituting the found expressions into Relations (15) and (16), we obtain

$$\big[1 + D_1(0,x)\big]\mu_1(x) - \big[1 + D_N(T,x)\big]\mu_N(x) = -F_1(0,x) + F_N(T,x) -$$
$$- H_1(0,x,w,u) + H_N(T,x,w,u), \quad x \in [0,\omega], \quad (19)$$

$$\big[1 + D_{p+1}(\theta_p,x)\big]\mu_{p+1}(x) - \big[1 + D_p(\theta_p,x)\big]\mu_p(x) = \varphi_p(x) + F_p(\theta_p,x) - F_{p+1}(\theta_p,x) +$$
$$+ H_p(\theta_p,x,w,u) - H_{p+1}(\theta_p,x,w,u), \quad p = \overline{1,N-1},\ x \in [0,\omega]. \quad (20)$$

Using the coefficients for $\mu_s(x)$, $s = \overline{1,N}$, on the left-hand sides of the system of the equations (19), (20), we compose an $N \times N$ matrix $Q(x)$ in the following form:

$$Q(x) = \begin{bmatrix} 1+D_1(0,x) & 0 & 0 & . & 0 & -1-D_N(T,x) \\ -1-D_1(\theta_1,x) & 1+D_2(\theta_1,x) & 0 & . & 0 & 0 \\ 0 & -1-D_2(\theta_2,x) & 1+D_3(\theta_2,x) & . & 0 & 0 \\ 0 & 0 & -1-D_3(\theta_3,x) & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & -1-D_{N-1}(\theta_{N-1},x) & 1+D_N(\theta_{N-1},x) \end{bmatrix}$$

Let us write down the system of the equations (19)–(20) in the next form:

$$Q(x)\mu(x) = -F_*(x) - H_*(x,w,u), \qquad x \in [0,\omega], \tag{21}$$

where the N vector functions $F_*(\Delta_N(\omega),x)$, $H_*(\Delta_N(\omega),x,w,u)$ have the forms

$$F_*(x) = \Big(F_1(0,x) - F_N(T,x), -\varphi_1(x) - F_1(\theta_1,x) + F_2(\theta_1,x), \dots,$$
$$- \varphi_{N-1}(x) - F_{N-1}(\theta_{N-1},x) + F_N(\theta_{N-1},x)\Big),$$

$$H_*(x,w,u) = \Big(H_1(0,x,w,u) - H_N(T,x,w,u), -H_1(\theta_1,x,w,u) + H_2(\theta_1,x,w,u), \dots$$
$$- H_{N-1}(\theta_{N-1},x,w,u) + H_N(\theta_{N-1},x,w,u)\Big).$$

3 Algorithm and Main result

If the functions $w(t,x)$ and $u(t,x)$ are known for all $(t,x) \in \Omega_s$, $s = \overline{1,N}$, then from the system of functional equations (21) we find $\mu(x)$ with components $\mu_s(x) \in C([0,\omega],\mathbb{R})$, $s = \overline{1,N}$. Then from the integral representation (18) and the differential equations (13), we define $\widetilde{v}_s(t,x)$ and its derivative $\frac{\partial \widetilde{v}_s}{\partial t}$ for all $(t,x) \in \Omega_s$, $\quad s = \overline{1,N}$.

Conversely, if $\widetilde{v}_s(t,x)$, $\frac{\partial \widetilde{v}_s}{\partial t}$, and $\mu_s(x) \in C([0,\omega],\mathbb{R})$ are known for all $(t,x) \in \Omega_s$, where $s = \overline{1,N}$, then from the integral equations (17) we can find the functions $u(t,x)$, $w(t,x)$ for all $(t,x) \in \Omega_s$, $\; s = \overline{1,N}$.

Since the function $\widetilde{v}_s(t,x)$ and its derivative $\frac{\partial \widetilde{v}_s}{\partial t}$, the functions $u(t,x)$, $w(t,x)$ and the functional parameters $\mu_s(x)$, $s = \overline{1,N}$, are unknown together, we use the iteration method to find a solution to the problem (13)–(17).

We determine a quadruple

$$\{\widetilde{v}^*([t],x), \mu^*(x), \; u^*(t,x), w^*(t,x)\},$$

with elements $\{\widetilde{v}_s^*(t,x),\, \mu_s^*(x),\, u^*(t,x), w^*(t,x)\}$, as a limit of sequence of quadruples

$$\{\widetilde{v}^{(k)}([t],x), \; \mu^{(k)}(x), \; u^{(k)}(t,x), \; w^{(k)}(t,x)\},$$

with elements $\{\widetilde{v}_s^{(k)}(t,x), \mu_s^{(k)}(x), u^{(k)}(t,x), w^{(k)}(t,x)\}$, $s=\overline{1,N}$, $k=0,1,2,...$ by the following *algorithm*:

Step 0. Assume that the $(N\times N)$ matrix $Q(x)$ is invertible for all $x\in[0,\omega]$.

1) Putting $u(t,x)=\psi(t)$, $w(t,x)=\dot{\psi}(t)$ on the right-hand side of system (21), we define initial approximation of functional parameter $\mu^{(0)}(x)=(\mu_1^{(0)}(x),\mu_2^{(0)}(x),...,\mu_N^{(0)}(x))$ with components $\mu_s^{(0)}(x)\in C([0,\omega],\mathbb{R})$ from the system of functional equations

$$Q(x)\mu(x)=-F_*(x)-H_*(x,\dot{\psi},\psi),\quad x\in[0,\omega].$$

2) Assuming on the right-hand side of the family of the differential equations (13) that $u(t,x)=\psi(t)$, $w(t,x)=\dot{\psi}(t)$, $\mu_s(x)=\mu_s^{(0)}(x)$, $s=\overline{1,N}$, and solving the family of Cauchy problems (13)–(14), we find $\widetilde{v}_s^{(0)}(t,x)$

$$\widetilde{v}_s^{(0)}(t,x)=D_s(t,x)\mu_s^{(0)}(x)+F_s(t,x)+H_s(t,x,\dot{\psi},\psi),\quad (t,x)\in\Omega_s,\ s=\overline{1,N},\tag{22}$$

and we determine its derivative

$$\frac{\partial\widetilde{v}_s^{(0)}}{\partial t}=A(t,x)\widetilde{v}_s^{(0)}(t,x)+[A(t,x)+A_0(t,x)]\mu_s^{(0)}(x)+f(t,x)+$$
$$+B(t,x)\dot{\psi}(t)+C(t,x)\psi(t),\quad (t,x)\in\Omega_s,\ s=\overline{1,N}.\tag{23}$$

3) From the integral equations (17) we define $u^{(0)}(t,x)$ and $w^{(0)}(t,x)$ as follows:

$$u^{(0)}(t,x)=\psi(t)+\int_0^x[\widetilde{v}_s^{(0)}(t,\xi)+\mu_s^{(0)}(\xi)]d\xi,\quad w^{(0)}(t,x)=\dot{\psi}(t)+\int_0^x\frac{\partial\widetilde{v}_s^{(0)}(t,\xi)}{\partial t}d\xi,$$
$$(t,x)\in\Omega_s,\ s=\overline{1,N}.\tag{24}$$

Step 1. 1) Putting $u(t,x)=u^{(0)}(t,x)$ and $w(t,x)=w^{(0)}(t,x)$ on the right-hand side of the system (21), we define the first approximation of the functional parameter $\mu^{(1)}(x)=(\mu_1^{(1)}(x),$ $\mu_2^{(1)}(x),\ldots,\mu_N^{(1)}(x))$ with components $\mu_s^{(1)}(x)\in C([0,\omega],\mathbb{R})$ from the system of the functional equations

$$Q(x)\mu(x)=-F_*(x)-H_*(x,w^{(0)},u^{(0)}),\qquad x\in[0,\omega].$$

2) Assuming on the right-hand side of the family of the differential equations (13)

$$u(t,x)=u^{(0)}(t,x),\ w(t,x)=w^{(0)}(t,x),\ \mu_s(x)=\mu_s^{(1)}(x),\ s=\overline{1,N},$$

and solving the family of Cauchy problems (13)–(14), we find $\widetilde{v}_s^{(1)}(t,x)$:

$$\widetilde{v}_s^{(1)}(t,x)=D_s(t,x)\mu_s^{(1)}(x)+F_s(t,x)+H_s(t,x,w^{(0)},u^{(0)}),\quad (t,x)\in\Omega_s,\ s=\overline{1,N},\tag{25}$$

and we determine its derivative

$$\frac{\partial \widetilde{v}_s^{(1)}}{\partial t} = A(t,x)\widetilde{v}_s^{(1)}(t,x) + [A(t,x) + A_0(t,x)]\mu_s^{(1)}(x) + f(t,x) + \\ + B(t,x)w^{(0)}(t,x) + C(t,x)u^{(0)}(t,x), \quad (t,x) \in \Omega_s, \ s = \overline{1,N}. \quad (26)$$

3) From the integral equations (17) we define $u^{(1)}(t,x)$ and $w^{(1)}(t,x)$ as follows:

$$u^{(1)}(t,x) = \psi(t) + \int_0^x [\widetilde{v}_s^{(1)}(t,\xi) + \mu_s^{(1)}(\xi)]d\xi, \quad w^{(1)}(t,x) = \dot{\psi}(t) + \int_0^x \frac{\partial \widetilde{v}_s^{(1)}(t,\xi)}{\partial t} d\xi, \\ (t,x) \in \Omega_s, \ s = \overline{1,N}. \quad (27)$$

And so on.

Step k. 1) Putting $u(t,x) = u^{(k-1)}(t,x)$ and $w(t,x) = w^{(k-1)}(t,x)$ on the right-hand side of the system (21), we define the kth approximation of the functional parameter $\mu^{(k)}(x) = (\mu_1^{(k)}(x), \mu_2^{(k)}(x), ..., \mu_N^{(k)}(x))$ with the components $\mu_s^{(k)}(x) \in C([0,\omega],\mathbb{R})$ from the system of the functional equations

$$Q(x)\mu(x) = -F_*(x) - H_*(x, w^{(k-1)}, u^{(k-1)}), \quad x \in [0,\omega].$$

2) Assuming on the right-hand side of the family of the differential equations (13) that

$$u(t,x) = u^{(k-1)}(t,x), \ w(t,x) = w^{(k-1)}(t,x), \ \mu_s(x) = \mu_s^{(k)}(x), \ s = \overline{1,N},$$

and solving the family of Cauchy problems (13)–(14), we find $\widetilde{v}_s^{(1)}(t,x)$

$$\widetilde{v}_s^{(k)}(t,x) = D_s(t,x)\mu_s^{(k)}(x) + F_s(t,x) + H_s(t,x,w^{(k-1)},u^{(k-1)}), \quad (t,x) \in \Omega_s, \ s = \overline{1,N}, \quad (28)$$

and determine its derivative

$$\frac{\partial \widetilde{v}_s^{(k)}}{\partial t} = A(t,x)\widetilde{v}_s^{(k)}(t,x) + [A(t,x) + A_0(t,x)]\mu_s^{(k)}(x) + f(t,x) + \\ + B(t,x)w^{(k-1)}(t,x) + C(t,x)u^{(k-1)}(t,x), \quad (t,x) \in \Omega_s, \ s = \overline{1,N}. \quad (29)$$

3) From the integral equations (17) we define $u^{(k)}(t,x)$ and $w^{(k)}(t,x)$ as follows:

$$u^{(k)}(t,x) = \psi(t) + \int_0^x [\widetilde{v}_s^{(k)}(t,\xi) + \mu_s^{(k)}(\xi)]d\xi, \quad w^{(k)}(t,x) = \dot{\psi}(t) + \int_0^x \frac{\partial \widetilde{v}_s^{(k)}(t,\xi)}{\partial t} d\xi, \\ (t,x) \in \Omega_s, \ s = \overline{1,N}. \quad (30)$$

Here $k = 1, 2, \ldots$

The developed method allows us to find unknown functions in three stages:

1) From the system of the functional equations (21) we determine the introduced functional parameters $\mu_s(x)$ for all $x \in [0, \omega]$, $s = \overline{1, N}$.

2) From the family of Cauchy problems (13), (14) we find unknown functions $\widetilde{v}_s(t, x)$ and its derivative $\frac{\partial \widetilde{v}_s(t,x)}{\partial t}$ for all $(t, x) \in \Omega_s$, $s = \overline{1, N}$.

3) From the integral equations (17) we define $u(t, x)$ and $w(t, x)$ for all $(t, x) \in \Omega_s$, $s = \overline{1, N}$.

Each of the problems has a unique solution under assumptions about the initial data. To implement the algorithm, it is necessary to establish the convergence of approximate solutions to the exact solution of the problem with parameters (13)–(17).

We use the following notations:

$$\begin{aligned}
\alpha(x) &= \max_{t \in [0,T]} ||A(t, x)||, \\
\alpha_0(x) &= \max_{t \in [0,T]} ||A_0(t, x)||, \\
\theta &= \max\Big\{ \max_{r=\overline{1,N}} (\theta_r - \zeta_{r-1}), \max_{r=\overline{1,N}} (\zeta_{r-1} - \theta_{r-1}) \Big\}.
\end{aligned}$$

The following theorem establishes conditions for the convergence of the proposed algorithm and the existence of a unique solution to the problem with parameters (13)–(17).

Theorem 1. *Assume that the $N \times N$ matrix $Q(x)$ is invertible for all $x \in [0, \omega]$. Then the problem with parameters (13)–(17) has a unique solution.*

From the equivalent problems (1)–(4) and (13)–(17) we have the following.

Theorem 2. *Assume that the $N \times N$ matrix $Q(x)$ is invertible for all $x \in [0, \omega]$. Then the periodic problem for the impulsive hyperbolic equation with discrete memory (1)–(4) has a unique solution.*

References

[1] Wiener J. *Generalized Solutions of Functional Differential Equations*, World Scientific, 1993.

[2] Samoilenko A.M., Perestyuk N.A. *Impulsive Differential Equations*, World Scientific, 1995.

[3] Akhmet M.U. *Integral manifolds of differential equations with piecewise constant argument of generalized type*, Nonl. Anal. 66:3 (2007), 367–383. http:dx.doi.org/10.1016/j.na.2005.11.032

[4] Akhmet M.U. *Almost periodic solutions of differential equations with piecewise constant argument of generalized type*, J. Math. Anal. Appl. 336:4 (2007), 646–663. http:dx.doi.org/10.1016/j.jmaa.2007.03.010

[5] Akhmet M.U. *Principles of discontinuous dynamical systems*, Springer, 2010.

[6] Akhmet M.U. *Nonlinear hybrid continuous/discrete-time models*, Atlantis Press, 2011.

[7] Akhmet M.U., Yilmaz E. *Neural Networks with Discontinuous/Impact Activations*, Springer, 2013.

[8] Nieto J.J., Rodriguez-Lopez R. *Green's function for second order periodic BVPs with piecewise constant argument*, J. Math. Anal. Appl. 304:1 (2005), 33–57. http:dx.doi.org/10.1016/j.jmaa.2004.09.023

[9] Assanova A.T. *Hyperbolic equation with piecewise-constant argument of generalized type and solving boundary value problems for it*, Lobachevskii J. Math. 42:15 (2021), 3584–3593. http:dx.doi.org/10.1134/S1995080222030040

[10] Assanova A.T., Uteshova R.E. *Solution of a nonlocal problem for hyperbolic equations with piecewise constant argument of generalized type*, Chaos Solitons & Fractals. 165-2 :12 (2022), art. no. 112816. http:dx.doi.org/10.1016/j.chaos.2022.112816

[11] Abildayeva A., Assanova A.T., Imanchiyev A. *A multi-point problem for a system of differential equations with piecewise-constant argument of generalized type as a neural network model*, Eurasian Math. J. 13:2 (2022), 8–17. http:dx.doi.org/10.32523/2077-9879-2022-13-2-08-17

[12] Assanova A.T., Kadirbayeva Z.M. *Periodic problem for an impulsive system of the loaded hyperbolic equations*, Electr. J. Differ. Equ. 2018 (2018), art. no. 72.

[13] Imanchiyev A.E., Assanova A.T., Molybaikyzy A. *Properties of a nonlocal problem for hyperbolic equations with impulse discrete memory*, Lobachevskii J. Math. 44:10 (2023), 4299–4309. http:dx.doi.org/10.1134/S1995080223100177

[14] Dzhumabayev D.S. *Criteria for the unique solvability of a linear boundary-value problem for an ordinary differential equation*, U.S.S.R. Comp. Math. Math. Phys. 29:1 (1989), 34–46.

Асанова А. Т., Молыбайқызы А. ДИСКРЕТ ЖАДЫЛЫ ИМПУЛЬСТІК ГИПЕРБОЛАЛЫҚ ТЕҢДЕУ ҮШІН ПЕРИОДТЫ ЕСЕПТІҢ ШЕШІМІ

Мақалада дискрет жадылы импульстік гиперболалық теңдеу үшін периодты есеп қарастырылады. Дискрет жадылы импульстік гиперболалық теңдеулер нейрондық желілердегі, үзілісті динамикалық жүйелердегі, гибрид жүйелердегі және т.т. физикалық үдерістерді сипаттауға арналған математикалық моделдер ретінде туындайды. Дискрет жадылы импульстік гиперболалық теңдеулер үшін периодты есептердің шешімдерінің бар болуы мен құру мәселелері үзілісті дербес туындылы дифференциалдық теңдеулер теориясының маңызды мәселесі болып қалып отыр. Осы есептің шешілімділік шарттарын табу үшін Джумабаевтың параметрлеу әдісі пайдаланылады. Дискрет жадылы импульстік гиперболалық теңдеу үшін периодты есептің шешімінің бар болуы мен жалғыздығының коэффициенттік шарттары орнатылған. Осы есептің жуық шешімін анықтау алгоритмі ұсынылған және дискрет жадылы импульстік гиперболалық теңдеу үшін периодты есептің дәл шешіміне жинақтылығы көрсетілген.

Түйін сөздер: гиперболалық теңдеу, импульс әсерлері, периодты шарт, дискрет жады, облысты бөліктеу, параметрлері бар есеп, шешілімділік шарттары.

Асанова А. Т., Молыбайкызы А. РЕШЕНИЕ ПЕРИОДИЧЕСКОЙ ЗАДАЧИ ДЛЯ ИМПУЛЬСНОГО ГИПЕРБОЛИЧЕСКОГО УРАВНЕНИЯ С ДИСКРЕТНОЙ ПАМЯТЬЮ

В статье рассматривается периодическая задача для импульсного гиперболического уравнения с дискретной памятью. Импульсные гиперболические уравнения с дискретной памятью возникают как математические модели, описывающие физические процессы в нейронных сетях, в разрывных динамических системах, в гибридных системах и т.д. Вопросы существования и построения решения периодических задач для импульсных гиперболических уравнений с дискретной памятью остаются важными проблемами теории разрывных дифференциальных уравнений в частных производных. Для нахождения условий разрешимости этой задачи используется метод параметризации Джумабаева. Установлены коэффициентные условия существования и единственности решения периодической для импульсного гиперболического уравнения с дискретной памятью. Предложен алгоритм для определения приближенного решения данной задачи и показана сходимость к точному решению периодической задачи для импульсного гиперболического уравнения с дискретной памятью.

Ключевые слова: гиперболическое уравнение, импульсные воздействия, периодическое условие, дскретная память, деление области, задача с параметрами, условия разрешимости.