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On the correctness of non-local boundary value problems with an integral condition for fourth-order quasi-hyperbolic equations

Bakytbek D. Koshanov[1], Gulzhazira D. Smatova[2], Zhanat B. Sultangazieva[3]

[1]Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
[2]Satbaev University, Almaty, Kazakhstan
[3]Al-Farabi Kazakh National University, Almaty, Kazakhstan

[1]koshanov@math.kz, [2]smatova1977@mail.ru, [3]zhanat_87@mail.ru

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Abstract. This paper presents results on the correctness (uniqueness and non-uniqueness, existence and non-existence of solutions) of non-local boundary value problems with partially integral conditions with respect to the time variable for fourth-order quasi-hyperbolic differential equations. The influence of the spectral parameter and the behavior of the integrand on the solvability of the non-local problem are also determined.

Keywords. Quasi-hyperbolic equations, non-local problems, integral conditions, regular solutions, uniqueness and non-uniqueness of a solution, existence and non-existence of a solution.

1 Introduction and formulation of the problem

Let Ω be the bounded area of the space $\mathbb{R}^n$ variables $x_1, x_2, \ldots, x_n$ with smooth compact boundary $\Gamma = \partial\Omega$. We consider the following differential operator in the cylindrical area $Q = \Omega \times (0,T)$, $S = \Gamma \times (0,T)$, where $0 < T < +\infty$,

$$Lu \equiv \frac{\partial^4 u}{\partial t^4} + \Delta u + c(x,t)u - \lambda u = f(x,t), \quad x \in \Omega, \quad t \in (0,T), \tag{1}$$

where $c(x,t)$ and $f(x,t)$ are given functions.

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Boundary value problem I: It is required to find a function $u(x,t)$ which is a solution to Equation (1) in the cylinder Q that satisfies the following conditions

$$u(x,t)\Big|_{S} = 0, \tag{2}$$

$$u(x,0) = \frac{\partial u}{\partial t}(x,0) = \frac{\partial^2 u}{\partial t^2}(x,0) = 0,\ x \in \Omega, \tag{3}$$

$$\int\limits_0^T N(t)u(x,t)dt = 0,\ x \in \Omega. \tag{4}$$

Boundary value problem II: It is required to find a function $u(x,t)$ which is a solution to Equation (1) in the cylinder Q that satisfies Conditions (2), (4) and

$$\frac{\partial u}{\partial t}(x,0) = \frac{\partial^2 u}{\partial t^2}(x,0) = \frac{\partial^3 u}{\partial t^3}(x,0) = 0,\ x \in \Omega. \tag{5}$$

The study of the solvability of boundary value problems for quasi-hyperbolic equations began, apparently, with the works by V.N. Vragov [1], [2]. Further results can be found in [3–5]. One of the main conditions for correctness in these studies was the condition that the parameter λ is non-negative. Investigations of non-local problems with integral conditions for linear parabolic equations, for differential equations of odd order, and for some classes of non-stationary equations have been actively carried out recently in the works by A.I. Kozhanov [4, 6, 7, 8]. In [5], the solvability of Problem (2), (3), (5) for fourth order quasi-hyperbolic equations with $p = 2$ is investigated. In [9], the Fredholm property and index of the generalized Neumann problem containing powers of normal derivatives in the boundary conditions are investigated. For these problems, sufficient conditions for the Fredholm solvability of the problem are obtained and formulas for the index of this problem are given. The papers [10] and [11] are devoted to investigations of the solvability of various boundary value problems of order $0 \le k_1 < k_2 < ... < k_l \le 2l-1$ for the polyharmonic equation in a multidimensional ball. In [12], a criterion for the uniqueness of solution for some differential-operator equations was obtained.

2 Supporting statement. Solvability of the non-local problem I

First, we discuss the effect of the parameter λ on the solvability of the non-local problem I in the case where $c(x,t) \equiv 0$. Let $\{w_k(x)\}_{k=1}^{\infty}$ and $\{\mu_k\}_{k=1}^{\infty}$ be the sequences of the eigenfunctions and their corresponding eigenvalues for the problem

$$\Delta w(x) = \mu w(x),\ x \in \Omega, \quad w(x)|_{\Gamma} = 0,$$

where the sequence $\{w_k(x)\}_{k=1}^{\infty}$ is orthonormal in the space $L_2(\Omega)$. It is known that the functions $w_k(x)$ belong to the space $W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$, form a basis in the space $L_2(\Omega)$, and that the values μ_k are all negative. They can be arranged in a monotonically decreasing sequence (which we shall assume has been done), and the sequence $\{\mu_k\}_{k=1}^{\infty}$ has a unique limit point at $-\infty$.

3 Main results

First, we consider the case $\lambda > \mu_1$.

We define the values $\gamma_k(\lambda)$ as positive numbers that satisfy $\gamma_1^4(\lambda) = \lambda - \mu_k$. Next, we define the function $\varphi_1(z)$ for $z \geq 0$ as follows:

$$\varphi_1(z) = e^z - e^{-z} - 2\sin z.$$

Theorem 1. *Let the function $N(t)$ be continuous on the interval $[0, T]$, and for a fixed value of λ in the interval $(\mu_1, +\infty)$, let the following condition hold:*

$$\int_0^T N(t)[e^{\gamma_k(\lambda)t} - e^{-\gamma_k(\lambda)t} - 2\sin(\gamma_k(\lambda)t)]dt \neq 0, \quad k = 1, 2, \ldots. \tag{6}$$

Then, in the case where $c(x,t) \equiv 0$, the non-local problem I cannot have more than one solution in the space $W_2^{2,4}(Q)$.

Proof. We assume that in the non-local problem I with identically zero function $c(x,t)$, we also have $f(x,t) \equiv 0$. The solution $u(x,t)$ of such a problem in the space $W_2^{2,4}(Q)$ can be represented as a Fourier series:

$$u(x,t) = \sum_{k=1}^{\infty} c_k(t) w_k(x),$$

where the functions $c_k(t)$ must satisfy the problem

$$c_k''''(t) - \gamma_k^4(\lambda)\, c_k(t) = 0, \tag{7}$$

$$c_k(0) = c_k'(0) = c_k''(0) = 0, \quad \int_0^T N(t) c_k(t) dt = 0. \tag{8}$$

In this problem, the differential equation for the functions $c_k(t)$ and the first three conditions give the equalities

$$c_k(t) = C_{1,k}\, \varphi_1(\gamma_k(\lambda)t), \quad k = 1, 2, \ldots.$$

Using the integral condition, we find that if Condition (6) holds and $\lambda > \mu_1$, then each constant $C_{1,k}$ must be equal to 0. This implies that in the case $f(x,t) \equiv 0$ and $c(x,t) \equiv 0$, the solution $u(x,t)$ of the non-local problem I in the space $W_2^{2,4}(Q)$ can only be the identically zero function. This means the uniqueness of a solution of the non-local problem I. □

Corollary 2. *Let $N(t)$ be a continuous non-negative function on the interval $[0,T]$, which is not identically zero. Then for $\lambda \in (\mu_1, +\infty)$, the non-local problem I in the case $c(x,t) \equiv 0$ cannot have more than one solution in the space $W_2^{2,4}(Q)$.*

Proof. For the function $\varphi_1(z)$, we have

$$\varphi_1(0) = 0, \ \varphi_1'(z) = e^z + e^{-z} - 2\cos z > 0 \quad \text{for } z > 0.$$

Therefore, the function $\varphi_1(z)$ is strictly positive for $z > 0$. Since $N(t)$ is continuous and non-negative and there exist points where it is positive, Condition (6) is satisfied due to the positivity of the function $\varphi_1(\gamma_k(\lambda)t)$ (for $t > 0$). This also establishes the uniqueness of solutions for the non-local problem I in the case where $c(x,t) \equiv 0$. □

The proof of the next corollary is straightforward.

Corollary 3. *Let the function $N(t)$ be continuous on the interval $[0,T]$, and for a fixed value of λ in the interval $(\mu_1, +\infty)$, let there exist a set of natural numbers $k_1, \ldots, k_m$ such that*

$$\int_0^T N(t)\varphi_1(\gamma_k(\lambda)t)dt \neq 0.$$

Then, in the case where $f(x,t) \equiv 0$ and $c(x,t) \equiv 0$, the non-local problem I has linearly independent solutions in the space $W_2^{2,4}(Q)$ on the interval $[0,T]$.

Now, we consider the case $\lambda \in (-\infty, \mu_1]$. We define the values $\delta_k(\lambda)$ for $k = 1, 2, \ldots$ and the function $\psi_1(z)$ as follows:

$$\delta_k(\lambda) = |\mu_k - \lambda|^{1/4}, \quad \delta_k > 0,$$

$$\psi_1(z) = (e^z - e^{-z})\cos z - (e^z + e^{-z})\sin z.$$

Furthermore, for values of λ in the interval $(-\infty, \mu_1]$, let $k_0(\lambda)$ be a natural number such that $\mu_{k_0(\lambda)+1} < \lambda \leq \mu_{k_0(\lambda)}$.

Theorem 4. *Let the function $N(t)$ be continuous on the interval $[0,T]$, and let the following conditions hold for a fixed value of λ in the interval $(-\infty, \mu_k)$:*

$$\lambda \neq \mu_k, \quad k = 1, 2, \ldots, \tag{9}$$

$$\int_0^T N(t)\,\psi_1(\delta_k(\lambda)\,t)dt \neq 0, \quad k = 1, \dots, k_0(\lambda), \tag{10}$$

$$\int_0^T N(t)\,\varphi_1(\gamma_k(\lambda)\,t)dt \neq 0, \quad k = k_0(\lambda)+1, k_0(\lambda)+2, \dots, \tag{11}$$

or the conditions

$$\lambda = \mu_{k_0}, \tag{12}$$

$$\int_0^T N(t)\,\psi_1(\delta_k(\lambda)\,t)dt \neq 0, \quad k = 1, \dots, k_0 - 1, \tag{13}$$

$$\int_0^T t^3\,N(t)dt \neq 0, \tag{14}$$

as well as Condition (11). Then, in the case where $c(x,t) \equiv 0$, the non-local problem I cannot have more than one solution in the space $W_2^{2,4}(Q)$.

Proof. Assume that Condition (9) holds. For a solution $u(x,t)$ of the non-local problem I in the case where $c(x,t) \equiv 0$ and $f(x,t) \equiv 0$, we apply the Fourier series representation:

$$u(x,t) = \sum_{k=1}^{\infty} c_k(t) w_k(x),$$

with $c_k(t)$ that either solve the boundary problem (7), (8) for $k = k_0(\lambda)+1$, $k_0(\lambda)+2$, $\dots$, or solve the problem

$$c_k''''(t) - \delta_k^4(\lambda)\,c_k(t) = 0, \tag{15}$$

$$c_k(0) = c_k'(0) = c_k''(0) = 0, \quad \int_0^T N(t) c_k(t)dt = 0 \tag{16}$$

for $k = 1, \dots, k_0(\lambda)$. Using Conditions (10) and (11), it is straightforward to show that all functions $c_k(t)$ must be identically zero on the interval $[0,T]$. This implies the uniqueness of a solution.

If Condition (12) now holds, then for $k = k_0(\lambda)+1$, $k_0(\lambda)+2$, $\dots$, we determine the functions $c_k(t)$ as solutions of Problem (7), (8); for $k = 1, \dots, k_0 - 1$, as solutions of Problem (15), (16); and for $k = k_0$, as a solution of the problem

$$c_k''''(t) = 0,$$

$$c_k(0) = c_k'(0) = c_k''(0) = 0, \quad \int\limits_0^T N(t)\, c_k(t) dt = 0.$$

Using the representations of the functions $c_k(t)$ and considering Conditions (11), (13), and (14), it is once again straightforward to conclude that the functions $c_k(t)$, and hence the function $u(x,t)$, are identically zero. □

Remark 5. *In the case where $k_0(\lambda) = 1$, Condition (13) is assumed to be absent. The proven theorems allow us to draw the following conclusions:*

1. *For any real number λ, there exists an infinite number of continuous functions $N(t)$ on the interval $[0,T]$ such that the given number λ will be an eigenvalue of the non-local problem I in the case $c(x,t) \equiv 0$ with any preassigned multiplicity.*
2. *Theorems 1 and 4 provide conditions under which the given real number λ is not an eigenvalue of the non-local problem I in the case $c(x,t) \equiv 0$.*

We now proceed to the discussion of the existence of solutions to the non-local problem I. We again consider initially the case $c(x,t) \equiv 0$.

Theorem 6. *Let the following conditions hold:*

$$c(x,t) \equiv 0 \quad \textit{for} \quad (x,t) \in \bar{Q}, \quad \lambda \in (\mu_1, +\infty), \tag{17}$$

$$N(t) \in C([0,T]),\ N(t) \geq N_0 t^m,\ N_0 > 0,\ m \geq 0,\ t \in [0,T]. \tag{18}$$

Then, for any function $f(x,t)$ in the space $L_2(0,T; W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega))$, the non-local problem I has a solution $u(x,t)$ belonging to the space $W_2^{2,4}(Q)$, and this solution is unique.

Proof. We consider the boundary value problem: find a function $\nu(x,t)$ that, within the cylinder Q, is a solution of the equation

$$\nu_{tttt} + \Delta \nu = \lambda \nu + f(x,t), \quad (x,t) \in Q \tag{19}$$

and satisfies the conditions

$$\nu(x,0) = \nu_t(x,0) = \nu_{tt}(x,0) = \nu_t(x,T) = 0, \quad x \in \Omega. \tag{20}$$

Using the techniques from [5], it is straightforward to show that if the condition $\lambda > \mu_1$ holds and the function $f(x,t)$ belongs to the space $L_2(0,T; W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega))$, then this problem has a solution $\nu(x,t)$ such that $\nu(x,t) \in W_2^{2,4}(Q)$, $\Delta\nu(x,t) \in L_\infty(0,T; L_2(\Omega))$. We define the function $\beta(x)$ as follows:

$$\beta(x) = -\int\limits_0^T N(t)\, \nu(x,t) dt.$$

Now we consider an additional auxiliary boundary value problem: find a function $w(x,t)$ that, within the cylinder Q, is a solution of the equation

$$w_{tttt} + \Delta w = \lambda w, \quad (x,t) \in Q, \tag{21}$$

and satisfies the conditions

$$w(x,0) = w_t(x,0) = w_{tt}(x,0) = 0, \quad \int\limits_0^T N(t) w(x,t) dt = \beta(x), \quad x \in \Omega. \tag{22}$$

We show, using the Fourier method, that under Conditions (17) and (18), this problem has a solution that belongs to the space $W_2^{2,4}(Q)$. For the function $\beta(x)$, there is a Fourier series representation:

$$\beta(x) = \sum_{k=1}^{\infty} \beta_k w_k(x), \quad \beta_k = \int\limits_0^T N(t) \left(\int\limits_\Omega \nu(x,t) w_k(x) dx \right) dt.$$

We will also look for the function $w(x,t)$ in the form of a Fourier series:

$$w(x,t) = \sum_{k=1}^{\infty} d_k(t) w_k(x),$$

the functions $d_k(t)$ must be solutions to the problem

$$d_k''''(t) - \gamma_k^4(\lambda) d_k(t) = 0, \quad t \in (0,T),$$

$$d_k(0) = d_k'(0) = d''(0) = 0, \quad \int\limits_0^T N(t) d_k(t) dt = \beta_k.$$

The following equality holds:

$$d_k(t) = \frac{\beta_k}{\alpha_k} \left[e^{\gamma_k(\lambda)t} - e^{-\gamma_k(\lambda)t} - 2\sin(\gamma_k(\lambda)t) \right].$$

Here, the numbers α_k are determined by the equalities

$$\alpha_k = \int\limits_0^T N(t) \left[e^{\gamma_k(\lambda)t} - e^{-\gamma_k(\lambda)t} - 2\sin(\gamma_k(\lambda)t) \right] dt, \quad k = 1, \ldots$$

Due to Condition (18), all numbers α_k are positive. We show that there exists a natural number k_0 such that $k > k_0$ and the following holds:

$$\alpha_k \geq \frac{N_1 e^{\gamma_k(\lambda)T}}{\gamma_k(\lambda)}. \tag{23}$$

Indeed, for the numbers α_k, the following inequality holds:

$$\alpha_k \geq N_0 \int\limits_0^T t^m \varphi_1(\gamma_k(\lambda)t)dt.$$

Next, we have

$$\varphi_1(\gamma_k(\lambda)t) = e^{\gamma_k(\lambda)t}\left[1 - e^{-2\gamma_k(\lambda)t} - 2e^{-\gamma_k(\lambda)t}\sin(\gamma_k(\lambda)t)\right].$$

Since the sequence $\{\gamma_k(\lambda)\}_{k=1}^{\infty}$ monotonically approaches $+\infty$, there exists a natural number k_1 such that

$$\varphi_1(\gamma_k(\lambda)t) \geq \frac{1}{2}e^{\gamma_k(\lambda)t} \quad \text{for} \quad k > k_1.$$

Taking these inequalities into account, we find that for the numbers α_k when $k > k_1$, the following inequality holds:

$$\alpha_k \geq \frac{1}{2}N_1 \int\limits_0^T t^m e^{\gamma_k(\lambda)t}dt. \tag{24}$$

For the integral on the right-hand side, we have the equality:

$$\int\limits_0^T t^m e^{\gamma_k(\lambda)t}dt \geq \frac{T^{m-1}e^{\gamma_k(\lambda)T}}{\gamma_k}\left[T - \frac{m}{\gamma_k}\right]. \tag{25}$$

Again, due to the monotonic increase of the sequence $\{\gamma_k(\lambda)\}_{k=1}^{\infty}$, there exists a natural number k_2 such that for $k > k_2$ the following inequality holds:

$$T - \frac{m}{\gamma_k} \geq \frac{T}{2}.$$

From the inequalities (23) and (24), it follows that for $k > k_0 = \max(k_1, k_2)$. Thus, the required estimate (25) holds. □

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Қошанов Б.Д., Сматова Г.Д., Сұлтанғазиева Ж.Б. ТӨРТІНШІ РЕТТІ КВАЗИГИПЕРБОЛАЛЫҚ ТЕҢДЕУЛЕР ҮШІН ИНТЕГРАЛДЫҚ ШАРТЫ БАР ЛОКАЛЬДЫ ЕМЕС ШЕКАРАЛЫҚ ЕСЕПТЕРДІҢ ДҰРЫСТЫҒЫ ТУРАЛЫ

Бұл жұмыста төртінші ретті квазигиперболалық дифференциалдық теңдеулер үшін уақыт айнымалысына қатысты ішінара интегралдық шарттары бар локальды емес шеттік есептердің дұрыстығы (шешімнің жалғыздығы мен жалғыз еместігі, шешімнің бар болуы және жоқтығы) бойынша нәтижелер көрсетілген. Сондай-ақ локальды емес есептің шешімділігінің спектрлік параметрден және интегралдың ішіндегі функциядан әсері анықталған.

Түйін сөздер: квазигиперболалық теңдеулер, локальды емес есептер, интегралдық шарттар, регулярлы шешімдер, шешімнің жалғыздығы мен жалғыз еместігі, шешімнің бар болуы және жоқтығы.

Кошанов Б.Д., Сматова Г.Д., Султангазиева Ж.Б. О КОРРЕКТНОСТИ НЕЛОКАЛЬНЫХ КРАЕВЫХ ЗАДАЧ С ИНТЕГРАЛЬНЫМ УСЛОВИЕМ ДЛЯ КВАЗИГИПЕРБОЛИЧЕСКИХ УРАВНЕНИЙ ЧЕТВЕРТОГО ПОРЯДКА

В данной работе представлены результаты о корректности (единственности и неединственности, существовании и несуществовании решений) нелокальных краевых задач с частично интегральными по временной переменной условиями для квазигиперболических дифференциальных уравнений четвертого порядка. А также определены влиянии спектрального параметра и поведении подынтегральной функции на разрешимость нелокальной задачи.

Ключевые слова: квазигиперболические уравнения, нелокальные задачи, интегральные условия, регулярные решения, единственность и неединственность решения, существование и несуществование решения.