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Evaluation of solutions of one class of finite-dimensional nonlinear equations. I

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Abstract. In this article, we obtain a theorem on a priori estimates for solutions of nonlinear equations in a finite-dimensional space. This theorem is proved under certain conditions which are borrowed from the conditions that are satisfied by finite-dimensional approximations of one class of nonlinear initial-boundary-value problems. The main result establishes sufficient conditions for the existence of a solution to $A(u) = f$, where A is a nonlinear operator. An example is given to illustrate the applicability of the main result to nonlinear analysis and mathematical physics.

Keywords. Differential operator, nonlinear equation, existence of a solution, uniqueness of a solution, a priori estimation of a solution.

1 Introduction and the origin of the problem

Many problems of mathematical physics, thanks to the law of energy conservation, allow us to prove the existence of a solution that satisfies an energy estimate. The energy estimate in the case where the number of spatial variables n is not less than 3, usually does not allow us to use perturbation theory.

Solutions that do not allow (or rather cannot allow) us to use perturbation theory are called (usually) "weak" solutions.

The ability to use perturbation theory is very important in mathematical physics problems. Therefore, in the theory of differential equations, there is great interest in the existence of a solution that allows us to use perturbation theory.

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A solution to an equation that allows us to use perturbation theory is mathematically called a "strong" solution (not always).

Many problems of mathematical physics can be written in "restricted notation" (in the form of an integral equation), usually of the following form

$$f(u) = u + L(u) = g, \tag{1}$$

where $L(u)$ is the nonlinear part. This equation is often studied in the metric of some Banach or Hilbert space H.

When moving to an "abbreviated notation" the energy estimate, usually performed for problems in mathematical physics, will turn into an a priori estimate of the following form

$$\|G(u)\| \leq C \cdot \|u + L(u)\| = C\|g\|, \tag{2}$$

where C is a constant number independent of $u \in H$, and G is a completely continuous operator in H.

An a priori estimate (2) usually does not allow the use of perturbation theory. Therefore, it becomes necessary to obtain an estimate of the following form

$$\|u\| \leq \varphi\,(\|f(u)\|), \tag{3}$$

where $\varphi(\cdot)$ is a continuous function on $[0, \infty)$.

The presence of an estimate of the form (3), as a rule, opens the possibility of using perturbation theory (with an appropriate choice of the space H).

A very important problem is the problem of the existence of a sequence of finite-dimensional approximations of the problem (1) (more precisely, approximations of the operation $u+L(u)$):

$$f_1(\cdot),\ f_2(\cdot),\ \dots\ ,\ f_n(\cdot),\ \dots \tag{4}$$

considered in the spaces

$$H_1,\ H_2,\ \dots\ ,\ H_n,\ \dots\ , \quad \dim H_n = n, \tag{5}$$

such that a priori estimates of the form (2) are satisfied and it is possible to obtain an estimate similar to (3).

It is implied that $H_n\,(n = 1, 2, \dots)$ is a subspace of H and the metric H_n is the metric induced from the metric of H.

The problem of describing the dynamics of an incompressible fluid, due to its theoretical and applied importance, attracts the attention of many researchers.

This work is devoted to the problem of the existence and smoothness of solutions to equations of mathematical physics [1].

The works [2–4] provide a fairly complete analysis of the current state of the problem and a review of the available literature, and propose methods to solve the problem. The articles [5–13] are devoted to the study of the solvability in general of equations of mathematical physics, the continuous dependence of the solution of a parabolic equation, and the smoothness of the solution.

This work arose as a result of numerous attempts by the authors to solve the problem of the existence of a strong solution to an equation of mathematical physics.

In this work, we obtain two theorems on a priori estimates of solutions to nonlinear equations in a finite-dimensional Hilbert space. The work consists of four sections. The first section is devoted to the introduction and origin of the problem. The second section provides the notation used and the formulation of the main results. The third section provides a proof of Theorem 1, which in the limit gives weak solvability of many problems of mathematical physics. In the fourth paragraph, we prove Theorem 2, which in the limit allows us to establish strong solvability of some problems of mathematical physics that admit perturbation theory. The conditions of the theorems are such that they can be used in studying a certain class of initial-boundary value problems to obtain strong a priori estimates in the presence of weak a priori estimates.

2 The conditions used and the formulation of the results

Let us derive uniform estimates for nonlinear problems in a finite-dimensional space. The equations under consideration are (usually) analogs of finite-dimensional approximations of equations of mathematical physics written in "abbreviated notation".

Throughout this section, H is a finite-dimensional real Hilbert space with a scalar product $\langle \cdot, \cdot \rangle$ and a norm $\|\cdot\|$.

We are interested in an equation of the following form

$$u + L(u) = g \in H, \tag{6}$$

where $L(\cdot)$ is a nonlinear continuous transformation, g is an element of the space H. The solution u of problem (6) is sought in H.

We are focused on such finite-dimensional equations of the form (6) that are finite-dimensional approximations of infinite-dimensional problems of the form (6) in an infinite-dimensional Hilbert space. In this case, it will turn out to be very important to obtain estimates that are independent of the approximation number and allow one to pass to the limit and obtain an a priori estimate in the limit for solving the infinite-dimensional problem.

It will be very important to obtain estimates that do not depend on the number of approximations, allowing one to pass to the limit and to obtain in the limit a priori estimate for solving an infinite-dimensional problem. Infinite-dimensional problems of the form (6), which we are focused on in what follows, are, as a rule, problems of mathematical physics written in a limited form.

Here and everywhere below, $f(u)$ will mean an operation of the form

$$f(u) := u + L(u). \tag{7}$$

If $\xi \in [0, +\infty)$ is a parameter and the vector $u(\xi)$ is a vector function continuously differentiable with respect to the parameter ξ, then we assume that the vector-function $L(u(\xi))$, is also continuously differentiable, as well as the expressions that arise from $L(u)$ and $f(u)$.

We introduce the notation L_u:

$$(L(u(\xi)))_\xi = L_{u(\xi)} u_\xi(\xi). \tag{8}$$

It is obvious that L_u (for each $u \in H$) will be a linear operator

$$L_u v = (L(u(\xi)))|_{u_\xi = v}. \tag{9}$$

We have

$$(f(u(\xi)))_\xi = u_\xi + L_u u_\xi = (E + L_u) u_\xi.$$

In what follows, if u_0, $v_0 \in H$, then the vector $L_{u_0} v_0$ is understood as follows: we take a continuously differentiable vector function $u(\xi)$ such that

$$u|_{\xi=0} = u_0, \quad u_\xi(\xi)|_{\xi=0} = v_0$$

and for $L_{u_0} v_0$ we take the vector

$$L_{u_0} v_0 = (L(u(\xi))_\xi \big|_{\xi=0}.$$

Here and everywhere in what follows, E is an identity operator.

We denote

$$D_u = E + L_u, \qquad D_u^* = E + L_u^*, \tag{10}$$

$$D_u^* f(u) = (E + L_u^*) f(u). \tag{11}$$

$$M_u a = \left(D^*_{u(\xi)} f(u(\xi))\right)_\xi \Bigg|_{\substack{u(\xi) = u \\ u_\xi(\xi) = a}} = M_u u_\xi \big|_{u_\xi = a} = M_u a. \tag{12}$$

Let us present the conditions used.

Condition U1: *For operators $L(\cdot)$, L_u, L_u^*, D_u, D_u^* the following conditions are satisfied*

$$\begin{cases} \|M_u - M_v\|_{H\to H} + \|L(u) - L(v)\| + \|L_u - L_v\|_{H\to H} + \|L_u^* - L_v^*\|_{H\to H} \leq \\ \qquad\qquad \leq \psi\left(\|u\|\right) \psi\left(\|v\|\right) \|u - v\|, \\ \|M_v u\| + \|D_v^* u\| + \|D_v u\| \leq \psi\left(\|v\|\right) \|u\|, \end{cases} \tag{13}$$

where $\|\cdot\| = \|\cdot\|_H$, $\psi(\cdot)$ *is a non-decreasing on* $[0,\infty)$ *positive continuous function.*

Condition U2: *There exist linear invertible operators* T *and* Q *such that*

$$\|T\| \le C_T, \quad \|Q\| \le C_T, \quad \|T^{-1}\| < \infty, \quad \|Q^{-1}\| < \infty, \tag{14}$$

and for any $u \in H$ *the inequalities hold*

$$\langle Tu, L(u)\rangle \ge 0, \quad \langle Tu, u\rangle \ge \|Qu\|^2. \tag{15}$$

In (14) C_T *is some fixed constant number.*

In what follows, C or c (uppercase or lowercase, with or without indices) will denote constant numbers (generally speaking, different in different places), independent of the adjacent factors.

Theorem 1. *Let condition U1 and condition U2 be satisfied. Then for any* $g \in H$ *the problem*

$$f(u) = g \tag{16}$$

has a solution $u \in H$, *satisfying the estimate*

$$\|Qu\|^2 \le C_T \,\|g\|^2, \tag{17}$$

where Q *is the operator from condition U2, and* C_T *is the constant from condition U2.*

The notations of the transformations $f(u)$, $L(u)$, the operators $L_u\cdot$, $D_u\cdot$, $M_u\cdot$ (defined for each $u \in H$, (see (6)–(11)) and their conjugates L_u^*, D_u^* and M_u will be used without reservations.

We will also introduce the following notations:

$$J(u) = \|u\|^2 \,\exp\left\{-\|f(u)\|^2\right\}, \tag{18}$$

$$N(u) = D_u^* f(u) - \gamma(u)\, u\,. \tag{19}$$

We often use the notations (18) and (19) without reservations, as well as the notations that arise in the formulations of conditions U1 and U2, and the notations that arise in the formulations of conditions U3 and U4 given below.

Condition U3: *There exists an invertible operator* G, *such that*

$$\|G\|_{H\to H} \le C_0 < \infty, \quad \|G^{-1}\| < \infty \tag{20}$$

and for any $u \in H$ *the inequality*

$$\|G\,u\|^2 \le d_0 \;\|f(u)\|^2, \tag{21}$$

where $d_0 > 0$ is a constant.

Condition U4: *If $0 \neq u_0 \in H$, $\gamma(u) > \|u\|^{-2}$ and $N(u) = 0$, then strict inequalities are satisfied*

$$\inf_{\{a\}} \frac{\langle M_u P_u a, P_u a\rangle - \gamma(u)\|P_u a\|^2}{\|P_u a\|^2} < 0 < \sup_{\{a\}} \frac{\langle M_u P_u a, P_u a\rangle - \gamma(u)\|P_u a\|^2}{\|P_u a\|^2}, \tag{22}$$

where $P_u a$ is an orthogonal projector.

The following theorem is true.

Theorem 2. *If conditions U1, U3, and U4 are satisfied, then for any $u \in H$ the a priori estimate holds:*

$$\|u\|^2 \leq C \exp\left\{\|f(u)\|^2\right\}. \tag{23}$$

Note that the estimate (23) is satisfied if conditions U1, U3 and the following condition U5 are satisfied.

Condition U5: *There exist constant numbers c_0, c_1, m and a self-adjoint operator T, such that if $\|u\| \geq 1$, then the inequalities are satisfied*

$$\|L(u)\| \geq c_0 \|Tu\|^m, \qquad \|u\| \leq c_1 \|u\|^m. \tag{24}$$

Remark 1. If the conditions of Theorem 1 are satisfied, that is, conditions U1 and U2 are satisfied, then condition U3 is also satisfied.

3 Proof of Theorem 1

The ways of proving theorems whose contents are similar to the statement of Theorem 1 are well known, and we could limit ourselves to a reference to them. However, for the sake of completeness of the presented results, we provide a proof of Theorem 1.

For the proof, we use one well-known technique (in a form convenient for us).

Let $g \in H$. Denote by $M(g)$ the set of vectors

$$M(g) = \{u \in H : \langle Tu, u\rangle \leq 16 \langle Tg, g\rangle\}, \tag{25}$$

where T is the operator from condition U2.

Assume that the equation $u + L(u) = g$ has no solution $u \in M(g)$. We define the transformation

$$F(u) = -\frac{u + L(u) - g}{\sqrt{\langle T(u + L(u) - g), u + L(u) - g\rangle}} 4\sqrt{\langle Tg, g\rangle}. \tag{26}$$

Since the equation $u+L(u) = g$ has no solution, then by virtue of condition U2 (see (15)), the transformation $F(\cdot)$ is continuous. It is easy to see that this transformation transforms the set $M(g)$ into itself. Therefore, by virtue of Browder's fixed point theorem and the finite dimensionality of H, the transformation $F(\cdot)$ has a fixed point $u_0 \in M(g)$, that is,

$$F(u_0) = u_0. \tag{27}$$

We apply the operator T to (27), and then scalar multiply the resulting equality by $u_0 + L(u_0) - g$. Then, using (26), we obtain

$$\begin{gathered} -A :\equiv -4\sqrt{\langle Tg, g\rangle}\sqrt{\langle T(u_0 + L(u_0) - g), u_0 + L(u_0) - g\rangle} = \\ = \langle Tu_0, u_0 + L(u_0) - g_0\rangle \geq \langle Tu_0, u_0\rangle - \langle Tu_0, g\rangle = \\ = \langle Tu_0, u_0\rangle - \tfrac{1}{2}\left(\langle Tu_0, g\rangle + \langle u_0, T^* g\rangle\right). \end{gathered} \tag{28}$$

In deriving (28) in the penultimate transition, (15) from condition U2 was used.

Since, according to condition U2, the inequality $\langle Tv, v\rangle \geq \|Qv\|^2$ is satisfied for any $v \in H$ and the operator Q is invertible, then the quantity $\langle Tu_0, u_0\rangle$ can be taken as the square of the norm of the vector u_0, and the quantity $\frac{1}{2}\left(\langle Tu_0, g\rangle + \langle u_0, T^* g\rangle\right)$ as the scalar product of the vectors u_0 and g in some real Hilbert space (provided by this norm). Therefore, we can use the well-known Cauchy inequality and obtain the inequality

$$-A \geq \langle Tu_0, u_0\rangle - (\varepsilon^{-1}\langle Tu_0, u_0\rangle + \varepsilon\langle Tg, g\rangle) = \langle Tu_0, u_0\rangle(1 - \varepsilon^{-1}) - \varepsilon\langle Tg, g\rangle. \tag{29}$$

From (27) and (26) we have

$$\langle Tu_0, u_0\rangle = \langle TF(u_0), F(u_0)\rangle = 16\ \langle Tg, g\rangle.$$

From here and from (29) we obtain

$$-A \geq [(1 - \varepsilon^{-1})16 - \varepsilon]\ \langle Tg, g\rangle.$$

Choosing here $\varepsilon = 2$, we obtain $-A \geq 6\ \langle Tg, g\rangle$.

Since the left-hand side of the inequality is a negative value, it contradicts the inequality (15) from condition U2. This shows that equation $u + Tu = g$ has a solution $u \in M(g)$.

To solve the equation $u + L(u) = g$, multiplying it scalarly by Tu, we have

$$\langle Tu, u\rangle + \langle Tu, L(u)\rangle = \langle Tu, g\rangle = \frac{1}{2}\left(\langle Tu, g\rangle + \langle u, T^* g\rangle\right).$$

Here on the right is the scalar product of the vectors Tu and g in some Hilbert space. Applying the well-known Cauchy inequality and taking into account the first inequality from (15) condition U2, we obtain:

$$\langle Tu, u\rangle \leq \left(\langle Tu, u\rangle\right)^{1/2}\left(\langle Tg, g\rangle\right)^{1/2}.$$

From here

$$\langle Tu, u\rangle \le \langle Tg, g\rangle \le \|Tg\|\|g\| \le C_T\|g\|^2,$$

where C_T is a constant from condition U2.

Now, from the second inequality (15) of condition U2 we obtain

$$\|Qu\|^2 \le C_T\|g\|^2.$$

Theorem 1 is proven.

Remark 2. Theorem 1 allows us to prove the existence of a "weak" solution to some problems of mathematical physics. To prove the existence of a "strong" solution, which allows us to use perturbation theory for some problems of mathematical physics, we need another finite-dimensional theorem, which will be proved under conditions U1, U3, and U4.

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Қошанов Б.Д., Өтелбаев М., Шыныбеков А.Н. АҚЫРЛЫ ӨЛШЕМДІ СЫЗЫҚТЫ ЕМЕС ТЕҢДЕУЛЕРДІҢ БІР КЛАСЫНЫҢ ШЕШІМДЕРІН БАҒАЛАУ. I

Осы мақалада шектелген өлшемді кеңістіктегі сызықты емес теңдеулердің шешімдеріне арналған алдын ала бағалау теоремасы алынады. Бұл теорема сызықты емес бастапқы-шектік есептер класының шектелген өлшемді жуықтаулары қанағаттандыратын шарттардан алынған белгілі бір шарттарда дәлелденеді. Негізгі нәтиже сызықты емес оператор A берілген $A(u) = f$ теңдеуінің шешімі бар болуына жеткілікті шарттарды анықтайды. Негізгі нәтижені сызықты емес талдау мен математикалық физикада қолдануға болатындығын көрсету үшін мысал келтіріледі.

Түйін сөздер: дифференциалдық оператор, сызықтық емес теңдеу, шешімнің бар болуы, шешімнің жалғыздығы, шешімнің априорлық бағалауы.

Кошанов Б.Д., Отелбаев М., Шыныбеков А.Н. ОЦЕНКА РЕШЕНИЙ ОДНОГО КЛАССА КОНЕЧНОМЕРНЫХ НЕЛИНЕЙНЫХ УРАВНЕНИЙ. I

В данной статье получена теорема об априорных оценках решений нелинейных уравнений в конечномерном пространстве. Доказана теорема об априорной оценке решения, что при условиях, заимствованных из условий, которые выполняются для конечномерных аппроксимаций одного класса нелинейных начально-краевых задач. Основной результат устанавливает достаточные условия существования решения уравнения $A(u) = f$, где A — нелинейный оператор. Приведён пример, иллюстрирующий применимость основного результата в нелинейном анализе и математической физике.

Ключевые слова: дифференциальный оператор, нелинейное уравнение, существование решения, единственность решения, априорная оценка решения.