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On $\mathcal{F}$-homogeneous models

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Abstract. We introduce the notion of an $\mathcal{F}$-homogeneous model and prove that if a theory has an infinite $\mathcal{F}$-homogeneous model, then it has an $\mathcal{F}$-homogeneous model of any infinite cardinality which is not less than the cardinality of the theory.

Keywords. $\mathcal{F}$-homogeneous model, interval homogeneous linear order.

1 Introduction

In the book [1, p. 56] the following definition is given.

Definition 1.1. An infinite linear order is said to be *interval homogeneous* if it is isomorphic to each of its non-singleton closed intervals.

It is also stated that the cardinality of any interval homogeneous linear order does not exceed the cardinality of continuum.

The question on the cardinality of an interval homogeneous linear order is quite natural. We will consider a more general version of this question, generalizing the notion of interval homogeneity to models of arbitrary purely predicate languages and thereby defining the notion of an $\mathcal{F}$-homogeneous model, and we will prove that a theory that has an infinite $\mathcal{F}$-homogeneous model has an $\mathcal{F}$-homogeneous model of any cardinality (that is not less than the cardinality of the language). In particular, it follows from this that there exist interval homogeneous linear orders of any infinite cardinalities. Moreover, for any uncountable cardinal λ there exist 2^λ non-isomorphic interval homogeneous linear orders of cardinality λ.

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2 Basic definitions

We will use more or less standard model-theoretic notation and definitions that correspond to [2] and [3].

Models will be denoted by $\mathcal{M}$ and $\mathcal{N}$, and their underlying sets (universes) will be denoted by M and N respectively. The cardinality of a set X will be denoted by $|X|$. The first infinite cardinal will denoted by ω, and the first uncountable cardinal will be denoted by ω_1. Elements of a model will be denoted by a, b, and tuples of elements will be denoted by $\bar{a}$, $\bar{b}$. Variables will be denoted by x, y, and tuples of variables will be denoted by $\bar{x}$, $\bar{y}$. By ${}^{\omega}2$ we will denote the set of functions from ω to $2 = \{0, 1\}$.

Let $\mathcal{M}$ be a model in a language L containing only predicate symbols. Let $\mathcal{F}$ be a set of L-formulas having at least one free variable.

Definition 2.1. (1) We will say that a submodel $\mathcal{N}$ of the model $\mathcal{M}$ is $\mathcal{F}$-*definable* if there exist a formula $\varphi(x, \bar{y}) \in \mathcal{F}$ and a tuple $\bar{a}$ of elements of the model $\mathcal{M}$ such that

$$N = \varphi(\mathcal{M}, \bar{a}) = \{b \in M : \mathcal{M} \vDash \varphi(b, \bar{a})\}.$$

(2) A model $\mathcal{M}$ will be called $\mathcal{F}$-*homogeneous* if it is isomorphic to each of its $\mathcal{F}$-definable submodel $\mathcal{N}$ of cardinality $|N| > 1$.

Example 2.1. If $\mathcal{F} = \{y_0 \leqslant x \leqslant y_1\}$ then an $\mathcal{F}$-homogeneous model of the theory of linear orders is precisely an interval homogeneous linear order.

3 Results

Theorem 3.1. *If a theory T has an infinite $\mathcal{F}$-homogeneous model, then for every cardinal $\lambda \geqslant |T|$ the theory T has an $\mathcal{F}$-homogeneous model of cardinality λ.*

Proof. Let $\mathcal{M}$ be an infinite $\mathcal{F}$-homogeneous model of the theory T. We extend the language L of the theory T to a language L^* by adding a predicate symbol for each $\mathcal{F}$-definable submodel of the model $\mathcal{M}$. Namely, for a submodel of $\mathcal{M}$ with an underlying set $\varphi(\mathcal{M}, \bar{a})$ we add an $(n+2)$-ary predicate symbol R_φ, where n is the length of the tuple $\bar{a}$.

Let us consider a theory T^* in the language L^* having the following axioms:

(1) the axioms of the theory T;

(2) for every formula $\varphi(x, \bar{y}) \in \mathcal{F}$ and every $\bar{y}$ if $|\varphi(\mathcal{M}, \bar{y})| > 1$, then the set

$$R_{\varphi,\bar{y}} = \{(u, v) : R_\varphi(\bar{y}, u, v)\}$$

is an isomorphism of the model $\mathcal{M}$ to the submodel with the underlying set $\varphi(\mathcal{M}, \bar{y})$.

Let us explain how to write (2) down as a set of formulas in a first-order language: for each formula $\varphi(x, \bar{y}) \in \mathcal{F}$ we write that $R_{\varphi,\bar{y}}$ is a bijection from M onto $\varphi(\mathcal{M}, \bar{y})$ and add for every predicate symbol S of the language L the universal closure of the following formula:

$$(\exists x)(\exists x')[x \neq x' \wedge \varphi(x, \bar{y}) \wedge \varphi(x', \bar{y})] \rightarrow$$

$$[S(u_1, \ldots, u_m) \leftrightarrow S(R_{\varphi,\bar{y}}(u_1), \ldots, R_{\varphi,\bar{y}}(u_m))].$$

It is clear that the model $\mathcal{M}$ can be expanded to a model of the theory T^*.

Since T^* has an infinite model, by the Löwenheim-Skolem-Tarski Theorem [2] for every cardinal $\lambda \geqslant |T^*| = |T|$ the theory T^* has a model $\mathcal{N}^*$ of cardinality λ. Then reduction of the model $\mathcal{N}^*$ to the language L will be an $\mathcal{F}$-homogeneous model of cardinality λ for the theory T. □

Corollary 3.2. *For every infinite cardinal λ there exists an interval homogeneous linear order of cardinality λ.*

Proof. The countable dense linear order with the maximal and minimal elements is an $\mathcal{F}$-homogeneous model of the theory of linear order, where $\mathcal{F} = \{y_0 \leqslant x \leqslant y_1\}$. □

Proposition 3.3. *Let T be an unsuperstable theory which has an infinite $\mathcal{F}$-homogeneous model.*

(1) For any cardinal $\lambda > |T|$ the theory T has 2^λ non-isomorphic $\mathcal{F}$-homogeneous models of cardinality λ.

(2) If $\lambda \geqslant \mu > |T|$ and μ is a regular cardinal, then T has at least 2^μ $\mathcal{F}$-homogeneous models such that any of them is not elementarily embeddable into any other.

Proof. Let us consider the theory $T^* \supseteq T$ from the proof of Theorem 3.1. It follows from the axioms of T^* that every model of T^* is $\mathcal{F}$-homogeneous. This implies that the pseudo-elementary class $\mathrm{PC}(T^*, T)$, which is the class of reducts of models of T^* of cardinality $\geqslant |T^*|$ to the language of T, consists of $\mathcal{F}$-homogeneous models of T of cardinality $\geqslant |T^*|$. Since T is not superstable, by the theorem of Shelah [3, Chapter VIII, § 2, Theorem 2.1] we have $I(\lambda, T^*, T) = 2^\lambda$ for any $\lambda > |T^*|$, where $I(\lambda, T^*, T)$ is the number of non-isomorphic models in $\mathrm{PC}(T^*, T)$ of cardinality λ. It remains to notice that $|T^*| = |T|$. Statement (1) is proved. Now we apply [3, Chapter VIII, § 2, Theorem 2.2] and get (2). □

Corollary 3.4. *(1) For any cardinal $\lambda > \omega$ there exist 2^λ non-isomorphic interval homogeneous linear orders of cardinality λ.*

(2) If $\lambda \geqslant \mu > \omega$ and μ is a regular cardinal, then there exist at least 2^μ interval homogeneous linear orders such that any of them is not elementarily embeddable into any other.

Proof. The theory of linear order is countable and unstable (and hence unsuperstable) and the countable dense linear order with the maximal and minimal elements is an $\mathcal{F}$-homogeneous model of the theory of linear order, where $\mathcal{F} = \{y_0 \leqslant x \leqslant y_1\}$. □

Proposition 3.5. *Any interval homogeneous linear order is dense.*

Proof. Follows from the definitions of interval homogeneity and density of linear orders. □

Therefore there exists only one (up to an isomorphism) countable interval homogeneous linear order.

Let us show that the theory of a certain linear order may have infinitely many (and even 2^ω) non-isomorphic countable $\mathcal{F}$-homogeneous models, for some $\mathcal{F}$.

Example 3.1. Let T be the theory of $(\mathbb{N},<)$, the set positive integers with its natural linear ordering. Every model of T is $\mathcal{F}$-homogeneous, where $\mathcal{F}=\{x\neq y\}$. Let us show that T has 2^ω non-isomorphic countable models.

For any $f\in{}^\omega 2$, let

$$M_f=(\mathbb{N},<)+\sum_{i<\omega}M_i,$$

the sum of linear orders, where

(a) if $i=2n$ and $f(n)=0$, then $M_i=(\mathbb{Z},<)$, the set of integers with its natural linear ordering;

(b) if $i=2n$ and $f(n)=1$, then $M_i=(\mathbb{Z},<)+(\mathbb{Z},<)$, the sum of linear orders;

(c) if $i=2n+1$ and $f(n)=1$, then $M_i=(\mathbb{Q},<)\times(\mathbb{Z},<)$, the lexicographic product of linear orders, where $(\mathbb{Q},<)$ is the set of rationals with its natural linear ordering.

If $f,f'\in{}^\omega 2$ and $f\neq f'$, then M_f is not isomorphic to $M_{f'}$. Therefore, the theory T has 2^ω non-isomorphic countable models.

Let us notice that if we take $\mathcal{F}=\{y<x\}$, then T has countably many non-isomorphic countable $\mathcal{F}$-homogeneous models:

$$(\mathbb{N},<),\quad (\mathbb{N},<)+((\mathbb{Q},<)\times(\mathbb{Z},<)),\quad (\mathbb{N},<)+\sum_{i<n}M_i,\ n\leqslant\omega,$$

where $M_i=(\mathbb{Z},<)$.

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[2] Chang C.C., Keisler H.J. Model theory. North-Holland, 1973.

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Құдайбергенов Қ.Ж. $\mathcal{F}$-БІРТЕКТІ МОДЕЛЬДЕР ТУРАЛЫ

Біз біртекті модель ұғымын енгіземіз және егер теорияның шексіз біртекті моделі болса, онда оның теория күшінен кем емес кез келген шексіз күштін біртекті моделі болатын дәлелдейміз.

Түйін сөздер: $\mathcal{F}$-біртекті модель, интервалдық біртекті сызықтық тәртіп.

Кудайбергенов К.Ж. ОБ $\mathcal{F}$-ОДНОРОДНЫХ МОДЕЛЯХ

Мы вводим понятие $\mathcal{F}$-однородной модели и доказываем, что если теория имеет $\mathcal{F}$-однородную модель, то она имеет $\mathcal{F}$-однородную модель любой бесконечной мощности не меньшей, чем мощность теории.

Ключевые слова: $\mathcal{F}$-однородная модель, интервально однородный линейный порядок.