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Dirichlet and Neumann problems for the heat equation on linear multilink thermal graphs and their solutions

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Abstract. We consider boundary value problems of thermal conductivity on a linear thermal graph, which can be used to study various structures under conditions of thermal heating (cooling). Here, based on the generalized function method, a unified technique has been developed for solving boundary value problems of thermal conductivity, typical for engineering applications. Generalized solutions to nonstationary and stationary boundary value problems of heat conduction on an edge and on a thermal linear graph are constructed under various boundary conditions at the ends of the graph and generalized Kirchhoff conditions at its node. Using the properties of the Fourier transformant of the fundamental solution, regular integral representations of solutions to boundary value problems are obtained in analytical form. The solutions obtained make it possible to simulate heat sources of various types, including using singular generalized functions. The method of generalized functions presented here makes it possible to solve a wide class of boundary value problems with local and connected boundary conditions at the ends of the edges of the graph and different transmission conditions at its nodes.

Keywords. Thermal conductivity, generalized functions, fundamental and generalized solution, Fourier transform, resolving boundary equations, linear graph.

1 Introduction

Graph theory has wide applications in subjects such as economics, logistics, sociology, optimal control, and navigation [1–2]. The properties of graphs are also actively used to solve

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boundary value problems (BVPs) on network-like structures, e.g., oil pipelines, gas pipelines, and electrical networks [3–10]. With the development of mechanical engineering, complex multi-link rod structures operating under various thermal conditions began to be actively used. They are widely used in structural mechanics, mechanical engineering, robotics, and many other fields.

Mathematical modeling of the thermodynamics of rod structures and the creation of information technologies based on it is one of the more effective and inexpensive methods for researching and designing such systems. An urgent scientific and technical task is to study the thermal state of network systems for various purposes under thermal influences, taking into account their construction and thermal influences, including impact types. This is necessary to analyze the strength and reliability of such objects, determine safe operating modes, and prevent disasters.

Here boundary value problems are considered on the linear multilink thermal graph (Fig. 1), which can be used to study various mesh structures under conditions of volume and thermal heating (cooling).

The novelty of the present work lies in the fact that a generalized function method is used to solve boundary value problems, leading to a differential equation solution with a singular right-hand side [11]. The solution is constructed as the convolution of the Green's function of the equation with the appropriate right-hand side. To determine the unknown boundary values of the solution and its derivatives on each segment, resolving boundary equations are constructed at the ends, employing the asymptotic properties of Green's function and its derivative at zero. To construct a closed system of equations, the obtained algebraic equations for each edge of the graph are supplemented with transmission conditions at the node and linear boundary conditions at its ends. These conditions can be either locally or not locally connected.

A resolving system of equations in the space of Fourier transforms over time and Fourier transforms of temperature on each link of the graph are constructed, which give a solution to stationary boundary value problems with oscillations with a fixed frequency. The inverse Fourier transform is used to construct the original. The obtained solutions give analytical formulas for calculating the temperature of such structures under thermal heating conditions, and can be used in the design of heating networks, as well as for solving boundary value problems in environments stratified by thermal graphs.

2 Statement of the boundary value problem on thermal linear graphs

We consider a thermal linear graph which contains N edges (A_{j-1}, A_j) of the length L_j, where $j = 1, 2, \ldots, N$ (Fig. 1). On each edge $S_j = \left\{x \in R^1 : 0 \leq x \leq L_j\right\}$ there is its own coordinate system (x_j, t) with the origin at the point A_{j-1}, that is, $x_j = 0$ at A_{j-1} and $x_j = L_j$ at A_j.

The temperature $\theta_j(x,t)$ satisfies the heat conduction equation at S_j:

$$\frac{\partial\theta_j}{\partial t} - \kappa_j \frac{\partial^2\theta_j}{\partial x^2} = F_j(x,t). \tag{1}$$

Here κ_j is the thermal diffusivity coefficient on the j-th segment, $F_j(x,t)$ describes the action of the heat source, $\theta_1^j(t)$ and $\theta_2^j(t)$ are the temperatures at the ends of the j-th edge.

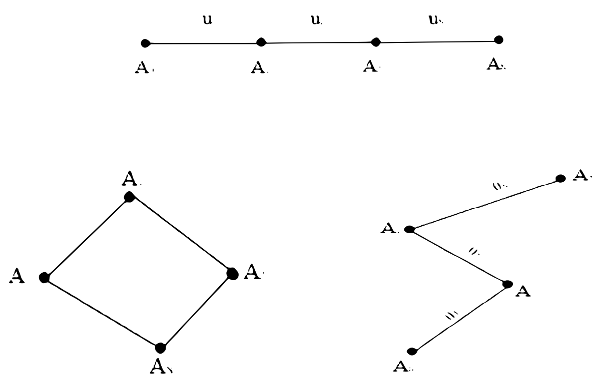

Figure 1. Linear graphs

The initial conditions at $t=0$ for the temperature of a graph are known:
(Cauchy conditions)

$$\begin{aligned} \theta_j(x,0) &= {\theta_0}^j(x), \quad 0 \le x \le L_j, \quad t=0, && (2)\\ \theta_j(0) &= \theta_0, && (3) \end{aligned}$$

where ${\theta_0}^j(x) \in C^2(R_+)$ for each j. Here we consider the two boundary value problems (BVP), $R_+^1 = \{t \in [0,\infty)\}$

Dirichlet conditions (BVP1). Temperature values are known at the ends of the graph:

$$\begin{array}{ll} \theta_1^1(t) \;=\; \theta_1(0,t) = \vartheta_1(t), & t \ge 0, \quad \vartheta_1(t) \in C(R_+^1),\\ \theta_2^N(t) = \theta_N(L_N,t) = \vartheta_2(t), & t \ge 0, \quad \vartheta_2(t) \in C(R_+^1). \end{array} \tag{4}$$

Here and further

$$\theta_j^1(t) = \theta_j(0,t),\;\; q_j^1(t) = \partial_x\theta_j(0,t),\;\; \theta_j^2(t) = \theta_j(L_J,t),\;\; q_j^2(t) = \partial_x\theta_j(L_J,t).$$

Neumann conditions (BVP 2). The values of the heat flows are known at the ends of the graph:

$$\begin{array}{ll} \kappa_1\, q_1^1\,(t) = \kappa_1 q_1(0,t) = \chi_1(t), & t \ge 0, \quad \chi_1(t) \in C(R_+^1),\\ \kappa_N q_2^N(t) = \kappa_N q_N(L_N,t) = \chi_2(t), & t \ge 0, \quad \chi_2(t) \in C(R_+^1). \end{array} \tag{5}$$

The following continuity conditions and generalized Kirchhoff conditions are specified in the common node A_0 of the graph.

Transmission conditions:

$$\begin{array}{l} \theta_2^j\,(t) = \theta_1^{j+1}(t), \quad j = 1, ..., N-1, \quad t \geq 0, \\ \theta_1^1\,(0) = \vartheta_1(0), \\ \theta_2^N(0) = \vartheta_2(0), \end{array} \tag{6}$$

$$\kappa_j q_2^j(t) = \kappa_{j+1} q_1^{j+1}(t) + Q_j(t), \quad j = 1, ..., N-1, \quad t \geq 0. \tag{7}$$

Here

$$\theta_1^j(t) = \theta_j(0,t), \; q_1^j(t) = \left.\frac{\partial \theta_j}{\partial x}\right|_{x=0}, \; q_2^j(t) = \left.\frac{\partial \theta_j}{\partial x}\right|_{x=L_j},$$

θ_0 is the initial temperature at the common node A_0.

We need to find the solutions of these two BVP on the heat linear graph by known $Q_j(t)$, where $j = 1, \ldots, N$, $\vartheta_1(t)$ and $\vartheta_N(t)$ (Dirichlet problem) or $\chi_1(t)$ and $\chi_N(t)$ (Neumann problem).

3 Statement of boundary value problem on a segment of a graph

At first we construct a solution of some boundary value on one graph segment. Let consider $\theta(x,t)$ on $[0, L]$, which is the solution of heat equation:

$$\frac{\partial \theta}{\partial t} - \kappa \frac{\partial^2 \theta}{\partial x^2} = F(x,t). \tag{8}$$

Initial conditions: the temperature is known at $t = 0$:

$$\theta(x,0) = \theta_0(x), \quad \theta_0(x) \in C\,\{0 \leq x \leq L\} \tag{9}$$

Here we consider solutions to BVPs with local and associated boundary conditions.
Local boundary conditions:

$$\left\{ \begin{array}{l} (\alpha_1\theta_1 + \beta_1\Pi_1(t))|_{x=0} = G_1(t), \\ (\alpha_2\theta_2 + \beta_2\Pi_2(t))|_{x=L} = G_2(t). \end{array} \right. \tag{10}$$

where α_j, β_j arbitrary constants, $\theta_j(t)$, $\Pi_j(t) = -k\left.\frac{\partial \theta}{\partial x}\right|_{x=x_j}$ $(j = 1, 2)$ are the temperature and heat flow at ends of the segment in points: $x = x_1 = 0$, $x = x_2 = L$. $G_j(t)$ are known functions which are integrated functions on $R_+^1 : G_j(t) \in L_1(R_+^1)$.

Connected boundary conditions:

$$\alpha_{1j}\theta_1(t) + \beta_{1j}\Pi_1(t) + \alpha_{2j}\theta_1(t) + \beta_{2j}\Pi_2(t) = D_j(t), \quad j = 1, 2. \tag{11}$$

Conditions for matching initial and boundary conditions:

$$\theta_1(t) = \theta(0,t), \quad \theta_2(t) = \theta(L,t), \quad \theta_j(t) \in C(R_1^+).$$

It is assumed that all functions defining boundary conditions also belong to Lebesgue space L_1. Relations (11) contain all classical formulations of heat BVPs if we take some $\alpha_{ij} = 0$, $\beta_{ij} = 0$. We find solutions to BVPs using the Generalized Function Method [14].

4 Generalized solution of boundary value problems on an graph segment. Generalized function method

To determine the solution on the graph, at first, we consider the BVP on the graph segment by using the general function method. For this, we consider the BVP for the heat equation on the segment $[0, L]$ in the space $S'(R^2) = \{\hat{f}(x,t),\, (x,t) \in R^2\}$ of generalized functions of slow growth [15]. To do this, we introduce a regular generalized function (we mark it with a cap):

$$\hat{\theta}(x,t) = \begin{cases} \theta(x,t), & (x,\, t) \in D^- \\ 0, & x \notin D^- \end{cases},$$

where $\theta(x,t)$ is the solution of BVP, $D^- = [0,L] \times [0,\infty)$. It can be represented in the form

$$\hat{\theta}(x,t) = \theta(x,t)H(L-x)H(x)H(t).$$

Here $H(x)$ is the Heaviside step function.

To construct the equation for $\hat{\theta}(x,t)$ in $S'(R^2)$, we calculate the generalized derivatives of $\hat{\theta}(x,t)$:

$$\frac{\partial\hat{\theta}}{\partial x} = \frac{\partial\theta}{\partial x}H(L-x)H(x)H(t) - \theta_2(t)\delta(L-x)H(t) + \theta_1(t)\delta(x)H(t),$$

$$\frac{\partial^2\hat{\theta}}{\partial x^2} = \frac{\partial^2\theta}{\partial x^2}H(L-x)H(x)H(t) - q_2(t)\delta(L-x)H(t) + q_1(t)\delta(x)H(t) + \\ + \theta_2(t)\delta'(L-x)H(t) + \theta_1(t)H(t)\delta'(x),$$

$$\frac{\partial\hat{\theta}}{\partial t} = \frac{\partial\theta}{\partial t}H(L-x)H(t) + \theta_0(x)H(L-x)\delta(t),$$

where $\delta\,(x)$ is a singular generalized δ-function, $q_j(t) = \frac{\partial\theta}{\partial x}\big|_{x=x_j}$, $j = 1, 2$.

The equation (7) in $S'(R^2)$ has the following form for $\hat{\theta}(x,t)$:

$$\frac{\partial\hat{\theta}}{\partial t} - \kappa\frac{\partial^2\hat{\theta}}{\partial x^2} = \hat{F}_2(x,t) + \kappa q_2(t)\delta(L-x)H(t) - \kappa q_1(t)\delta(x)H(t) - \\ - \kappa\theta_2(t)\delta'(L-x)H(t) - \kappa\theta_1(t)\delta'(x)H(t) + \theta_0(x)H(L-x)H(x)\delta(t). \quad (12)$$

Note that the right side of this equation includes all initial and boundary temperatures $\theta_j(t)$ and heat flows $\Pi_j(t) = \kappa q_j(t)$ (j=1, 2).

Throughout the paper, we denote the partial derivative $\frac{\partial U}{\partial x}$ by $U_{,x}(x,t)$.

According to the theory of generalized functions [15], the solution of (12) can be represented as a convolution of the fundamental solution of the heat equation (8) with the right-hand side of this equation:

$$\begin{aligned}\hat{\theta}(x,t) = \hat{F}_2(x,t) * U(x,t) + \kappa q_2(t)H(t) \underset{t}{*} U(L-x,t) - \\ - \kappa q_1(t)H(t) \underset{t}{*} U(x,t) - \kappa\theta_2(t)H(t) \underset{t}{*} U_{,x}(L-x,t) - \\ - \kappa\theta_1(t)H(t) \underset{t}{*} U_{,x}(x,t) + \theta_0(x)H(L-x)H(x) \underset{x}{*} U(x,t). \quad (13)\end{aligned}$$

Here, $U(x,t)$ is the fundamental solution of the heat equation (1) by $F(x,t) = \delta(x,t) = \delta(x)\delta(t)$. It decays at ∞ and has the form [15]:

$$U(x,t) = \frac{1}{\sqrt{2\pi\kappa t}} \exp(-x^2/4\kappa t)H(t). \quad (14)$$

We denote $\hat{F}(x,t) = F(x,t)H(x)H(L-x)H(t)$. If it is a regular function, then relation (13) can be represented in the next integral form:

$$\begin{aligned}&\theta(x,t)H(L-x)H(x)H(t) = \\ &= H(t)\int_0^t d\tau \int_{-\infty}^{+\infty} U(x-y,t-\tau)\,F_2(y,\tau)dy + \kappa H(x)H(t)\int_0^t q_2(t-\tau)U(L-x,\tau)d\tau - \\ &- \kappa H(L-x)H(t)\int_0^t U(x-y,t-\tau)q_1(\tau)d\tau - \kappa H(x)H(t)\int_0^t \theta_2(t-\tau)U_{,x}(L-x,\tau)d\tau - \\ &- \kappa H(L-x)H(t)\int_0^t U_{,x}(x,t-\tau)\theta_1(\tau)d\tau + \int_0^L U(x-y,t)\theta_0(y)H(L-y)H(y)dy. \quad (15)\end{aligned}$$

Formula (15) determines the temperature inside a segment by known temperature and heat flows at its ends and is very useful for engineering applications. However, for correctly posed boundary value problems, out of 4 boundary functions on the right side of formula (15), only 2 are known. To determine two unknown boundary functions, resolving boundary equations should be constructed using boundary conditions at the ends of the segment.

5 Solving boundary value problem in Fourier transformation space in time

To construct the resolving system of equations, we use Fourier transformation in time:

$$\begin{gathered}\bar{\theta}(x,\omega) = \mathrm{F}\left[\hat{\theta}(x,t)\right] = H(x)H(L-x)\int_0^\infty \theta(x,t)e^{i\omega t}dt,\\ \hat{\theta}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{\theta}(x,\omega)e^{-i\omega t}d\omega.\end{gathered} \tag{16}$$

To define the Fourier transform of the generalized solution (11) we use the property of Fourier transform of convolution [15]:

$$\begin{aligned}\hat{\theta}(x,\omega) = \bar{F}_2(x,\omega) \underset{x}{*} \bar{U}(x,\omega) + \theta_0(x)H(L-x)H(x)\underset{x}{*}\bar{U}(x,\omega)+\\ + \kappa\bar{q}_2(\omega)H(x)\bar{U}(L-x,\omega) - \kappa\bar{q}_1(\omega)H(L-x)\bar{U}(x,\omega)-\\ - \kappa\bar{\theta}_2(\omega)H(x)\bar{U}_{,x}(L-x,\omega) - \kappa\bar{\theta}_1(\omega)H(L-x)\bar{U}_{,x}(x,\omega).\end{aligned} \tag{17}$$

Here, a variable under the sign of convolution $\left(\underset{x}{*}\right)$ shows the convolution is applied only over the variable x. The integral representation of Equation (17) has the form:

$$\begin{aligned}&\bar{\theta}(x,\omega)H(L-x)H(x)H(\omega) =\\ &= H(x)\int_0^L \bar{U}\left(x-y,\omega\right)F_2(y,\omega)dy + \kappa H(x)\int_0^L \bar{U}(x-y,\omega)\theta_0(y)dy+\\ &+ \kappa\bar{q}_2(\omega)H(x)\bar{U}(L-x,\omega) - \kappa\bar{q}_1(\omega)H(L-x)\bar{U}(x,\omega) -\\ &- \kappa\bar{\theta}_2(\omega)H(x)\bar{U}_{,x}(L-x,\omega) - \kappa\bar{\theta}_1(\omega)H(L-x)\bar{U}_{,x}(x,\omega).\end{aligned} \tag{18}$$

Fourier transform of Green's function of the heat equation is equal to

$$\bar{U}(x,\omega) = -\frac{\sin\left(k\,|x|\right)}{2k\,\kappa}, \tag{19}$$

where $k = \sqrt{i\omega\,\kappa^{-1}} = e^{i\pi/4}\sqrt{\omega\,\kappa^{-1}} = (1+i)\sqrt{\frac{\omega}{2\kappa}}$. It satisfies the equation:

$$\frac{d^2\bar{U}}{dx^2} + i\,\omega\,\kappa^{-1}\bar{U} = \delta(x).$$

Its derivative has the gap in point $x=0$ and equal to

$$\bar{U}_{,x}(x,\omega) = -\frac{\mathrm{sgn}x}{2\kappa}\cos(\kappa\,|x|),\ \mathrm{sgn}x = \begin{cases}\ \ 1,\ x>0,\\ -1,\ x<0.\end{cases}$$

There are the following symmetry conditions:

$$\bar{U}(x,\omega) = \bar{U}(-x,\omega),\ \bar{U}_{,x}(\pm 0,\omega) = \mp\frac{1}{2\kappa}. \tag{20}$$

We use these properties for solving BVP.

6 Resolving equations of boundary value problems

To find unknown boundary functions, we pass in relation (18) to the limit at $x \to 0+\varepsilon$, where $\varepsilon > 0$:

$$\bar{\theta}_1(\omega) = \lim_{\varepsilon\to 0}\bar{\theta}(0+\varepsilon,\omega) = \bar{F}(x,\omega) \underset{x}{*} \bar{U}(x,\omega)\big|_{x=0} + \theta_0(x)H(L-x)H(x) \underset{x}{*} \bar{U}(x,\omega)\big|_{x=0} +$$

$$+\kappa\bar{q}_2(\omega)H(x)\bar{U}(L-0-\varepsilon,\omega) - \kappa\bar{q}_1(\omega)H(L-x)\bar{U}(0+\varepsilon,\omega) -$$

$$-\kappa\bar{\theta}_2(\omega)H(x)\bar{U}_x(L-0-\varepsilon,\omega) - \kappa\bar{\theta}_1(\omega)H(L-x)\bar{U}_{,x}(0+\varepsilon,\omega).$$

Next, we move the last term to the left side and take into account the right limit of $\bar{U}_{,x}(x,\omega)$ at zero (20). We obtain the next equation on the left end of the segment:

$$\frac{1}{2}\bar{\theta}_1(\omega) = \bar{F}(x,\omega) \underset{x}{*} \bar{U}(x,\omega)\big|_{x=0} + \theta_0(x)H(L-x)H(x) \underset{x}{*} \bar{U}(x,\omega)\big|_{x=0} +$$
$$+ \kappa\bar{q}_2(\omega)H(x)\bar{U}(L,\omega) - \kappa\bar{q}_1(\omega)\bar{U}(0,\omega) - \kappa\bar{\theta}_2(\omega)H(x)\bar{U}_{,x}(L,\omega) \tag{21}$$

Similarly, we consider the limit at $x \to L-\varepsilon,\ \varepsilon > 0$.

$$\bar{\theta}_2(\omega) = \lim_{\varepsilon\to 0}\bar{\theta}(L-\varepsilon,\omega) = \bar{F}(x,\omega) \underset{x}{*} \bar{U}(x,\omega)\big|_{x=L} + \theta_0(x)H(L-x)H(x) \underset{x}{*} \bar{U}(x,\omega)\big|_{x=L} -$$
$$- \kappa\bar{q}_1(\omega)\bar{U}(L-\varepsilon,\omega) - \kappa\bar{\theta}_1(\omega)\bar{U}_{,x}(L-\varepsilon,\omega) - \kappa\bar{\theta}_2(\omega)H(x)\bar{U}_{,x}(\varepsilon,\omega) \tag{22}$$

We move the last term to the left side, and obtain the second boundary equation:

$$\frac{1}{2}\bar{\theta}_2(\omega) = \bar{F}(x,\omega) \underset{x}{*} \bar{U}(x,\omega)\big|_{x=L} + \theta_0(x)H(L-x)H(x) \underset{x}{*} \bar{U}(x,\omega)\big|_{x=L} -$$
$$- \kappa\bar{q}_1(\omega)\bar{U}(L,\omega) - \kappa\bar{\theta}_1(\omega)\bar{U}_{,x}(L,\omega) \tag{23}$$

We formulate the obtained results in the form of this theorem.

Theorem 1. *The Fourier time transformants of boundary functions of boundary value problems* (7)–(10) *satisfy the system of linear algebraic equations of the form:*

$$
\begin{bmatrix} 0,5 & 0 \\ \kappa\bar{U}_{,x}(L,\omega) & \kappa\bar{U}(L,\omega) \end{bmatrix} \begin{bmatrix} \bar{\theta}_1(\omega) \\ \bar{q}_1(\omega) \end{bmatrix} + \\
+ \begin{bmatrix} \kappa\bar{U}_{,x}(L,\omega) & -\kappa\bar{U}(L,\omega) \\ 0,5 & 0 \end{bmatrix} \begin{bmatrix} \bar{\theta}_2(\omega) \\ \bar{q}_2(\omega) \end{bmatrix} = \begin{bmatrix} \bar{Q}_1(0,\omega) \\ \bar{Q}_2(L,\omega) \end{bmatrix}, \quad (24)
$$

where

$$
\begin{aligned}
\bar{Q}_1(0,\omega) &= \bar{F}(x,\omega) \underset{x}{*} \bar{U}(x,\omega)\big|_{x=0} + \theta_0(x)H(L-x)H(x) \underset{x}{*} \bar{U}(x,\omega)\big|_{x=0}, && (25) \\
\bar{Q}_2(L,\omega) &= \bar{F}(x,\omega) \underset{x}{*} \bar{U}(x,\omega)\big|_{x=L} + \theta_0(x)H(L-x)H(x) \underset{x}{*} \bar{U}(x,\omega)\big|_{x=L}. && (26)
\end{aligned}
$$

The resulting system (20) makes it possible to solve BVP for any given two boundary functions of temperature and heat flow at the ends of a segment of four boundary functions. To solve all temperature BVPs, it is convenient to consider the extended system of equations in the form of a matrix equation:

$$
\mathrm{A}(\omega) \cdot \mathrm{B}(\omega) = \mathrm{C}(\omega), \quad (27)
$$

where

$$
\mathrm{A}(\omega) = \begin{pmatrix} 0,5 & 0 & \kappa\bar{U}_{,x}(L,\omega) & -\kappa\bar{U}(L,\omega) \\ \kappa\bar{U}_{,x}(L,\omega) & \kappa\bar{U}(L,\omega) & 0,5 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix},
$$

$$
B(\omega) = \left(\bar{\theta}_1(\omega), \bar{q}_1(\omega), \bar{\theta}_2(\omega), \bar{q}_2(\omega)\right),
$$

$$
\mathrm{C}(\omega) = (\bar{Q}_1(0,\omega), \bar{Q}_2(L,\omega), \bar{b}_3(\omega), \bar{b}_4(\omega)).
$$

The last two equations in the system (27) are determined by boundary conditions at the ends of the segment, which are known for BVP:

$$
\begin{bmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \begin{bmatrix} \bar{\theta}_1(\omega) \\ \bar{q}_1(\omega) \end{bmatrix} + \begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} \bar{\theta}_2(\omega) \\ \bar{q}_2(\omega) \end{bmatrix} = \begin{bmatrix} \bar{b}_3(\omega) \\ \bar{b}_4(\omega) \end{bmatrix}. \quad (28)
$$

By given coefficients a_{ij} and right-hand side $b_i(\omega)$, we have four equations (27) for definition of four boundary functions. The solution of Eqs (27) has the form:

$$
B(\omega) = \mathrm{A}^{-1}(\omega) \times C(\omega), \quad (29)
$$

where $\mathbf{A}^{-1}(\omega)$ is the inverse matrix of $\mathrm{A}(\omega)$.

So, all boundary functions are defined; therefore, the Fourier transform (17) for solving the boundary value problem is constructed. Using the inverse Fourier transform (16), we obtain the original $\theta(x,t)$ on the segment [0, L].

We use the solution (17) and Eqs (24) for constructing the solution of BVP on the linear graph.

7 Algebraic equations for determining unknown boundary functions on a heat linear graph

We return to the consideration of BVP for the heat equation on a heat linear graph (Fig. 1). On each segment L_j of the graph, we have the system of linear algebraic equations for determining four boundary functions:

$$\begin{pmatrix} 1 & 0 & -\cos(k_j L_j) & \frac{\sin(k_j L_j)}{k_j(\omega)} \\ -\cos(k_j L_j) & -\frac{\sin(k_j L_j)}{k_j(\omega)} & 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{\theta}_1^j(\omega) \\ \bar{q}_1^j(\omega) \\ \bar{\theta}_2^j(\omega) \\ \bar{q}_2^j(\omega) \end{pmatrix} = \begin{pmatrix} \bar{F}_1^j(\omega) \\ \bar{F}_2^j(\omega) \end{pmatrix}, \tag{30}$$
$$k_j(\omega) = (1+i)\sqrt{\frac{\omega}{2\kappa_j}}, \quad j = 1, ..., N.$$

Here, j denotes the number of the corresponding graph segment, and $\bar{F}_1^j(\omega) = 2\bar{Q}_1^j(0,\omega)$, $\bar{F}_2^j(\omega) = 2\bar{Q}_2^j(L,\omega)$. So, we have $2N$ equation for the determination of $4N$ boundary functions at every edge: $B(\omega) = \left(\bar{\theta}_1^1, \bar{q}_1^1, \bar{\theta}_2^1, \bar{q}_2^1, ..., \bar{\theta}_1^N, \bar{q}_1^N, \bar{\theta}_2^N, \bar{q}_2^N\right)$. Also, we have 2 conditions on the ends of the graph (4) or (5) and $2(N-1)$ transmission conditions at the node points of this graph (6). So we have the full system $4N$ equations for determination of $4N$ boundary functions at every edge.

Theorem 2. *Resolving system of equations of Dirichlet boundary value problem* (2), (4), (6) *on a heat linear graph with N different segments has the form:*

$$\mathbf{\Lambda}1(\omega) \times B(\omega) = C(\omega), \tag{31}$$

Resolving system of equations of Neumann boundary value problem (2), (4), (6) *on a heat linear graph with N different segments has the form:*

$$\mathbf{\Lambda}2(\omega) \times B(\omega) = C(\omega), \tag{32}$$

Here the matrices $\mathbf{\Lambda}1(\omega)$, $\mathbf{\Lambda}2(\omega)$ *have the following dimensions* $4N \times 4N$.

The first $2N$ *lines along the diagonal* $\mathbf{\Lambda}1(\omega)$, $\mathbf{\Lambda}2(\omega)$ *contain the connection matrices* (30) *of unknown boundary functions of edges. The remaining elements are zero:*

$$\{\Lambda_{ij}\} = \begin{pmatrix} \Lambda_1(\omega) & O_{2\times4} & O_{2\times4} & ... & & ... & ... & ... & O_{2\times4} \\ O_{2\times4} & \Lambda_2(\omega) & O_{2\times4} & ... & & ... & ... & ... & O_{2\times4} \\ O_{2\times4} & ... & ... & ... & . & ... & ... & ... & O_{2\times4} \\ O_{2\times4} & O_{2\times4} & O_{2\times4} & ... & & ... & ... & ... & \Lambda_N(\omega) \end{pmatrix},$$

$$i = 1, ..., 2N, \quad j = 1, ..., 4N.$$

Here

$$\Lambda_j(\omega) = \begin{pmatrix} 1 & 0 & -\cos(k_j L_j) & \frac{\sin(k_j L_j)}{k_j(\omega)} \\ -\cos(k_j L_j) & -\frac{\sin(k_j L_j)}{k_j(\omega)} & 1 & 0 \end{pmatrix},$$

$$O_{2\times 4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The next $N-1$ *rows of the matrices* $\mathbf{\Lambda}1(\omega)$, $\mathbf{\Lambda}2(\omega)$ *contain the continuity conditions* (6) *at node points*

$$\{\Lambda_{ij}\} = \begin{pmatrix} 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & ... & ... & ... & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & ... & ... & ... & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ... & ... & ... & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ... & ... & ... & ... & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$i = 2N+1, ..., 3N, \quad j = 1, ..., 4N$$

The next $N-1$ *rows of these matrices contain the conditions* (7) *at node points:*

$$\{\Lambda_{ij}\} = \begin{pmatrix} 0 & 0 & 0 & \kappa_1 & 0 & -\kappa_2 & 0 & 0 & ... & ... & ... & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \kappa_2 & ... & ... & ... & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ... & ... & ... & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ... & ... & ... & ... & 0 & -\kappa_N & 0 & 0 \end{pmatrix},$$

$$i = 3N-1, ..., 4N-2, \quad j = 1, ..., 4N.$$

The last two rows of the matrix are the boundary conditions at the ends of the graph. For the Dirichlet problem, this is condition (4)*:*

$$\{\Lambda_{ij}\} = \left\{ \begin{matrix} 1 & 0 & 0 & 0 & ... & ... & ... & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & ... & ... & ... & 0 & 0 & 1 & 0 \end{matrix} \right\},$$

$$i = 4N-1, 4N; \quad j = 1, ..., 4N$$

For the Neumann problem, this is condition (5)*:*

$$\{\Lambda_{ij}\} = \left\{ \begin{matrix} 0 & \kappa_1 & 0 & 0 & ... & ... & ... & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & ... & ... & ... & 0 & 0 & 0 & \kappa_N \end{matrix} \right\},$$

$$i = 4N-1, 4N; \quad j = 1, ..., 4N$$

Theorem 4 follows from Theorem 3.

Theorem 3. *The solution to boundary value problems* (1)–(6) *on the thermal graph has the form:*

$$\bar{\theta}_j(x_j,\omega)H(L-x_j)H(x_j) =$$
$$= H(x_j)\int_0^{L_j} \bar{U}\,(x_j-y,\omega)\,F_2^j(y,\omega)dy + \kappa_j H(x_j)\int_0^{L} \bar{U}_j(x_j-y,\omega)\theta_0^j(y)dy+$$
$$+\,\kappa_j\bar{q}_2^j(\omega)H(x)\bar{U}_j(L_j-x_j,\omega) - \kappa_j\bar{q}_1^j(\omega)H(L_j-x_j)\bar{U}_j(x_j,\omega)-$$
$$-\,\kappa_j\bar{\theta}_2^j(\omega)H(x_j)\bar{U}_{j,x}(L_j-x_j,\omega) - \kappa_j\bar{\theta}_{j,j}^j(\omega)H(L_j-x_j)\bar{U}_{j,x_j}(x_j,\omega). \quad (33)$$

Here $\left(\bar{\theta}_1^1(\omega),\bar{q}_1^1(\omega),\bar{\theta}_2^1(\omega),\bar{q}_2^1(\omega),......,\bar{\theta}_1^N(\omega),\bar{q}_1^N(\omega),\bar{\theta}_2^N(\omega),\bar{q}_2^N(\omega)\right) = B(\omega)$, *where* $B(\omega)$ *are the solution of resolving system equations:*

for Dirichlet problem: $B(\omega) = \mathbf{\Lambda}1^{-1}(\omega)\times C(\omega)$,

for Neumann problem: $B(\omega) = \mathbf{\Lambda}2^{-1}(\omega)\times C(\omega)$.

So we defined the Fourier transformant of the solution of BVPs on the thermal graph. Then by using the formula of inverse Fourier transformations (16) we calculate the original solution—-the temperature at every point of the graph. So, both BVPs have been solved.

Conclusion

Using the method of generalized functions, we solved the boundary value problems of thermal conductivity on the thermal linear graph, which can be used to study various network-like structures under conditions of thermal heating (cooling). A unified technique has been developed for solving various boundary value problems typical for practical applications.

The action of heat sources can be modeled by both regular and singular generalized functions under various boundary conditions at the ends of the graph. The obtained regular integral representations of generalized solutions make it possible to determine the temperature and heat flows on each element of the graph, at any point of it, for stationary oscillations with a constant frequency and in the case of periodic oscillations.

For nonstationary processes, performing the inverse Fourier transform in time, we obtain the original solution in the original space-time. The construction of the original depends on the boundary conditions and the type of functions that determine them and should be considered separately for a specific boundary value problem. The generalized function method presented here makes it possible to solve a wide class of boundary value problems with local and connected boundary conditions at the ends of the edges of the graph and various transmission conditions at its nodes and can be extended to network structures of very different types. It distinguishes this method from all others that are used to solve similar problems.

It should be noted that if we change the transmission condition (6), setting $Q_j(t) = 0$ and $q_j^2(t) = q_{j+1}^1(t)$, i.e. introduce the continuity condition of the derivatives with respect to

x at the nodes of the linear graph, then the solution to this problem for the heat equation with discontinuous coefficients is also constructed by this method. Only in the rows of the matrix of the resolving system of equations that contain this transmission condition, should we put 1 instead of κ_j. The issues of the correctness of setting such problems for parabolic equations with discontinuous coefficients on a certain class of functions were considered in [16], [17], [18].

In [19], a boundary value problem for the heat equation with a piecewise constant thermal conductivity coefficient with one discontinuity point under homogeneous boundary conditions with the condition of equality of heat fluxes at the discontinuity point was considered.

The generalized function method presented here makes it possible to solve a wide class of boundary value problems with local and connected boundary conditions at the ends of the edges of the graph and various transmission conditions at its node and can be extended to network structures of very different types. It distinguishes this method from all others that are used to solve similar problems.

The proposed method applies to a wide range of BVPs, including those on mesh structures.

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Айнакеева Н.Ж., Алексеева Л.А., Приказчиков Д.А. Жылулық теңдеу үшін Дирихле және Нейман есептері сызықтық көп буынды жылу графтары және олардың шешімдері

Жылулық қыздыру (салқындату) жағдайында әртүрлі құрылымдарды зерттеу үшін пайдаланылуы мүмкін сызықтық жылулық граф бойынша жылу өткізгіштіктің шекаралық есептері қарастырылады. Мұнда жалпыланған функциялық әдіс негізінде инженерлік қолданбаларға тән жылу өткізгіштіктің шекаралық есептерін шешудің бірыңғай әдістемесі әзірленді. Жиекте және жылу сызығының графында жылу өткізгіштіктің стационарлы емес және стационар шекаралық есептерінің жалпыланған шешімдері графтың шеттерінде әртүрлі шекаралық шарттарда және оның түйінінде жалпыланған Кирхгоф шарттарында құрастырылады. Негізгі шешімнің Фурье түрлендірушісінің қасиеттерін пайдалана отырып, аналитикалық түрде шекаралық есептердің шешімдерінің тұрақты интегралдық бейнелері алынады. Алынған шешімдер әртүрлі типтегі жылу көздерін модельдеуге мүмкіндік береді, соның ішінде сингулярлы жалпыланған функцияларды пайдаланады. Мұнда келтірілген жалпыланған функциялар әдісі графтың шеттерінің шеттеріндегі жергілікті және байланысқан шекаралық шарттармен және оның түйіндеріндегі әртүрлі берілу жағдайларымен шекаралық есептердің кең класын шешуге мүмкіндік береді.

Түйін сөздер: жылу өткізгіштік, жалпыланған функциялар, іргелі және жалпылама шешім, Фурье түрлендіруі, шекаралық теңдеулерді шешу, сызықтық граф.

Айнакеева Н.Ж., Алексеева Л.А., Приказчиков Д.А. Задача Дирихле и Неймана для уравнения теплопроводности линейных многозвенных тепловых графов и их решения

Рассматриваются краевые задачи теплопроводности на линейном тепловом графе, которые могут быть использованы для исследования различных конструкций в условиях теплового нагрева (охлаждения). Здесь на основе метода обобщенных функций разработана единая методика решения краевых задач теплопроводности, типичная для инженерных приложений. Построены обобщенные решения нестационарных и стационарных краевых задач теплопроводности на ребре и на линейном тепловом графе при различных граничных условиях на концах графа и обобщенных условиях Кирхгофа в его узле. Используя свойства трансформанты Фурье фундаментального решения, получены регулярные интегральные представления решений краевых задач в аналитическом виде. Полученные решения позволяют моделировать источники тепла различных типов, в том числе с использованием сингулярных обобщенных функций. Представленный здесь метод обобщенных функций позволяет решать широкий класс краевых задач с локальными и связанными граничными условиями на концах ребер графа и различными условиями пропускания в его узлах.

Ключевые слова: теплопроводность, обобщенные функции, фундаментальное и обобщенное решение, преобразование Фурье, разрешение граничных уравнений, линейный граф.