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Weighted and Logarithmic Caffarelli-Kohn-Nirenberg type inequalities on stratified groups and applications

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Abstract. The classical Caffarelli–Kohn–Nirenberg inequalities, originally established in Euclidean space in the 1980s, provide a unified framework for interpolation between Sobolev and Hardy inequalities. Their extension to stratified (or homogeneous Carnot) Lie groups began in the early 2000s, motivated by subelliptic analysis and geometric measure theory, revealing rich interactions between group structure, dilation symmetry, and functional inequalities. In this paper, we establish the weighted and logarithmic Caffarelli-Kohn-Nirenberg type inequalities on a stratified Lie group. As a consequence, we can apply it to prove the weighted ultracontractivity of positive strong solutions to

$$d^{\alpha}\frac{\partial u}{\partial t} = \mathcal{L}_p(d^{\alpha}u)^m,$$

where $\mathcal{L}_p f = \nabla_H(|\nabla_H f|^{p-2}\nabla_H f)$ is a p-sub-Laplacian, d is a homogeneous norm associated with a fundamental solution for sub-Laplacian and $\alpha \in \mathbb{R}$, $1 < p < Q$.

Keywords. Caffarelli-Kohn-Nirenberg inequality; logarithmic Caffarelli-Kohn-Nirenberg inequality; stratified Lie group.

1 Introduction

In 1984, Caffarelli, Kohn and Nirenberg published a celebrated work [4] where they derived the general case of inequalities such as Gagliardo-Nirenberg inequalities, Sobolev inequalities, Hardy-Sobolev inequalities, Nash's inequalities and Hardy's inequalities in the following form:

Theorem 1 (Caffarelli-Kohn-Nirenberg inequality [4]). *Let $p, q, r, \alpha, \beta, \sigma$ and a be fixed pa-*

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rameters in $\mathbb{R}$ *satisfying*

$$p, q \geq 1, \ r > 0, \ 0 \leq a \leq 1, \tag{1}$$

$$\frac{1}{p} + \frac{\alpha}{n}, \ \frac{1}{q} + \frac{\beta}{n}, \ \frac{1}{r} + \frac{\gamma}{n} > 0, \tag{2}$$

where

$$\gamma = a\sigma + (1-a)\beta. \tag{3}$$

There exists a positive constant C *such that the following inequality holds for all* $f \in C_0^\infty(\mathbb{R}^n)$

$$|||x|^\gamma f||_{L^r(\mathbb{R}^n)} \leq C|||x|^\alpha |\nabla f|||^a_{L^p(\mathbb{R}^n)} |||x|^\beta f||^{1-a}_{L^q(\mathbb{R}^n)}, \tag{4}$$

if and only if the following relations hold:

$$\frac{1}{r} + \frac{\gamma}{n} = a\left(\frac{1}{p} + \frac{\alpha - 1}{n}\right) + (1-a)\left(\frac{1}{q} + \frac{\beta}{n}\right), \tag{5}$$

(this is dimensional balance),

$$0 \leq \alpha - \sigma \ \textit{if}\ a > 0, \tag{6}$$

$$\alpha - \sigma \leq 1 \ \textit{if}\ a > 0 \ \textit{and}\ \frac{1}{p} + \frac{\alpha - 1}{n} = \frac{1}{r} + \frac{\gamma}{n}. \tag{7}$$

Note that when $a = 1$, $\alpha = 0$, $\gamma = -s/r$, $0 \leq s \leq p < n$ and $p^*(s) = r = \frac{p(n-s)}{n-p}$, Theorem 1 gives the Hardy-Sobolev inequality that is the interpolation of Sobolev's inequality ($s = 0$) and Hardy's inequality ($s = p$) such as

$$\left(\int_{\mathbb{R}^n} \frac{|f|^{p^*(s)}}{|x|^s} dx\right)^{\frac{1}{p^*(s)}} \leq C(n, p, s) \left(\int_{\mathbb{R}^n} |\nabla f|^p dx\right)^{\frac{1}{p}}. \tag{8}$$

Since then, the Caffarelli-Kohn-Nirenberg (CKN) inequality has been extended in different directions. For instance, in the Euclidean setting, the sharpness of constants and extremal functions in the CKN inequality was investigated by many authors such as Bouchez-Willem [3], Catrina-Wang [5], Chou-Chu [6], Del Pino-Dolbeault [7], Dolbeault-Esteban-Laptev-Loss [8], Lin-Wang [17], and Liu-Zhao [18]. In recent years, CKN inequality has been actively investigated in the setting of the Heisenberg group, stratified Lie group, and homogeneous groups. For example, Garofalo-Lanconelli [9], Feng-Niu-Qiao [13], Han [11], Zhang-Han-Dou [28], Han-Niu-Zhang [12] on Heisenberg group, Ruzhansky-Suragan [23, 26], Ruzhansky-Suagan-Yessirkegenov [24, 25], S.-Suragan [27] on stratified groups, Ozawa-Ruzhansky-Suragan [20] on homogeneous groups.

Motivated by results in [11], [19] and [13], in this paper, we investigate the weighted and logarithmic Caffarelli-Kohn-Nirenberg type inequalities on a stratified Lie group. As a consequence, we can apply it to prove the weighted ultracontractivity of positive strong solutions to the equation of the form

$$d^\alpha \frac{\partial u}{\partial t} = \mathcal{L}_p (d^\alpha u)^m, \tag{9}$$

where $\mathcal{L}_p f = \nabla_H(|\nabla_H f|^{p-2}\nabla_H f)$ is a p-sub-Laplacian, d is a homogeneous norm associated with a fundamental solution for sub-Laplacian and $\alpha \in \mathbb{R}$, $1 < p < Q$. The outline of this work has the following form:

Section 2 is devoted to obtaining a weighted Caffarelli-Kohn-Nirenberg type inequality with respect to the homogeneous norm associated with a fundamental solution for sub-Laplacian. First, we introduce the propositions of Hardy and Sobolev inequalities on a stratified group. Combining these inequalities, we prove a weighted Hardy-Sobolev-type inequality with a homogeneous norm associated with a fundamental solution for sub-Laplacian. As a result, we could derive a weighted Caffarelli-Kohn-Nirenberg-type inequality.

Section 3 presents the logarithmic and parametric logarithmic Caffarelli-Kohn-Nirenberg inequalities with a homogeneous norm associated with a fundamental solution for sub-Laplacian on a stratified group. There are obtained the weighted and logarithmic Hölder inequalities. Using those inequalities and the weighted Caffarelli-Kohn-Nirenberg type inequalities in the case $\gamma = \alpha = \beta$, we prove the logarithmic and parametric-logarithmic Caffarelli-Kohn-Nirenberg inequalities.

Section 4 is dedicated to proving the weighted ultracontractivity of positive strong solutions to a kind of evolution equation (9) by using the parametric logarithmic Caffarelli-Kohn-Nirenberg inequality.

1.1. Preliminaries

Let $\mathbb{G}$ be a stratified Lie group (or a homogeneous Carnot group), with dilation structure δ_λ and Jacobian generators $X_1, \dots, X_N$, so that N is the dimension of the first stratum of $\mathbb{G}$. We refer to [15] and [2], or to a recent book [22] for extensive discussions of stratified Lie groups and their properties. Let Q be the homogeneous dimension of $\mathbb{G}$. The sub-Laplacian on $\mathbb{G}$ is given by

$$\mathcal{L} = \sum_{k=1}^{N} X_k^2. \tag{10}$$

It was shown by Folland [15] that the sub-Laplacian has a unique fundamental solution ε,

$$\mathcal{L}\varepsilon = \delta,$$

where δ denotes the Dirac distribution with singularity at the neutral element 0 of $\mathbb{G}$. The fundamental solution $\varepsilon(x,y)=\varepsilon(y^{-1}x)$ is homogeneous of degree $-Q+2$ and can be written in the form

$$\varepsilon(x,y)=[d(y^{-1}x)]^{2-Q}, \tag{11}$$

for some homogeneous d which is called the $\mathcal{L}$-gauge. Thus, the $\mathcal{L}$-gauge is a symmetric homogeneous (quasi-) norm on the stratified group $\mathbb{G}=(\mathbb{R}^n,\circ,\delta_\lambda)$, that is,

- $d(x)>0$ if and only if $x\neq 0$,
- $d(\delta_\lambda(x))=\lambda d(x)$ for all $\lambda>0$ and $x\in\mathbb{G}$,
- $d(x^{-1})=d(x)$ for all $x\in\mathbb{G}$.

We also recall that the standard Lebesgue measure dx on $\mathbb{R}^n$ is the Haar measure for $\mathbb{G}$ (see, e.g. [16, Proposition 1.6.6]). The left-invariant vector field X_j has an explicit form and satisfies the divergence theorem, see e.g. [16] for the derivation of the exact formula: more precisely, we can write

$$X_k=\frac{\partial}{\partial x'_k}+\sum_{l=2}^{r}\sum_{m=1}^{N_l}a_{k,m}^{(l)}(x',...,x^{(l-1)})\frac{\partial}{\partial x_m^{(l)}}, \tag{12}$$

with $x=(x',x^{(2)},\ldots,x^{(r)})$, where r is the step of $\mathbb{G}$ and $x^{(l)}=(x_1^{(l)},\ldots,x_{N_l}^{(l)})$ are the variables in the l^{th} stratum, see also [16, Section 3.1.5] for a general presentation. The horizontal gradient is given by

$$\nabla_H:=(X_1,\ldots,X_N),$$

and the horizontal divergence is defined by

$$\mathrm{div}_H f:=\nabla_H\cdot f.$$

The p-sub-Laplacian is defined by

$$\mathcal{L}_p f:=\mathrm{div}_H(|\nabla_H f|^{p-2}\nabla_H f),\quad 1<p<\infty. \tag{13}$$

2 Weighted Caffarelli-Kohn-Nirenberg-type inequalities

Proposition 2 ([11]). *For all vectors $v_1,v_2\in\mathbb{R}^n$, we have the following expressions such as*

- *For $p\leq 2$ we have*

$$|v_1+v_2|^p-|v_1|^p-p|v_1|^{p-2}\langle v_1,v_2\rangle\leq C(p)|v_2|^p.$$

- *For $p > 2$ we have*

$$|v_1 + v_2|^p - |v_1|^p - p|v_1|^{p-2}\langle v_1, v_2\rangle \leq \frac{p(p-1)}{2}(|v_1| + |v_2|)^{p-2}|v_2|^2,$$

where $\langle v_1, v_2\rangle$ is the inner product.

Proposition 3 (Hardy type inequality)**.** *Let $\mathbb{G}$ be a stratified Lie group and let $d = \varepsilon^{\frac{1}{2-Q}}$, where ε is the fundamental solution to the sub-Laplacian $\mathcal{L}$. Suppose that $Q \geq 3$ then for every $u \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have the following Hardy type inequality*

$$\int_{\mathbb{G}} \frac{|\nabla_H d|^p}{d^p}|u|^p dx \geq \left(\frac{p}{Q-p}\right)^p \int_{\mathbb{G}} |\nabla_H u|^p dx. \tag{14}$$

Note that the Hardy type inequality (14) was obtained by Garofalo-Lanconelli [9], D'Ambrosio [1], Goldstein-Kombe-Yener [10], and authors [21].

Proposition 4 (Sobolev inequality)**.** *Let $\mathbb{G}$ be a stratified Lie group, and let C be a positive constant. Then for every function $u \in C_0^\infty(\mathbb{G})$ we have*

$$\left(\int_{\mathbb{G}} |u|^{p^*} dx\right)^{\frac{1}{p^*}} \leq C\left(\int_{\mathbb{G}} |\nabla_H u|^p dx\right)^{\frac{1}{p}}, \tag{15}$$

where $p^* = \frac{pQ}{Q-p}$ with $1 < p < Q$.

Note that in the setting of the Heisenberg group and stratified Lie group, Sobolev inequalities (15) were obtained by Folland-Stein [14] and Folland [15], respectively.

Lemma 5 (Hardy-Sobolev type inequality)**.** *Let $\mathbb{G}$ be a stratified Lie group and let $d = \varepsilon^{\frac{1}{2-Q}}$, where ε is the fundamental solution to the sub-Laplacian $\mathcal{L}$. Then there exists a positive constant $C_1(s, p, Q)$ such that for all functions $u \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have*

$$\int_{\mathbb{G}} \frac{|\nabla_H d|^s}{d^s}|u|^{p_*(s)} dx \leq C_1\left(\int_{\mathbb{G}} |\nabla_H u|^p dx\right)^{\frac{Q-s}{Q-p}}, \tag{16}$$

where $p_*(s) = \frac{p(Q-s)}{Q-p}$, $0 \leq s \leq p$ and $1 < p < Q$.

Proof of Lemma 5. The outline of the proof is to apply the Hölder inequality with $p_*(s) = \left(1 - \frac{s}{p}\right)p^* + \frac{s}{p}p$, the Hardy type inequality (14), and the Sobolev inequality (15), respectively.

Then we have

$$\begin{aligned}\int_{\mathbb{G}} \frac{|\nabla_H d|^s}{d^s}|u|^{p_*(s)}dx &\leq \left(\int_{\mathbb{G}} |u|^{p^*}dx\right)^{1-\frac{s}{p}} \left(\int_{\mathbb{G}} \frac{|\nabla_H d|^p}{d^p}|u|^p dx\right)^{\frac{s}{p}} \\ &\leq \left(C^{p^*}\left(\int_{\mathbb{G}} |\nabla_H u|^p dx\right)^{\frac{p^*}{p}}\right)^{1-\frac{s}{p}} \left(\left(\frac{p}{Q-p}\right)^p \int_{\mathbb{G}} |\nabla_H u|^p dx\right)^{\frac{s}{p}} \\ &= C_1 \left(\int_{\mathbb{G}} |\nabla_H u|^p dx\right)^{\frac{Q-s}{Q-p}},\end{aligned}$$

where $C_1 = C^{p^*(1-s/p)} \left(\frac{p}{Q-p}\right)^p$. This proves Lemma 5. □

Now, we prove the following weighted Hardy-Sobolev inequality on a stratified Lie group.

Theorem 6 (Weighted Hardy-Sobolev inequality)**.** *Let $\mathbb{G}$ be a stratified Lie group and let $d = \varepsilon^{\frac{1}{2-Q}}$, where ε is the fundamental solution to the sub-Laplacian $\mathcal{L}$. Then*

$$p_*(p,s,Q) = \frac{p(Q-s)}{Q-p}, \text{ and } 1 < p < Q \text{ with } Q \geq 3,$$
$$0 \leq s \leq p \text{ and } \alpha \geq \frac{p-Q}{p},$$

there exists a positive constant $C(p,s,\alpha,Q)$ such that for all functions $u \in W^{1,p}_\alpha(\mathbb{G}\backslash\{0\})$ we have

$$\int_{\mathbb{G}} \frac{|\nabla_H d|^s}{d^s}|d^\alpha u|^{p_*(p,s,Q)}dx \leq C\left(\int_{\mathbb{G}} |d^\alpha \nabla_H u|^p dx\right)^{\frac{Q-s}{Q-p}}, \tag{17}$$

where $W^{1,p}_\alpha(\mathbb{G}\backslash\{0\})$ is the closure of $C_0^\infty(\mathbb{G}\backslash\{0\})$ with respect to the norm

$$||u||_{W^{1,p}_\alpha(\mathbb{G})} := \left(\int_{\mathbb{G}} |d^\alpha \nabla_H u|^p dx\right)^{\frac{1}{p}}.$$

Remark 7. *Note that Lemma 5 is inferred from Theorem 6 when $\alpha = 0$.*

Proof of Theorem 6. The proof is divided in the cases such as $1 < p \leq 2$ and $2 < p < Q$. In each case, we make use of Proposition 2 and Lemma 5.

Case $1 < p \leq 2$. The outline of proof consists of the following steps: we take $v = d^\alpha u$, then apply Proposition 2, the integral by parts, the divergence theorem, and inequality (14), respectively. Then for $v_1 = \alpha v d^{-1}\nabla_H d$ and $v_2 = \nabla_H v - \alpha v d^{-1}\nabla_H d$ in Proposition 2, we have

$$
\begin{aligned}
C(p)\int_{\mathbb{G}} |d^{\alpha}\nabla_H u|^p dx &= C(p)\int_{\mathbb{G}} \left|\nabla_H v - \frac{\alpha v \nabla_H d}{d}\right|^p dx \\
\geq \int_{\mathbb{G}} &\left[|\nabla_H v|^p - |\alpha|^p \frac{|v|^p|\nabla_H d|^p}{d^p}\right] dx \\
&- p\alpha|\alpha|^{p-2}\int_{\mathbb{G}} \frac{v|v|^{p-2}}{d^{p-1}}|\nabla_H d|^{p-2}\langle \nabla_H d, \nabla_H v - \alpha d^{-1} v \nabla_H d\rangle dx \\
= \int_{\mathbb{G}} &\left[|\nabla_H v|^p + (p-1)|\alpha|^p \frac{|\nabla_H d|^p}{d^p}|v|^p\right] dx - p\alpha|\alpha|^{p-2}\int_{\mathbb{G}} \frac{v|v|^{p-2}}{d^{p-1}}|\nabla_H d|^{p-2}\langle \nabla_H d, \nabla_H v\rangle dx \\
= \int_{\mathbb{G}} &\left[|\nabla_H v|^p + (p-1)|\alpha|^p \frac{|\nabla_H d|^p}{d^p}|v|^p\right] dx - \alpha|\alpha|^{p-2}\int_{\mathbb{G}} \frac{|\nabla_H d|^{p-2}}{d^{p-1}}\langle \nabla_H d, \nabla_H |v|^p\rangle dx \\
= \int_{\mathbb{G}} &\left[|\nabla_H v|^p + (p-1)|\alpha|^p \frac{|\nabla_H d|^p}{d^p}|v|^p\right] dx + \alpha|\alpha|^{p-2}\int_{\mathbb{G}} |v|^p \mathrm{div}_H(|\nabla_H d|^{p-2} d^{1-p}\nabla_H d) dx \\
\geq \int_{\mathbb{G}} &\left[|\nabla_H v|^p + (p-1)|\alpha|^p \frac{|\nabla_H d|^p}{d^p}|v|^p\right] dx + \alpha|\alpha|^{p-2}(Q-p)\int_{\mathbb{G}} \frac{|\nabla_H d|^p}{d^p}|v|^p dx \\
= \int_{\mathbb{G}} &|\nabla_H v|^p dx + \alpha|\alpha|^{p-2}(Q-p+\alpha(p-1))\int_{\mathbb{G}} \frac{|\nabla_H d|^p}{d^p}|v|^p dx \\
\geq &\left(1 + \alpha|\alpha|^{p-2}(Q-p+\alpha(p-1))\left(\frac{p}{Q-p}\right)^p\right)\int_{\mathbb{G}} |\nabla_H v|^p dx.
\end{aligned}
$$

So we arrive at

$$
C(p)\int_{\mathbb{G}} |d^{\alpha}\nabla_H u|^p dx \geq C_1(\alpha, Q, p)\int_{\mathbb{G}} |\nabla_H v|^p dx. \tag{18}
$$

By applying Lemma 5 on the right hand side of inequality (18) and $v = d^{\alpha}u$, then we obtain

$$
C(\alpha, s, p, Q)\left(\int_{\mathbb{G}} |d^{\alpha}\nabla_H u|^p dx\right)^{\frac{Q-s}{Q-p}} \geq \int_{\mathbb{G}} \frac{|\nabla_H d|^s}{d^s}|d^{\alpha}u|^{p_*(s)} dx.
$$

This proves inequality (17) in the case $1 < p \leq 2$.

Case $2 < p < Q$. The outline of proof is to estimate in both the upper and lower bound of the following expression

$$
2^{p-2}\frac{p(p-1)}{2}\int_{\mathbb{G}} (|\alpha d^{-1} v \nabla_H d| + |\nabla_H v|)^{p-2}|\nabla_H v - \alpha d^{-1} v \nabla_H d|^2 dx. \tag{19}
$$

The upper bound estimate is obtained by applying Hölder inequality, Minkowski inequality and Hardy type inequality (14), respectively. On the other hand, lower bound estimate is acquired by making use of Proposition 2 with $v_1 = \alpha d^{-1} v \nabla_H d$ and $v_2 = \nabla_H v - \alpha d^{-1} v \nabla_H d$, integral by parts, the divergence theorem and Hardy type inequality (14), respectively.

Let us estimate the upper bound of expression (19), by denoting $C_p = 2^{p-2}\frac{p(p-1)}{2}$ we have

$$\begin{aligned}
&C_p \int_{\mathbb{G}} (|\alpha d^{-1} v \nabla_H d| + |\nabla_H v|)^{p-2} |\nabla_H v - \alpha d^{-1} v \nabla_H d|^2 dx \\
\leq & C_p \left(\int_{\mathbb{G}} (|\alpha d^{-1} v \nabla_H d| + |\nabla_H v|)^p dx \right)^{\frac{p-2}{p}} \left(\int_{\mathbb{G}} |\nabla_H v - \alpha d^{-1} v \nabla_H d|^p dx \right)^{\frac{2}{p}} \\
\leq & C_p \left[\left(\int_{\mathbb{G}} |\alpha d^{-1} v \nabla_H d|^p dx \right)^{\frac{1}{p}} + \left(\int_{\mathbb{G}} |\nabla_H v|^p dx \right)^{\frac{1}{p}} \right]^{p-2} \left(\int_{\mathbb{G}} |\nabla_H v - \alpha d^{-1} v \nabla_H d|^p dx \right)^{\frac{2}{p}} \\
\leq & C_p \left(1 + |\alpha \frac{p}{Q-p}| \right)^{p-2} \left(\int_{\mathbb{G}} |\nabla_H v|^p dx \right)^{\frac{p-2}{p}} \left(\int_{\mathbb{G}} |\nabla_H v - \alpha d^{-1} v \nabla_H d|^p dx \right)^{\frac{2}{p}}. \qquad (20)
\end{aligned}$$

Now we estimate the lower bound of expression (19), by using Proposition 2. We have

$$\begin{aligned}
&2^{p-2} \frac{p(p-1)}{2} (|\alpha d^{-1} v \nabla_H d| + |\nabla_H v|)^{p-2} |\nabla_H v - \alpha d^{-1} v \nabla_H d|^2 \\
\geq & \frac{p(p-1)}{2} (2|\alpha d^{-1} v \nabla_H d| + |\nabla_H v|)^{p-2} |\nabla_H v - \alpha d^{-1} v \nabla_H d|^2 \\
\geq & \frac{p(p-1)}{2} (|\alpha d^{-1} v \nabla_H d| + |\nabla_H v - \alpha d^{-1} v \nabla_H d|)^{p-2} |\nabla_H v - \alpha d^{-1} v \nabla_H d|^2 \\
\geq & |\nabla_H v|^p - |\alpha d^{-1} v \nabla_H d|^p - p |\alpha d^{-1} v \nabla_H d|^{p-2} \langle \alpha d^{-1} v \nabla_H d, \nabla_H v - \alpha d^{-1} v \nabla_H d \rangle \\
= & |\nabla_H v|^p + (p-1)|\alpha|^p \frac{|v|^p}{d^p} |\nabla_H d|^p - p\alpha |\alpha|^{p-2} v |v|^{p-2} |\nabla_H d|^{p-2} \langle \nabla_H d, \nabla_H v \rangle.
\end{aligned}$$

By integrating both side of the above inequality, we arrive at

$$\begin{aligned}
&C_p \int_{\mathbb{G}} (|\alpha d^{-1} v \nabla_H d| + |\nabla_H v|)^{p-2} |\nabla_H v - \alpha d^{-1} v \nabla_H d|^2 dx \\
\geq & \int_{\mathbb{G}} |\nabla_H v|^p dx + (p-1)|\alpha|^p \int_{\mathbb{G}} \frac{|v|^p}{d^p} |\nabla_H d|^p dx - \alpha |\alpha|^{p-2} \int_{\mathbb{G}} |\nabla_H d|^{p-2} \langle \nabla_H d, \nabla_H |v|^p \rangle dx \\
= & \int_{\mathbb{G}} |\nabla_H v|^p dx + (p-1)|\alpha|^p \int_{\mathbb{G}} \frac{|v|^p}{d^p} |\nabla_H d|^p dx + \alpha |\alpha|^{p-2} \int_{\mathbb{G}} |v|^p \mathrm{div}_H (|\nabla_H d|^{p-2} \nabla_H d) dx \\
\geq & \int_{\mathbb{G}} |\nabla_H v|^p dx + \alpha |\alpha|^{p-2} (Q - p + \alpha(p-1)) \int_{\mathbb{G}} \frac{|\nabla_H d|^p}{d^p} |v|^p dx \\
\geq & C_1 \int_{\mathbb{G}} |\nabla_H v|^p dx. \qquad (21)
\end{aligned}$$

By combining (20) and (21), we obtain

$$C(p) \int_{\mathbb{G}} |d^\alpha \nabla_H u|^p dx \geq C_1(\alpha, Q, p) \int_{\mathbb{G}} |\nabla_H v|^p dx. \qquad (22)$$

By applying Lemma 5 on the right hand side of inequality (22) and $v = d^{\alpha}u$, then we obtain

$$C(\alpha, s, p, Q)\left(\int_{\mathbb{G}} |d^{\alpha}\nabla_H u|^p dx\right)^{\frac{Q-s}{Q-p}} \geq \int_{\mathbb{G}} \frac{|\nabla_H d|^s}{d^s}|d^{\alpha}u|^{p_*(s)}dx.$$

This proves inequality (17) in the case $2 < p \leq Q$. □

Theorem 8 (Weighted Caffarelli-Kohn-Nirenberg inequality)**.** *Let $\mathbb{G}$ be a stratified group. Let*

$$\begin{aligned} &1 < p < Q,\ q \geq 1,\ r > 0,\ 0 \leq a \leq 1, \\ &\frac{1}{p} + \frac{\alpha}{Q} > 0,\ \frac{1}{q} + \frac{\beta}{Q} > 0,\ \frac{1}{r} + \frac{\gamma}{Q} > 0, \end{aligned} \tag{23}$$

where $\gamma = a\sigma + (1-a)\beta$. Then there exists a positive constant C such that

$$|||\nabla_H d|^{\alpha-\gamma} d^{\gamma} u||_{L^r(\mathbb{G})} \leq C||d^{\alpha}|\nabla_H u|||^a_{L^p(\mathbb{G})}|||\nabla_H d|^{\alpha-\beta}d^{\beta}u||^{1-a}_{L^q(\mathbb{G})}, \tag{24}$$

holds for all functions $u \in C_0^{\infty}(\mathbb{G})$, and if and only if the following conditions hold:

$$\frac{1}{r} + \frac{\gamma}{Q} = a\left(\frac{1}{p} + \frac{\alpha-1}{Q}\right) + (1-a)\left(\frac{1}{q} - \frac{\beta}{Q}\right), \tag{25}$$

$$0 \leq \alpha - \sigma \leq 1,\ \textit{if } a > 0 \textit{ and } \frac{1}{p} + \frac{\alpha-1}{Q} = \frac{1}{r} + \frac{\gamma}{Q}. \tag{26}$$

Note that in Theorem 8 if we choose $a = 1$ then the condition (26) has the following form

$$0 \leq \alpha - \sigma = \alpha - \gamma \leq 1,\ \text{ and }\ r = \left(\frac{1}{p} + \frac{\alpha-\sigma-1}{Q}\right)^{-1}, \tag{27}$$

since $\gamma = \sigma$. Furthermore, we have $p \leq r \leq p^* = \frac{Qp}{Q-p}$ that allows to write r as follows

$$r = tp + (1-t)p^* = \frac{p(Q-tp)}{Q-p}. \tag{28}$$

By combining (27), (28) and $(\alpha-\sigma)r = tp$ we obtain the following relations:

$$\frac{p(Q-tp)}{Q-p} = \left(\frac{1}{p} + \frac{\alpha-\sigma-1}{Q}\right)^{-1}, \tag{29}$$

$$tp = (\alpha-\sigma)\left(\frac{1}{p} + \frac{\alpha-\sigma-1}{Q}\right)^{-1}. \tag{30}$$

Note that tp equals s from Theorem 6. If we insert (29) and (30) into inequality (17) then we arrive at

$$\int_{\mathbb{G}} \left(\frac{|\nabla_H d|}{d}\right)^{(\alpha-\sigma)\left(\frac{1}{p}+\frac{\alpha-\sigma-1}{Q}\right)^{-1}} |d^{\alpha}u|^{\left(\frac{1}{p}+\frac{\alpha-\sigma-1}{Q}\right)^{-1}} dx \leq C\left(\int_{\mathbb{G}} |d^{\alpha}\nabla_H u|^p dx\right)^{\frac{1}{p}\left(\frac{1}{p}+\frac{\alpha-\sigma-1}{Q}\right)^{-1}}. \tag{31}$$

Here, we showed that Theorem 8 implies Theorem 6 and inequality (31) will be used in the proof of Theorem 8.

Proof of Theorem 8. First, we calculate the following term

$$\begin{aligned}
\int_{\mathbb{G}} |\nabla_H d|^{(\alpha-\gamma)r} d^{\gamma r}|u|^r dx &= \int_{\mathbb{G}} \left[|\nabla_H d|^{\alpha-a\sigma-(1-a)\beta} d^{a\sigma+(1-a)\beta}|u|\right]^r dx \qquad (32)\\
&= \int_{\mathbb{G}} \left[|\nabla_H d|^{a(\alpha-\sigma)+(1-a)(\alpha-\beta)} d^{a\sigma+(1-a)\beta}|u|\right]^r dx\\
&= \int_{\mathbb{G}} \left[|\nabla_H d|^{(\alpha-\sigma)} d^{\sigma}|u|\right]^{ar} \left[|\nabla_H d|^{(\alpha-\beta)} d^{\beta}|u|\right]^{r(1-a)} dx\\
&\geq \left(\int_{\mathbb{G}} \left[|\nabla_H d|^{(\alpha-\sigma)} d^{\sigma}|u|\right]^{\left(\frac{1}{p}+\frac{\alpha-\sigma-1}{Q}\right)^{-1}} dx\right)^{ra\left(\frac{1}{p}+\frac{\alpha-\sigma-1}{Q}\right)}\\
&\quad\times \left(\int_{\mathbb{G}} \left[|\nabla_H d|^{(\alpha-\beta)} d^{\beta}|u|\right]^q dx\right)^{\frac{r(1-a)}{q}}.
\end{aligned}$$

Now we prove inequality (24) by applying the inequalities (32) and (31),

$$\begin{aligned}
\int_{\mathbb{G}} |\nabla_H d|^{(\alpha-\gamma)r} d^{\gamma r}|u|^r dx &= \left(\int_{\mathbb{G}} |\nabla_H d|^{(\alpha-\gamma)r} d^{\gamma r}|u|^r dx\right)^{\frac{1}{r}} \left(\int_{\mathbb{G}} |\nabla_H d|^{(\alpha-\gamma)r} d^{\gamma r}|u|^r dx\right)^{\frac{r-1}{r}}\\
&\geq \left(\int_{\mathbb{G}} \left[|\nabla_H d|^{(\alpha-\sigma)} d^{\sigma-\alpha}|d^{\alpha}u|\right]^{\left(\frac{1}{p}+\frac{\alpha-\sigma-1}{Q}\right)^{-1}} dx\right)^{a\left(\frac{1}{p}+\frac{\alpha-\sigma-1}{Q}\right)}\\
&\quad\times \left(\int_{\mathbb{G}} |\nabla_H d|^{q(\alpha-\beta)} d^{\beta q}|u|^q dx\right)^{\frac{1-a}{q}} \left(\int_{\mathbb{G}} |\nabla_H d|^{(\alpha-\gamma)r} d^{\gamma r}|u|^r dx\right)^{\frac{r-1}{r}}\\
&\geq C\left(\int_{\mathbb{G}} |d^{\alpha}\nabla_H u|^p dx\right)^{\frac{a}{p}} \left(\int_{\mathbb{G}} |\nabla_H d|^{(\alpha-\beta)q} d^{\beta q}|u|^q dx\right)^{\frac{1-a}{q}}\\
&\quad\times \left(\int_{\mathbb{G}} |\nabla_H d|^{(\alpha-\gamma)r} d^{\gamma r}|u|^r dx\right)^{\frac{r-1}{r}}.
\end{aligned}$$

We arrive at

$$\left(\int_{\mathbb{G}} |\nabla_H d|^{(\alpha-\gamma)r} d^{\gamma r}|u|^r dx\right)^{\frac{1}{r}} \geq C\left(\int_{\mathbb{G}} |d^{\alpha}\nabla_H u|^p dx\right)^{\frac{a}{p}} \left(\int_{\mathbb{G}} |\nabla_H d|^{(\alpha-\beta)q} d^{\beta q}|u|^q dx\right)^{\frac{1-a}{q}}.$$

We finish the proof. □

When $\alpha = \gamma = \beta$ in Theorem 8, we obtain the following Corollary.

Corollary 9. *Let $\mathbb{G}$ be a stratified group. Let*

$$\begin{gathered} 1 < p < Q,\ q \geq 1,\ r > 0,\ 0 \leq a \leq 1, \\ \frac{1}{p} + \frac{\alpha}{Q} > 0,\ \frac{1}{q} + \frac{\alpha}{Q} > 0,\ \frac{1}{r} + \frac{\alpha}{Q} > 0. \end{gathered} \tag{33}$$

Then, there exists a positive constant C such that

$$||d^{\alpha}u||_{L^r(\mathbb{G})} \leq C||d^{\alpha}|\nabla_H u|||^a_{L^p(\mathbb{G})}||d^{\alpha}u||^{1-a}_{L^q(\mathbb{G})}, \tag{34}$$

holds for all functions $u \in C_0^{\infty}(\mathbb{G})$, and if and only if the following conditions hold:

$$\frac{1}{r} = a\left(\frac{1}{p} - \frac{1}{Q}\right) + \frac{1-a}{q}. \tag{35}$$

Note that Corollary 9 is a main ingredient to acquire the logarithmic Caffarelli-Kohn-Nirenberg type inequalities in the next section.

3 Logarithmic Caffarelli-Kohn-Nirenberg type inequalities

Lemma 10. *Let $\mathbb{G}$ be a stratified group. Let the parameters*

$$1 < p \leq r \leq q \leq \infty,\ \theta \in [0,1],$$

satisfy

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q},$$

then we have the following Hölder inequality

$$||d^{\alpha}u||_{L^r(\mathbb{G})} \leq ||d^{\alpha}u||^{\theta}_{L^p(\mathbb{G})}||d^{\alpha}u||^{1-\theta}_{L^q(\mathbb{G})}, \tag{36}$$

for $d^{\alpha}u \in L^p(\mathbb{G}) \cap L^q(\mathbb{G})$.

Remark 11. *Note that the Hölder inequality* (36) *is equivalent to the convexity of the function*

$$\phi\left(\frac{1}{r}\right)=\frac{1}{r}\log\left(\int_{\mathbb{G}}(d^{\alpha}u)^{r}dx\right),$$

that is

$$\phi\left(\frac{1}{r}\right)\leq\theta\phi\left(\frac{1}{p}\right)+(1-\theta)\phi\left(\frac{1}{q}\right),$$

for $\frac{1}{r}=\frac{\theta}{r}+\frac{1-\theta}{q}$.

Proof of Lemma 10. A simple computation gives

$$\begin{aligned}\int_{\mathbb{G}}(d^{\alpha}u)^{r}dx&=\int_{\mathbb{G}}(d^{\alpha}u)^{\theta r}(d^{\alpha}u)^{(1-\theta)r}dx\\&\leq\left(\int_{\mathbb{G}}(d^{\alpha}u)^{p}dx\right)^{\frac{\theta r}{p}}\left(\int_{\mathbb{G}}(d^{\alpha}u)^{q}dx\right)^{\frac{(1-\theta)r}{q}},\end{aligned}$$

since

$$1=\frac{r\theta}{p}+\frac{(1-\theta)r}{q}.$$

□

Lemma 12 (A logarithmic Hölder inequality)**.** *Let* $\mathbb{G}$ *be a stratified group. Let* $1<p<q\leq\infty$, *then we have*

$$\int_{\mathbb{G}}\frac{(d^{\alpha}u)^{p}}{||d^{\alpha}u||^{p}_{L^{p}(\mathbb{G})}}\log\left(\frac{(d^{\alpha}u)^{p}}{||d^{\alpha}u||^{p}_{L^{p}(\mathbb{G})}}\right)dx\leq\frac{p}{q-p}\log\left(\frac{\int_{\mathbb{G}}(d^{\alpha}|u|)^{q}dx}{\int_{\mathbb{G}}(d^{\alpha}|u|)^{p}dx}\right),\tag{37}$$

for $d^{\alpha}u\in L^{p}(\mathbb{G})\cap L^{q}(\mathbb{G})$.

Proof of Lemma 12. Observe that the derivative of convexity of the function

$$\phi(h)=h\log\left(\int_{\mathbb{G}}(d^{\alpha}|u|)^{\frac{1}{h}}dx\right),$$

that is,

$$\frac{d\phi(h)}{dh}=\log\left(\int_{\mathbb{G}}(d^{\alpha}|u|)^{\frac{1}{h}}dx\right)-\frac{1}{h}\frac{\int_{\mathbb{G}}(d^{\alpha}|u|)^{\frac{1}{h}}\log(d^{\alpha}|u|)dx}{\int_{\mathbb{G}}(d^{\alpha}|u|)^{\frac{1}{h}}dx}.$$

Then convexity of function ϕ in $(0,\infty)$ is equivalent to

$$\frac{d\phi(h)}{dh}\geq\frac{\phi(h_{1})-\phi(h)}{h_{1}-h},$$

for $h > h_1 \geq 0$. Now by taking $h := \frac{1}{p}$ and $h_1 := \frac{1}{q}$, we derive to

$$\frac{\int_{\mathbb{G}}(d^\alpha|u|)^p \log(d^\alpha|u|)dx}{\int_{\mathbb{G}}(d^\alpha|u|)^p dx} - \frac{1}{p}\log\left(\int_{\mathbb{G}}(d^\alpha|u|)^p dx\right) \leq \frac{p}{q-p}\log\left(\frac{\int_{\mathbb{G}}(d^\alpha|u|)^q dx}{\int_{\mathbb{G}}(d^\alpha|u|)^p dx}\right).$$

The left side of the above inequality can be rearranged in the following form

$$\begin{aligned}
&\frac{\int_{\mathbb{G}}(d^\alpha|u|)^p \log(d^\alpha|u|)dx}{\int_{\mathbb{G}}(d^\alpha|u|)^p dx} - \frac{1}{p}\log\left(\int_{\mathbb{G}}(d^\alpha|u|)^p dx\right)\\
&= \frac{\int_{\mathbb{G}}(d^\alpha|u|)^p \log(d^\alpha|u|)dx}{\int_{\mathbb{G}}(d^\alpha|u|)^p dx} - \frac{\int_{\mathbb{G}}(d^\alpha|u|)^p \log(||d^\alpha u||_{L^p(\mathbb{G})})dx}{\int_{\mathbb{G}}(d^\alpha|u|)^p dx}\\
&= \int_{\mathbb{G}} \frac{(d^\alpha|u|)^p}{||d^\alpha u||^p_{L^p(\mathbb{G})}} \log\left(\frac{(d^\alpha|u|)^p}{(||d^\alpha u||^p_{L^p(\mathbb{G})})}\right)dx.
\end{aligned}$$

The proof of Lemma 12 is finished. □

Now we state the logarithmic Caffarelli-Kohn-Nirenberg inequality on $\mathbb{G}$.

Theorem 13 (Logarithmic Caffarelli-Kohn-Nirenberg inequality)**.** *Let $\mathbb{G}$ be a stratified group. For a positive constant C, the inequality*

$$\int_{\mathbb{G}} \frac{(d^\alpha u)^q}{||d^\alpha u||^q_{L^q(\mathbb{G})}} \log\left(\frac{(d^\alpha u)^q}{||d^\alpha u||^q_{L^q(\mathbb{G})}}\right)dx \leq \frac{1}{1-\frac{q}{p^*}}\log\left(C^q \frac{||d^\alpha|\nabla_H u|||^q_{L^q(\mathbb{G})}}{||d^\alpha u||^q_{L^q(\mathbb{G})}}\right) \tag{38}$$

is valid for parameters

$$1 < q < p^*,\ 1 < p < Q,\ \alpha p + Q > 0,\ \alpha q + Q > 0,$$

for every function $d^\alpha|\nabla_H u| \in L^p(\mathbb{G})$ and $d^\alpha u \in L^q(\mathbb{G})$.

Proof of Theorem 13. We use Lemma 37 with $q = p$, $p = r$, $1 < q < r \leq \infty$ and inequality (34). This gives

$$\begin{aligned}
\int_{\mathbb{G}} \frac{(d^\alpha u)^q}{||d^\alpha u||^q_{L^q(\mathbb{G})}} \log\left(\frac{(d^\alpha u)^q}{||d^\alpha u||^q_{L^q(\mathbb{G})}}\right)dx &\leq \frac{1}{1-q/r}\left[\log\left(||d^\alpha u||^q_{L^r(\mathbb{G})}\right) - \log\left(||d^\alpha u||^q_{L^q(\mathbb{G})}\right)\right]\\
&\leq \frac{1}{1-q/r}\left[\log\left(C^q ||d^\alpha|\nabla_H u|||^{aq}_{L^p(\mathbb{G})}||d^\alpha u||^{(1-a)q}_{L^q(\mathbb{G})}\right) - \log\left(||d^\alpha f||^q_{L^q(\mathbb{G})}\right)\right]\\
&= \frac{a}{1-q/r}\log\left(C^q \frac{||d^\alpha|\nabla_H u|||^q_{L^p(\mathbb{G})}}{||d^\alpha u||^q_{L^q(\mathbb{G})}}\right)\\
&= \frac{1}{1-q/p^*}\log\left(C^q \frac{||d^\alpha|\nabla_H u|||^q_{L^p(\mathbb{G})}}{||d^\alpha u||^q_{L^q(\mathbb{G})}}\right).
\end{aligned}$$

In the last line, we have used $1 - q/r = a(p^* - q)/p^*$ from $1/r = a/p^* + (1-a)/q$. That finishes the proof. □

Theorem 14 (Parametric logarithmic Caffarelli-Kohn-Nirenberg inequality)**.** *Let $\mathbb{G}$ be a stratified group. Suppose*

$$1 < p < Q,\ p^* = \frac{pQ}{Q-p},\ 1 < p^2/q < p^*,$$
$$\alpha p + Q > 0,\ \alpha p^2/q + Q > 0,\ \mu > 0.$$

There exists a positive constant C such that

$$\int_{\mathbb{G}} \frac{|d^\alpha u|^p}{||d^\alpha u||^p_{L^p(\mathbb{G})}} \log\left(\frac{|d^\alpha u|^p}{||d^\alpha u||^p_{L^p(\mathbb{G})}}\right) dx - \mu \frac{||\nabla_H (d^{\alpha u})^{q/p}||^p_{L^p(\mathbb{G})}}{||d^\alpha u||^p_{L^p(\mathbb{G})}} \tag{39}$$
$$\leq \frac{p}{q - p^2/p^*} \log\left(\frac{pC^p}{e(q-p^2/p^*)\mu}\right) + \frac{p}{q-p^2/p^*} \log ||d^\alpha u||^{p-q}_{L^p(\mathbb{G})},$$

for all functions $u \in L^p(\mathbb{G})$ and $\nabla_H (d^\alpha u)^{q/p} \in L^p(\mathbb{G})$.

Proof of Theorem 14. When $\alpha = 0$ in the logarithmic Caffarelli-Kohn-Nirenberg inequality (38), we have

$$\int_{\mathbb{G}} \frac{u^q}{||u||^q_{L^q(\mathbb{G})}} \log\left(\frac{u^q}{||u||^q_{L^q(\mathbb{G})}}\right) dx \leq \frac{1}{1 - \frac{q}{p^*}} \log\left(C^q \frac{|||\nabla_H u|||^q_{L^q(\mathbb{G})}}{||u||^q_{L^q(\mathbb{G})}}\right).$$

By taking p^2/q instead of q and substituting u with $(d^\alpha u)^{q/p}$ in above inequality, we arrive at

$$\int_{\mathbb{G}} \frac{|d^\alpha u|^p}{||d^\alpha u||^p_{L^p(\mathbb{G})}} \log\left(\frac{|d^\alpha u|^p}{||d^\alpha u||^p_{L^p(\mathbb{G})}}\right) dx \leq \frac{p}{q-p^2/p^*} \log\left(C^p \frac{||\nabla_H (d^\alpha u)^{q/p}||^p_{L^p(\mathbb{G})}}{||d^\alpha u||^p_{L^p(\mathbb{G})}}\right)$$
$$+ \frac{p}{q-p^2/p^*} \log ||d^\alpha u||^{p-q}_{L^p(\mathbb{G})}.$$

In the last line, we drag p/q from inside the expression of *log* and arrange to have a form as (40). Now we add the following term

$$-\mu \frac{||\nabla_H (d^\alpha u)^{q/p}||^p_{L^p(\mathbb{G})}}{||d^\alpha u||^p_{L^p(\mathbb{G})}} \tag{40}$$

to both sides of the above inequality, then we get

$$\begin{aligned}
&\int_{\mathbb{G}} \frac{|d^\alpha u|^p}{||d^\alpha u||^p_{L^p(\mathbb{G})}} \log\left(\frac{|d^\alpha u|^p}{||d^\alpha u||^p_{L^p(\mathbb{G})}}\right) dx - \mu \frac{||\nabla_H (d^\alpha u)^{q/p}||^p_{L^p(\mathbb{G})}}{||d^\alpha u||^q_{L^p(\mathbb{G})}} \\
&\leq \frac{p}{q-p^2/p^*} \log\left(C^p \frac{||\nabla_H (d^\alpha u)^{q/p}||^p_{L^p(\mathbb{G})}}{||d^\alpha u||^p_{L^p(\mathbb{G})}}\right) - \mu \frac{||\nabla_H (d^\alpha u)^{q/p}||^p_{L^p(\mathbb{G})}}{||d^\alpha u||^q_{L^p(\mathbb{G})}} \\
&+ \frac{p}{q-p^2/p^*} \log ||d^\alpha u||^{p-q}_{L^p(\mathbb{G})} \\
&= \frac{p}{q-p^2/p^*} \log\left(C^p z\right) - \mu z + \frac{p}{q-p^2/p^*} \log ||d^\alpha u||^{p-q}_{L^p(\mathbb{G})},
\end{aligned}$$

where

$$z = \frac{||\nabla_H (d^\alpha u)^{q/p}||^p_{L^p(\mathbb{G})}}{||d^\alpha u||^p_{L^p(\mathbb{G})}}.$$

If we maximize the right-hand side of the above inequality with respect to z, we get

$$\begin{aligned}
&\int_{\mathbb{G}} \frac{|d^\alpha u|^p}{||d^\alpha u||^p_{L^p(\mathbb{G})}} \log\left(\frac{|d^\alpha u|^p}{||d^\alpha u||^p_{L^p(\mathbb{G})}}\right) dx - \mu \frac{||\nabla_H (d^\alpha u)^{q/p}||^p_{L^p(\mathbb{G})}}{||d^\alpha u||^q_{L^p(\mathbb{G})}} \\
&\leq \frac{p}{q-p^2/p^*} \log\left(C^p \frac{p}{e\mu(q-p^2/p^*)}\right) + \frac{p}{q-p^2/p^*} \log ||d^\alpha u||^{p-q}_{L^p(\mathbb{G})}.
\end{aligned}$$

That proves inequality (39). □

4 Application

Theorem 15. *Let $\mathbb{G}$ be a stratified group. Suppose*

$$1 < p < Q, \ \frac{1}{p-1} < m < a_0 < \infty, \ t > 0.$$

Let $u(t)$ be a positive strong solution to $d^\alpha \dot{u} = \mathcal{L}_p (d^\alpha u)^m$. Then for a function $d^\alpha u(0) \in L^{a_0}(\mathbb{G})$ and $d^\alpha u(t) \in L^\infty(\mathbb{G})$, we have

$$||d^\alpha u(t)||_{L^\infty(\mathbb{G})} \leq C(Q,p,m,a_0) ||d^\alpha u(0)||^{\frac{a_0 p}{a_0 p + Q(m(p-1)-1)}}_{L^{a_0}(\mathbb{G})} t^{-\frac{Q}{a_0 p + Q(m(p-1)-1)}}, \tag{41}$$

for such $C(Q,p,m,a_0)$ is a positive constant.

Proof of Theorem 15. Suppose that $a_0 \leq r(t) < \infty$ and $\dot{r}(t) > 0$ hold for $r(t)$. A straightforward calculation gives

$$\frac{d}{dt}\left(\int_{\mathbb{G}} d^{\alpha r(t)} u^{r(t)} dx\right)^{\frac{1}{r(t)}} = ||d^\alpha u||_{L^{r(t)}(\mathbb{G})}\left(\frac{-\dot{r}(t)}{r^2(t)}\right)\log\left(\int_{\mathbb{G}} d^{\alpha r(t)} u^{r(t)} dx\right)$$
$$+\frac{||d^\alpha u||^{1-r(t)}_{L^{r(t)}(\mathbb{G})}}{r(t)}\int_{\mathbb{G}}\Big((d^\alpha u(t))^{r(t)}\log(d^\alpha u(t))\dot{r} + r(t)(d^\alpha u(t))^{r(t)-1} d^\alpha \dot{u}(t)\Big)dx$$
$$= ||d^\alpha u||_{L^{r(t)}(\mathbb{G})}\frac{\dot{r}(t)}{r^2(t)}\left[\frac{r(t)}{||d^\alpha u||^{r(t)}_{L^{r(t)}(\mathbb{G})}}\int_{\mathbb{G}}[d^\alpha u(t)]^{r(t)}\log(d^\alpha u(t))dx\right.$$
$$\left.-\log||d^\alpha u(t)||^{r(t)}_{L^{r(t)}(\mathbb{G})} + \frac{r^2(t)}{\dot{r}(t)||d^\alpha u||^{r(t)}_{L^{r(t)}(\mathbb{G})}}\int_{\mathbb{G}} d^\alpha \dot{u}(t)(d^\alpha u(t))^{r(t)-1}dx\right]$$
$$= ||d^\alpha u||_{L^{r(t)}(\mathbb{G})}\frac{\dot{r}^2(t)}{r^2(t)}\left[\int_{\mathbb{G}}\frac{(d^\alpha u(t))^{r(t)}}{||d^\alpha u||^{r(t)}_{L^{r(t)}(\mathbb{G})}}\log\left(\frac{(d^\alpha u(t))^{r(t)}}{||d^\alpha u||^{r(t)}_{L^{r(t)}(\mathbb{G})}}\right)dx\right.$$
$$\left.+\frac{r^2(t)}{\dot{r}(t)||d^\alpha u||^{r(t)}_{L^{r(t)}(\mathbb{G})}}\int_{\mathbb{G}}(d^\alpha u(t))^{r(t)-1}d^\alpha \dot{u}(t)dx\right].$$

$$\int_{\mathbb{G}}(d^\alpha u(t))^{r(t)-1}d^\alpha \dot{u}(t)dx = \int_{\mathbb{G}}(d^\alpha u(t))^{r(t)-1}\mathcal{L}_p(d^\alpha u(t))^m dx$$
$$= -\int_{\mathbb{G}}(|\nabla_H(d^\alpha u(t))^m|^{p-2}\nabla_H(d^\alpha u(t))^m)\cdot\nabla_H((d^\alpha u(t))^{r(t)-1})dx$$
$$= -m^{p-1}(r(t)-1)\int_{\mathbb{G}}(d^\alpha u(t))^{(m-1)(p-1)+r(t)-2}|\nabla_H(d^\alpha u(t))|^p dx$$
$$= -\frac{p^p m^{p-1}(r(t)-1)}{(r(t)+m(p-1)-1)^p}\int_{\mathbb{G}}\left||\nabla_H(d^\alpha u)|^{\frac{r(t)+m(p-1)-1}{p}}\right|^p dx.$$

Let us have a function v such that $(d^\alpha v)^r = (d^\alpha u)^p$, then we get

$$||d^\alpha u||^r_{L^r(\mathbb{G})} = ||d^\alpha v||^p_{L^p(\mathbb{G})}.$$

We insert to

$$
\begin{aligned}
&\int_{\mathbb{G}} \frac{(d^\alpha u(t))^r}{||d^\alpha u||^r_{L^r(\mathbb{G})}} \log\left(\frac{(d^\alpha u(t))^r}{||d^\alpha u||^r_{L^r(\mathbb{G})}}\right) dx \\
&- \frac{p^p m^{p-1}(r(t)-1)}{(r(t)+m(p-1)-1)^p} \frac{r^2}{\dot{r}||d^\alpha u||^r_{L^r(\mathbb{G})}} \int_{\mathbb{G}} \left| |\nabla_H (d^\alpha u)|^{\frac{r(t)+m(p-1)-1}{p}} \right|^p dx \\
= &\int_{\mathbb{G}} \frac{(d^\alpha v)^p}{||d^\alpha v||^p_{L^p(\mathbb{G})}} \log\left(\frac{(d^\alpha v)^p}{||d^\alpha v||^p_{L^p(\mathbb{G})}}\right) dx \\
&- \frac{p^p m^{p-1}(r(t)-1)}{(r(t)+m(p-1)-1)^p} \frac{r^2}{\dot{r}||d^\alpha v||^p_{L^p(\mathbb{G})}} \int_{\mathbb{G}} |\nabla_H (d^\alpha v)^{\frac{q}{p}}|^p dx,
\end{aligned}
$$

where $q = p\frac{r+m(p-1)-1}{r}$. Here a parameter μ from the parametric logarithmic Caffarelli-Kohn-Nirenberg type inequalities is given by $\mu = \frac{r^2}{\dot{r}} \frac{p^p m^{p-1}(r-1)}{(r+m(p-1)-1)^p}$. By applying Theorem 39, we obtain

$$
\begin{aligned}
&\frac{d}{dt}||d^\alpha u||_{L^r(\mathbb{G})} \\
&= ||d^\alpha u||_{L^r(\mathbb{G})} \frac{\dot{r}}{r^2} \left[\int_{\mathbb{G}} \frac{(d^\alpha v)^p}{||d^\alpha v||^p_{L^p(\mathbb{G})}} \log\left(\frac{(d^\alpha v)^p}{||d^\alpha v||^p_{L^p(\mathbb{G})}}\right) dx - \frac{\mu}{||d^\alpha v||^p_{L^p(\mathbb{G})}} \int_{\mathbb{G}} |\nabla_H (d^\alpha v)^{\frac{q}{p}}|^p dx \right] \\
&\leq ||d^\alpha u||_{L^r(\mathbb{G})} \frac{\dot{r}}{r^2} \frac{p}{q - p^2/p^*} \left[\log\left(C^p \frac{p}{e\mu(q-p^2/p^*)} \right) + \frac{r(p-q)}{p} \log ||d^\alpha u||_{L^r(\mathbb{G})} \right]. \qquad (42)
\end{aligned}
$$

Then inequality (42) can be seen as

$$
\dot{h} \leq F(t) h \log(h) + G(t) h, \qquad (43)
$$

where

$$
\begin{aligned}
h(t) &:= ||d^\alpha u(t)||_{L^r(\mathbb{G})}, \\
F(t) &:= \frac{\dot{r}}{r} \frac{p-q}{q-p^2/p^*}, \\
G(t) &:= \frac{\dot{r}}{r^2} \frac{p}{q-p^2/p^*} \log\left(C^p \frac{p}{e\mu(q-p^2/p^*)} \right).
\end{aligned}
$$

The solving (43) is a simple calculation but very long, so we refer to [19] where Q instead of n in our case. Then for $a_0 < b < \infty$ we have

$$
||d^\alpha u(t)||_{L^b(\mathbb{G})} \leq C(Q,p,m,a_0,b) ||d^\alpha u(0)||^{\frac{a_0(bp+Q(m(p-1)-1))}{b(a_0 p + Q(m(p-1)-1))}}_{L^{a_0}(\mathbb{G})} t^{-\frac{(b-a_0)Q}{b(a_0 p + Q(m(p-1)-1))}}, \qquad (44)
$$

where $C(Q,p,m,a_0,b) = C\left(\frac{b-m_1}{bm_2}\right)^{\frac{b-m_3}{bm_4}}$ with $m_1 = m_3 = a_0$, and

$$m_2 = \left(1-\frac{p}{p^*}\right)\left(a_0\left(1-\frac{p}{p^*}\right)+m(p-1)-1\right),\ m_4 = a_0\left(1-\frac{p}{p^*}\right)+m(p-1)-1.$$

Making use of L'Hospital, it can be shown that for $b \to \infty$ the constant is $C(Q,p,m,a_0,b) < \infty$. This proves

$$||d^\alpha u(t)||_{L^\infty(\mathbb{G})} \leq C(Q,p,m,a_0)||d^\alpha u(0)||_{L^{a_0}(\mathbb{G})}^{\frac{a_0 p}{a_0 p+Q(m(p-1)-1)}} t^{-\frac{Q}{a_0 p+Q(m(p-1)-1)}}.$$

□

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Сәбитбек Б.М. Салмақталған және логарифмдік Каффарелли-Кон-Ниренберг типті теңсіздіктер стратификацияланған топтар мен қолданбалар бойынша

Классикалық Каффарелли–Кона–Ниренберг теңсіздіктері алғаш рет 1980-жылдары Евклид кеңістігінде құрылып, Соболев және Харди теңсіздіктері арасындағы интерполяция үшін біртұтас негіз ұсынды. Бұл теңсіздіктерді стратификацияланған (немесе біртекті Карно) Ли топтарына жалпылау 2000-жылдардың басында субэллиптикалық талдау мен геометриялық өлшем теориясының ықпалымен басталды. Бұл кеңейту топтың құрылымы, масштабтау симметриясы және функционалдық теңсіздіктер арасындағы бай байланыстарды ашты. Бұл мақалада біз стратификацияланған Ли тобында салмақталған және логарифмдік Каффарелли-Кон-Ниренберг типті теңсіздіктерді орнатамыз. Нәтижесінде, біз оны келесі теңдеудің оң күшті шешімдерінің салмақталған ультра контрактивтілік (ultracontractivity) дәлелдеу үшін қолдана аламыз:

$$d^{\alpha}\frac{\partial u}{\partial t} = \mathcal{L}_p(d^{\alpha}u)^m,$$

мұндағы $\mathcal{L}_p f = \nabla_H(|\nabla_H f|^{p-2}\nabla_H f)$ - p-суб-Лапласиан, d суб-Лаплациан үшін іргелі шешімімен байланысты біртекті норма, және $\alpha \in \mathbb{R}$, $1 < p < Q$.

Түйін сөздер: Кафарелли-Кон-Ниренберг теңсіздігі; логарифмдік Каффарелли-Кон-Ниренберг теңсіздігі; стратификацияланған Ли тобы.

Сабитбек Б.М. Весовое и логарифмические неравенства типа Каффарелли-Кона-Ниренберга на стратифицированных группах и их приложения

Классические неравенства Каффарелли–Кона–Ниренберга, впервые установленные в евклидовом пространстве в 1980-х годах, предоставили единую основу для интерполяции между неравенствами Соболева и Харди. Их обобщение на стратифицированные (или однородные Карно) группы Ли началось в начале 2000-х годов под влиянием субэллиптического анализа и теории геометрической меры, выявив богатые взаимосвязи между структурой группы, симметрией растяжений и функциональными неравенствами. В этой статье мы устанавливаем весовые и логарифмические неравенства типа Каффарелли-Кона-Ниренберга на стратифицированной группе Ли. Как следствие, мы можем применить их для доказательства весовой ультрасжимаемости положительных сильных решений

$$d^{\alpha}\frac{\partial u}{\partial t} = \mathcal{L}_p(d^{\alpha}u)^m,$$

где $\mathcal{L}_p f = \nabla_H(|\nabla_H f|^{p-2}\nabla_H f)$ — p-суб-Лапласиан, d — однородная норма, связанная с фундаментальным решением для суб-Лапласиана и $\alpha \in \mathbb{R}$, $1 < p < Q$.

Ключевые слова: Неравенство Каффарелли-Кона-Ниренберга; логарифмическое неравенство Каффарелли-Кона-Ниренберга; стратифицированная группа Ли.