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The properties of amalgamation and joint embedding in the meaning of positive Jonsson theories

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Abstract. In this paper, we study special subclasses of theories based on the connection between the amalgamation property and the joint embedding property, as well as between the h-amalgamation property and the joint continuation property. Our results are presented in both the classical first-order logic and the positive logic, which exhibit a parallel structure. We establish sufficient conditions under which the amalgamation property implies the joint embedding property, and conversely; the h-amalgamation property implies the joint continuation property and vice versa. Furthermore, we investigate the preservation of these subclass links under extensions of the given theory.

Keywords. Existentially closed models, amalgamation property, joint embedding property, positive model theory, positively closed models, h-amalgamation property, joint continuation property, positively existentially prime Jonsson theories.

Introduction

This paper relates to both the so-called "East" direction of Model Theory that originated from Abraham Robinson's work [1] and the positive model theory, which was first studied by Itai Ben Yaacov and Bruno Poizat in [2].

Model theory, a fundamental branch of mathematical logic, has evolved through two distinct historical and methodological traditions. The first, often referred to as the "Western" tradition, originated from the pioneering works of Alfred Tarski and Robert Vaught. This approach primarily emphasizes the study of complete theories, and classification of models

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via stability and stability hierarchies, and techniques based on compactness and completeness theorems. A crucial aspect of this tradition is the use of elementary embeddings as the primary morphisms, ensuring that the logical structure is preserved precisely. It has been deeply connected with algebraic geometry, topological model theory, and, more recently, the development of o-minimality and geometric stability theory.

In contrast, the "Eastern" tradition of model theory, rooted in the works of Abraham Robinson and Anatoly Maltsev, focuses on methods derived from algebra and non-classical logic, particularly the model-theoretic study of algebraic structures through the lens of syntactical constructs. This tradition places a strong emphasis on non-elementary classes, Robinsonian concepts such as model-completeness, and the use of methods from universal algebra to investigate the nature of mathematical structures. Unlike the Western tradition, which predominantly operates within first-order logic with complete theories and elementary embeddings, the Eastern tradition explores weaker axiomatizations where the morphisms under consideration are more general, often allowing isomorphic embeddings. The research presented in this article is primarily aligned with the latter tradition.

A significant development within the Eastern tradition is the emergence of positive logic and positive model theory, introduced by Itai Ben Yaacov and Bruno Poizat [2]. Unlike classical first-order logic, where negation plays a central role, positive logic restricts itself to the study of theories that are preserved under positive embeddings—embeddings that respect existentially quantified formulae without involving negation. Instead of considering isomorphic embeddings, positive logic focuses on homomorphisms restricted to positive formulae. This perspective leads to a reformation of classical notions: the concept of inductive theories is generalized to h-inductive theories, existentially closed models are replaced by positively closed models, the amalgamation property transforms into the h-amalgamation property, and the joint embedding property is reformulated as the joint continuation property. These modifications provide a more flexible and structurally rich approach to model theory, particularly in contexts where classical elementary embeddings are too restrictive.

Further advancements in this area include the work of B. Poizat and A.R. Yeshkeyev [3] on positive Jonsson theories, whose attributes are h-inductiveness, h-amalgamation property, and joint continuation property. In this development, the authors extend the classical Jonsson theory framework into the realm of positive logic, redefining key Robinsonian notions to fit within this weaker logical setting. Their work provides new insights into the behavior of models under positive conditions, enriching the Eastern tradition's approach to structural analysis. Thus, the main object of this study is positive Jonsson theories.

More studies in positive model theory have been conducted in the framework of the Robinsonian tradition by Itai Ben Yaacov [4], Bruno Poizat and Aibat Yeshkeyev [5], Almaz Kungozhin [6] and Mohammed Belkasmi [7, 8, 9, 10, 11, 12].

Previously, Aibat Yeshkeyev defined subclasses of inductive theories regarding the amalgamation and joint embedding properties. It is well known that the amalgamation property and joint embedding property are independent of each other; however, there are cases where one implies the other, depending on the specificity of the class of models of the theory under consideration. In that work, we considered these classes of theories in both a classical first-order logic and a positive logic context. In this paper, we provide some sufficient conditions for inductive theories to belong to some of the distinguished classes, and show when the property of the connection between amalgamation and joint embedding can be preserved in extensions of the given theory. We also generalize these notions for positive model theory and similar results in which the h-amalgamation property implies the joint continuation property and vice versa.

Paper structure. This paper is structured as follows. The introduction is followed by two main sections and a reference list. In Section 1, we provide an overview of fundamental concepts in Robinsonian model theory, describe the specific subclasses of inductive theories concerning the amalgamation and joint embedding properties, and show some results in the framework of the presented notions. Section 2 describes the necessary concepts of positive model theory and introduces new subclasses of h-inductive theories based on the connection between the h-amalgamation property and the joint continuation property, along with key results characterizing these notions.

1 Amalgamation and joint embedding properties in classical logic

As mentioned in Introduction, model theory can be studied in various frameworks. In classical Robinsonian model theory, the central objects of interest include the specific axiomatization of theories under consideration, key properties of embeddings such as the amalgamation and joint embedding properties, the existence of specific models such as existentially closed models, algebraically prime models, and some others regarding the considered types of embeddings. In contrast, positive model theory provides a different perspective, modifying fundamental notions in restricted signature while preserving key structural aspects. In this section, we outline these concepts in the Robinsonian framework and give related results before comparing them to their positive counterparts in the next section.

Let us start with the notion of an inductive theory, which plays a crucial role in the classical "eastern" tradition of model theory.

Definition 1. [13, p. 62] A theory T is called inductive if it is closed under inductive unions, that is, whenever $(M_i)_{i\in I}$ is a chain of models of T, their union $\bigcup_{i\in I} M_i$ is also a model of T.

It is well known that a theory T is inductive iff it is $\forall\exists$-axiomatizable. Another well-known fact on inductive theories is related with the existentially closed models of such theories. Firstly, we recall the definition of an existentially closed model of a theory.

Definition 2. [13, p. 97] A model M of a theory T is said to be existentially closed if for every embedding $M \to N$ into another model N of T, and every existential formula $\varphi(x)$ with parameters from M that is satisfiable in N, is also satisfiable in M.

The following fact in existentially closed models is well-known from [14].

Theorem 3. *Let T be an L-theory, $M \in Mod(T)$, and let $T_\forall$ be the set of all universal L-consequences of T. Then the following conditions are equivalent:*

1. *M is existentially closed over T;*
2. *M is existentially closed over $T_\forall$.*

It is known that for any inductive theory T, for any model A of T, there is a model M existentially closed over A such that A is embedded in M, and this fact emphasizes the significance of the notion of existentially closed models in the study of inductive theories. However, another considerable class of models is the class of algebraically prime models. While existentially closed models ensure the maximal satisfaction of existential conditions within embeddings, algebraically prime models represent the minimal elements in the class of models of a theory. Recall the definition of an algebraically prime model.

Definition 4. [15] A model M of a theory T is called algebraically prime if for every model N of T, there is an embedding of M into N.

If a theory T has algebraically prime models, the class of all such models of T is denoted by $\mathcal{A}_T$. Similarly, $\mathcal{E}_T$ denotes the class of all existentially closed models of T.

Note that the notion of an algebraically prime model generalizes the concept of a prime model [13, p. 85], where elementary embeddings are considered.

Given a theory T, the existence of existentially closed models and algebraically prime models provides the special properties of T, and there are cases when a theory admits a model that is both existentially closed and algebraically prime. In this way, the concept of existentially prime theory was defined by A. Yeshkeyev in [16].

Definition 5. [16] An L-theory T is called an existentially prime theory if there is a model M of T such that $M \subseteq \mathcal{A}_T \cap \mathcal{E}_T$, that is, M is both algebraically prime and existentially closed.

Another key aspect of studying models of a theory is the structure of embeddings between them. Two fundamental properties in this regard are the amalgamation property (AP) and the joint embedding property (JEP), which play a significant role in this study.

Definition 6. [13, p. 80] A theory T has the amalgamation property if for any three models M_1, M_2, M_3 of T such that there exist elementary embeddings $M_1 \to M_2$ and $M_1 \to M_3$, there exists a model M_4 of T and embeddings $M_2 \to M_4$ and $M_3 \to M_4$ that make the corresponding diagram commute.

Definition 7. [13, p. 80] A theory T has the joint embedding property if for any two models M_1 and M_2 of T, there exists a model M_3 of T into which both M_1 and M_2 can be embedded.

One of the classical results on theories admitting JEP is the following theorem:

Theorem 8. *[17, p. 365] Suppose T is an L-theory that admits JEP. Let A and B be existentially closed models of T. Then each $\forall\exists$-sentence that is true in A is true in B as well.*

Generally, AP and JEP are independent of each other, which is supported by counterexamples of W. Forrest in [18]. However, there are partial cases where the specific construction of the class of models of a theory provides the implication of JEP from AP and vice versa. In this manner, the following definitions were introduced in [19] by A. Yeshkeyev:

Definition 9. Let K be a class of L-theories. We call this class (or a theory from K, for short, when the class can be recovered from the context)

1. an AP-class (an AP-theory), if each theory from K, which has the amalgamation property (AP), also admits the joint embedding property.

2. a JEP-class (a JEP-theory), if each theory from K, which admits the joint embedding property (JEP), also satisfies the amalgamation property (AP).

There are examples for each type of the theories. As mentioned in [19], the group theory, the theory of fields of a fixed characteristic, the theory of differential fields of characteristic 0, the theory of differentially perfect fields of characteristic p are strongly convex theories, which means that the class of strongly convex theories is an AP class. The class of complete inductive theories, which are also model complete, is an example of a JEP-class. This class contains theories such as the theory of dense linear orders without endpoints, the theory of algebraically closed fields of a fixed characteristic, and the theory of differentially closed fields of a fixed characteristic.

In this paper, we demonstrate some sufficient conditions of being an AP-theory or JEP-theory for an inductive L-theory. In this context, the following two propositions describe AP-theories and JEP-theories.

Proposition 10. *Let T be an L-theory such that $\mathcal{A}_T \neq \varnothing$ and T admits AP. Then T is an AP-theory, that is, the class of all L-theories, which have algebraically prime models, is an AP-class.*

Proof. We need to show that, under the given conditions, T has JEP. Let A and B be two arbitrary models of T. Since T has an algebraically prime model, there is a model M such that M is embedded both into A and B. Then due to the fact that T admits the amalgamation property, there is a model N such that both A and B are embedded into N. Therefore, T has the joint embedding property. $\square$

Proposition 11. *Let T be an L-theory such that T admits JEP and for any two models A and B of T, if there is an embedding $f : A \to B$ then f is unique. Then T is an AP-theory.*

Proof. Here we need to show that T admits AP. Let A, B and C be models of T such that there are embeddings $f_1 : A \to B$ and $f_2 : A \to C$. Since T has the joint embedding property, there exists a model $D \in Mod(T)$ and embeddings $g_1 : B \to D$ and $g_2 : C \to D$. In force of the fact that every embedding of models in $Mod(T)$ is unique, we obtain that $f_1 \cdot g_1 = f_2 \cdot g_2$. Thus, T has the amalgamation property. □

The given conditions demonstrate the semantic specificity of the connection of AP and JEP within the class of models of a single theory. However, the property of being an AP-theory (JEP-theory) can be obtained for the extensions of the given theories in L. In this context, we consider the case of two theories T and T' such that $T \subseteq T'$. To specify the link between the classes of models of T and T', we also restrict this case to mutually model consistent theories. Recall the definition of mutually model consistent theories.

Definition 12. [13, p. 157] Let T_1 and T_2 be L-theories. T_1 and T_2 are called mutually model consistent, if for any model A of T_1, there is a model B of T_2 such that there exists an embedding $A \to B$, and vice versa.

The following fact on mutually model consistent theories is well-known:

Proposition 13. [13, p. 158] *If T_1 and T_2 are mutually model consistent then $T_{1\forall} = T_{2\forall}$, where $T_{i\forall}$ is the set of all universal L-sentences that are deduced from T_i.*

We apply Propositions 10 and 11 to the case of two mutually model consistent theories and obtain the following results.

Theorem 14. *Let T be an inductive existentially prime L-theory, and let T' be an inductive L-theory such that $T \subseteq T'$ and T' is mutually model consistent with T. Then if T is an AP-theory, T' is also an AP-theory. In other words, let K be a class of inductive existentially prime L-theories, and let K' extend K in the following way: if an inductive theory T' contains some $T \in K$ and T' is mutually model consistent with T, then $T' \in K'$; then K' is an AP-class.*

Proof. Firstly, let us show that $\mathcal{E}_T = \mathcal{E}_{T'}$. Note that T and T' are mutually model consistent, which means that $T_\forall = T'_\forall$. Let $A \in \mathcal{E}_T$, then, according to Theorem 3, A is existentially closed over $T_\forall = T'_\forall$ and, consequently, over T'. Therefore, $A \in \mathcal{E}_{T'}$. Conversely, if $B \in \mathcal{E}_{T'}$, B is an existentially closed structure of $T'_\forall = T_\forall$ and T. Hence, $\mathcal{E}_T = \mathcal{E}_{T'}$.

Now, let M be an existentially closed algebraically prime model of T, that is $M \in \mathcal{E}_T \cap \mathcal{A}_T$. As we showed, $\mathcal{E}_T = \mathcal{E}_{T'}$, therefore $M \in \mathcal{E}_{T'}$. Since $T \subseteq T'$, $Mod(T') \subseteq Mod(T)$, then M is an existentially closed model of T' that is embedded into any model of T', which means that T' is an existentially prime L-theory. Hence, $\mathcal{A}_{T'} \neq \varnothing$.

Now, we show that T' admits AP. Let A, B, C be models of T' such that there exist embeddings $f_1 : A \to B$ and $f_2 : A \to C$. Note that $A, B, C \in Mod(T)$. Then there is a model $D \in Mod(T)$ and embeddings $g_1 : B \to D$ and $g_2 : C \to D$ and the diagram of these embeddings commutes, as T admits AP according to the condition of the theorem. If D is a model of T', then T' has AP. If D is not, there is an existentially closed model N of T such that $D \to N$. Since T is an AP-theory, T has JEP, and according to Theorem 8, M and N satisfy the same $\forall\exists$-sentences in L. Since T' is inductive and any inductive theory is $\forall\exists$-axiomatizable, $N \in Mod(T')$. Let $g : D \to N$. Then the embeddings $g_1 \cdot g : B \to N$ and $g_2 \cdot g : C \to N$ complete the diagram of the amalgamation of the models A, B, C, N in $Mod(T')$, and this diagram commutes. Therefore, T' admits AP.

Owing to Theorem 10, T' is an AP-theory. □

Theorem 15. *Let T be an inductive L-theory such that for any embedding $f : A \to B$, where A and $B \in Mod(T)$, f is unique. Let T' be an L-theory such that $T \subseteq T'$ and T is mutually model consistent with T'. Then if T is a JEP-theory, then T' is also a JEP-theory.*

Proof. Let $A, B \in Mod(T')$. Since $T \subseteq T'$, the inclusion $Mod(T') \subseteq Mod(T)$ holds; hence any embedding $g : A \to B$ is unique.

Now we show that T' admits JEP. Let $A, B \in Mod(T')$. It is clear that A and B are also models of T. Then there is a model C of T and embeddings $f : A \to C$ and $g : B \to C$. If C is a model of T', then T' is also has JEP. If C is not, we may consider an existentially closed model M of T such that C is embedded into M. Since T and T' are mutually model consistent, $E_T = E_{T'}$; therefore M is a model of T'. Thus, A and B are embedded in a model of T', and T' admits JEP.

Applying Theorem 11, we obtain that T' is a JEP-theory. □

2 The connection of APh and JCP for h-inductive

In this section, we present the results on model-theoretic link between h-amalgamation property and joint continuation property that are positive-logic analogues of the results of the previous section.

First, we give some fundamental definitions and facts on positive model theory.

During this article, we will use the denotation L^+ for a language in positive logic by the meaning of [3].

Let L^+ be a countable language involving individual constants, functions, and relations. L^+ also contains the binary relation of equality and a 0-ary symbol $\perp$ denoting antilogy.

Unlike classical Robinsonian model theory, where embeddings are typically isomorphic in nature, positive model theory focuses on homomorphisms as the primary type of model-theoretic inclusion. This shift reflects the broader semantic framework of positive logic, which emphasizes the preservation of positive formulae rather than isomorphisms and elementary equivalence.

Definition 16. [3] A map h from an L^+-structure M to an L^+-structure N is called a homomorphism between M and N, if for every individual constant c, every function symbol f and every relation symbol r of L^+, and every tuple $\bar{a} = (a_1, \ldots, a_n)$ of elements of M the following holds:

1. $h(c_M) = h(c_N)$;
2. $h(f_M(a_1, ...a_n)) = f_N(h(a_1), ...h(a_n))$;
3. if $M \models r_M(a_1, ...a_n)$ then $N \models r_N(h(a_1), ...h(a_n))$.

When there exists a homomorphism from M to N, we say that N is a continuation of M. Note that a continuation of M is nothing but a model of the positive diagram $\text{Diag}^+(\text{M})$ of M, which is the set of atomic sentences satified by M in the language L^+ obtained by adding to the language individual constants naming the elements of M.

According to [3], a positive formula is obtained from the atomic formulae by the use of $\vee, \wedge$ and $\exists$. Note that there are no universal quantifiers. A positive formula can be written in prenex form as $(\exists \bar{x})\varphi(\bar{x})$, where φ is positive quantifier-free; φ in turn can be written as a finite disjunction of finite conjunctions of atomic formulae.

Definition 17. [3] Let M and N be L^+-structures, and let h be a homomorphism between M and N. If every tuple $\bar{a}$ in M satisfies the same positive formulae as its image $h(\bar{a})$ in N, we say that h is a pure homomorphism, or an immersion.

The next definition presents a positive version of the concept of an existentially closed model.

Definition 18. [3] An L^+-structure M is positively closed inside a class Γ of L^+-structures if every homomorphism from M to any N in Γ is an immersion.

We denote the class of all positively closed models of a theory T by PC_T.

To define a positive analogue of an inductive theory, we need the following definition.

Definition 19. [3] A sentence is said to be an *h-inductive sentence* if it is equivalent to a finite conjunction of sentences each of them declaring that a certain positively defined set is included into another. Such a simple h-inductive sentence has the form $(\forall \bar{x})(\exists \bar{y})\varphi(\bar{x}, \bar{y})$.

In positive logic only h-inductive sentences are under consideration.

In [3], B. Poizat and A. Yeshkeyev defined an inductive limit of a chain of models, where the authors considered homomorphisms, possibly not injective. The following definition presents the analogue of the concept of an inductive theory and given via the notion of inductive limits in the sense of positive logic.

Definition 20. [3] An L^+-theory T is said an h-inductive theory, if it is equivalent to a set of inductive L^+-sentences.

It is known that the class of models of any h-inductive theory is h-inductive, that is closed under the union of chains in sense of inductive limits. Moreover, in an h-inductive class, every point can be continued into a positively closed element.

The following definitions generalizes the notion of mutually model consistent theories in the context of positive logic.

Definition 21. [3] Two h-inductive L^+-theories T and T' are called companion, if every model of one of them can be continued into a model of the other.

Similarly to the classical fact in first-order model theory, companion theories admit the following property concerning positively closed models:

Proposition 22. [3] *Let T and T' be L^+-theories that are companion. Then $PC_T = PC_{T'}$, that is, the class of all positively closed models of T is equal to the class of all positively closed models of T'.*

Just as algebraically prime models serve as distinguished representatives in classical model theory, positive model theory features an analogous notion of canonicity. These models capture the minimal structural essence of a theory within the framework of homomorphisms under consideration.

Definition 23. [3] Let T be a theory in L^+. A model $A \in Mod(T)$ is called a prime model of T, if for any model $B \in Mod(T)$, there is a homomorphism $f : A \to B$.

We denote the class of all prime models of T by P_T.

By analogy with the concept of existentially prime theories in first-order logic, the following concept was defined by A. Yeshkeyev:

Definition 24. An L^+-theory T is called a positively existentially prime theory, if there is a model M of T such that $M \subseteq P_T \cap PC_T$.

Note that if M is a positively closed prime model of T, for any $A \in Mod(T)$, a homomorphism $f : M \to A$ is an immersion.

We now turn to two fundamental properties in positive model theory: the h-amalgamation property and the joint continuation property. These properties, originally studied in classical model theory as the amalgamation property and the joint embedding property, respectively, take on a distinct character in the positive setting, where homomorphic rather than isomorphic embeddings govern the structure of models.

Definition 25. [2] An h-inductive theory T has the h-amalgamation property (APh) if, whenever there are two homomorphisms $f : A \to B$ and $g : A \to C$, where A, B, and $C \in Mod(T)$, there is a model $D \in Mod(T)$, and homomorphisms $f' : B \to D$ and $g' : C \to D$ such that $f \cdot f' = g \cdot g'$.

Definition 26. [2] An h-inductive theory T has the joint continuation property (JCP) if for any two models $A, B \in Mod(T)$ there is a model $C \in Mod(T)$ and homomorphisms $f : A \to C$ and $g : B \to C$.

In terms of positive model theory, A. Yeshkeyev introduced specific subclasses of h-inductive theories that are distinguished by the connection between the h-amalgamation property and the joint continuation property. Understanding when one of these properties implies the other within a given class of models provides valuable insight into the structural behavior of positive theories.

Definition 27. Let K^+ be a class of L^+-theories T. K^+ is called

1. an APh-class, if any $T \in K^+$, which admits h-amalgamation property, admits also joint continuation property.
2. a JCP-class, if each theory $T \in K^+$, which admits joint continuation property, also has h-amalgamation property.

We call a theory T an APh-theory (JCP-theory), if $T \in K^+$, where K^+ is an APh-class (JCP-class) in cases when then class K^+ can be recovered by the context.

The following fact was observed in [3]:

Theorem 28. [3] *Let T be an L^+-theory such that $P_T \neq \varnothing$ and T admits APh. Then T admits JCP.*

Thus, Theorem 28 states that an L^+-theory T is an APh-theory, if it admits APh and has a prime model.

We now present our main results. First, we provide a sufficient condition for a theory to be a JCP-theory.

Theorem 29. *Let T be an L^+-theory such that T admits JCP and for any two models A and B of T, if there is a homomorphism $h : A \to B$ then h is unique. Then T is an APh-theory.*

Proof. First, we prove that T admits APh. Let A, B and C be models of T such that there are homomorphisms $h_1 : A \to B$ and $h_2 : A \to C$. T has joint continuation property; therefore, there exists a model $D \in Mod(T)$ and homomorphisms $h'_1 : B \to D$ and $h'_2 : C \to D$. In force of the fact that every homomorphism of models in $Mod(T)$ is unique, we obtain that $h_1 \cdot h'_1 = h_2 \cdot h'_2$. We obtain that T admits h-amalgamation property. □

Next, we examine the preservation of these properties under companion extensions. Specifically, we establish conditions under which a theory's status as an APh-theory or a JCP-theory is preserved by its companion extension.

Theorem 30. *Let T be an h-inductive positively existentially prime L^+-theory, and let T' be an h-inductive L^+-theory such that $T \subseteq T'$ and T' is a companion of T. Then if T is an APh-theory, T' is also an APh-theory.*

Proof. Suppose that M is a positively closed prime model of T. Due to the condition of the theorem, T and T' are h-inductive theories that are companion. According to Proposition 22, $PC_T = PC_{T'}$; then $M \in PC_{T'}$. Since $T \subseteq T'$, $Mod(T') \subseteq Mod(T)$, then M is a positively closed model of T' that is continued in any model of T'; hence, T' is a positively existentially prime L^+-theory. Therefore, $P_{T'} \neq \varnothing$.

Now we show that T' has APh. Let A, B, C be models of T' such that there exist homomorphisms $h_1 : A \to B$ and $h_2 : A \to C$. Note that A, B, $C \in Mod(T)$. The theory T admits APh; therefore there is a model $D \in Mod(T)$ and homomorphisms $h_1' : B \to D$ and $h_2' : C \to D$ such that the diagram of these continuations commutes. Suppose that D is a model of T', then T' has APh. In case when $\nvDash T'$, there is a positively closed model N of T such that D is continued in N. Since $PC_T = PC_{T'}$, $N \in Mod(T')$. Let $h : D \to N$. Then we may consider homomorphisms $h_1' \cdot h : B \to N$ and $h_2' \cdot h : C \to N$ that complete the diagram of the h-amalgamation of the models A, B, C, N in $Mod(T')$; moreover, this diagram commutes. Therefore, T' has APh. Thus, according to Theorem 28, T' is an APh-theory. □

Theorem 31. *Let T be an h-inductive L^+-theory such that for any homomorphism $h : A \to B$, where $A, B \in Mod(T)$, h is unique. Let T' be an L^+-theory such that $T \subseteq T'$ and T is a companion of T'. Then if T is a JCP-theory, then T' is also a JCP-theory.*

Proof. Let A and B be two arbitrary models of T'. It is clear that $A, B \in Mod(T)$, then any continuation $h : A \to B$ is unique.

Let us show that T' has JCP. Since T admits JCP, there exists a model C of T and homomorphisms $h_1 : A \to C$ and $h_2 : B \to C$. In case when C is a model of T', then T' is also has JCP. If $C \nvDash T'$, we consider a positively closed model M of T such that C is continued in M. Since T and T' are companion theories, $PC_T = PC_{T'}$ according to Proposition 22; hence, $M \models T'$. We obtain that A and B are continued in a model of T'; therefore T' has JCP. Finally, we apply Theorem 29 and obtain that T' is a JCP-theory. □

References

[1] Robinson A. *Introduction to model theory and to the metamathematics of algebra*, North-Holland, 1986.

[2] Ben Yaacov I., Poizat B. *Fondements de la logique positive*, Journal of Symbolic Logic, 72:4 (2007), 1141–1162. https://doi.org/10.2178/jsl/1203350777

[3] Poizat B., Yeshkeyev A. *Positive Jonsson theories*, Logica Universalis. 12:1 (2018), 101–127. https://doi.org/10.1007/s11787-018-0185-8

[4] Ben Yaacov I. *Positive model theory and compact abstract theories*, Journal of Mathematical Logic. 3:1 (2003), 85–118. https://doi.org/10.1142/S0219061303000212

[5] Poizat B., Yeshkeyev A. *Back and Forth in Positive Logic*, In: Logic in Question. Studies in Universal Logic. Birkhäuser, Cham, 2022. https://doi.org/10.1007/978-3-030-94452-0_31

[6] Kungozhin A. *Existentially closed and maximal models in positive logic*, Algebra and Logic. 51:6 (2013), 496–506. https://doi.org/10.1007/s10469-013-9209-x

[7] Belkasmi M. *Positive Model Theory and Amalgamations*, Notre Dame J. Formal Logic. 55:2 (2014), 205–230. https://doi.org/10.1215/00294527-2420648

[8] Belkasmi M. *Algebraically closed structures in positive logic*, Annals of Pure and Applied Logic. 171:9 (2020).

[9] Belkasmi M. *Positive Amalgamation*, Logica Universalis. 14 (2020), 243–258. https://doi.org/10.1007/s11787-018-0216-5

[10] Belkasmi M. *Weakly and locally positive Robinson theories*, Mathematical Logic Quarterly. 67:3 (2021), 342–353. https://doi.org/10.1002/malq.202000078

[11] Belkasmi M. *Almost Existentially Closed Models in Positive Logic*, International Journal of Mathematics and Mathematical Sciences. 2024:1 (2024). 1–8. https://doi.org/10.1155/2024/5595281

[12] Belkasmi M. *Positive Complete Theories and Positive Strong Amalgamation Property*, Bulletin of the Section of Logic. 53:3 (2024). 301–319 https://doi.org/10.18778/0138-0680.2024.10

[13] Barwise J. *Teoriya modelei: spravochnaia kniga po matematicheskoi logike. Chast' 1 [Model theory: Handbook of mathematical logic. Part 1]*, Izdatel'stvo Nauka, 1982. [in Russian].

[14] Simmons H. *Existentially closed structures*, The journal of Symbolic Logic. 37:2 (1972), 293–310.

[15] Baldwin J., Kueker D. *Algebraically Prime Models*, Annals of Mathematical Logic. 20 (1981), pp. 289–330. https://doi.org/10.1016/0003-4843(81)90007-3

[16] Yeshkeyev A.R. *Existential Prime Convex Jonsson Theories and Their Models*, Bulletin of the Karaganda University. Mathematics Series. 1 (2016), pp. 41–45. http://rep.ksu.kz:80//handle/data/4131

[17] Hodges W.H. *Model Theory*, Cambridge University Press, 2008.

[18] Forrest W. K. *Model theory for universal classes with the amalgamation property: A study in the foundations of model theory and algebra*, Annals of Mathematical Logic, 11:3 (1977), 263–366. https://doi.org/10.1016/0003-4843(77)90001-8

[19] Yeshkeyev, A.R., Tungushbayeva, I.O., Kassymetova, M.T. *Connection between the amalgam and joint embedding properties*, Bulletin of the Karaganda University. Mathematics Series. 1051 (2022), pp. 127–135.

Аманбеков С.М., Онерхаан А., Тунгушбаева И.О. ПОЗИТИВТІ ЙОНСОНДЫҚ ТЕОРИЯЛАРДЫҢ АЯСЫНДА АМАЛЬГАМА ЖӘНЕ БІРЛЕСКЕН ЕНГІЗУ ҚАСИЕТТЕРІ

Бұл мақалада амальгама қасиеті мен бірлескен енгізу қасиетінің, сондай-ақ h-амальгама мен бірлескен жалғастыру қасиетінің өзара байланысына негізделген теориялардың арнайы ішкі кластары зерттеледі. Қарастырылған нәтижелер бірінші ретті классикалық логикада да, позитивті логикада да тұжырымдалады, алынған нәтижелер бойынша олардың құрылымы ұқсас болып қалады. Амальгама қасиеті бірлескен енгізу қасиетін және керісінше, h-амальгама бірлескен жалғастыру қасиетін тудыратын жеткілікті шарттар орнатылады. Сонымен қатар, осы қасиеттердің зерттелетін теориялардың кеңейтулерінде сақталу мәселесі қарастырылады.

Түйін сөздер: экзистенциалды тұйық модель, амальгама қасиеті, бірлескен енгізу қасиеті, позитивті модельдер теориясы, позитивті йонсондық теориялар, позитивті тұйық модельдер, h-амальгама қасиеті, бірлескен жалғастыру қасиеті, позитивті тұйық жай йонсондық теориялар.

Аманбеков С.М., Онерхаан А., Тунгушбаева И.О. СВОЙСТВА МАЛЬГАМИРОВАНИЯ И СОВМЕСТНОГО ВЛОЖЕНИЯ В КОНТЕКСТЕ ПОЗИТИВНЫХ ЙОНСОНОВСКИХ ТЕОРИЙ

В данной статье исследуются специальные подклассы теорий, определяемые связью между свойством амальгамирования и свойством совместного вложения, а также между h-амальгамированием и свойством совместного продолжения. Рассмотренные результаты формулируются как в классической логике первого порядка, так и в позитивной логике, причем структура результатов представляется аналогичной. Нами показаны достаточные условия, при которых свойство амальгамирования влечет свойство совместного вложения, и наоборот, а также условия, при которых h-амальгамирование влечет свойство совместного продолжения, и наоборот. Кроме того, исследуется вопрос сохранения принадлежности теории к данным подклассам при расширении рассматриваемой теории.

Ключевые слова: экзистенциально замкнутая модель, свойство амальгамирования, свойство совместного вложения, позитивная теория моделей, позитивные йонсоновские теории, позитивно замкнутые модели, свойство h-амальгамирования, свойство совместного продолжения, позитивно экзистенциально простые йонсоновские теории.