Kazakh Mathematical Journal ISSN 1682–0525

25:2 (2025) 19–24

On T-pseudofinite models of universal theories T

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Communicated by: Bektur S. Baizhanov

Received: 05.05.2025 ⋆ Accepted/Published Online: 12.05.2025 ⋆ Final Version: 05.05.2025

Abstract. The concept of pseudofinite structures emerged in the 1960s as part of efforts to understand infinite structures that behave, in certain respects, like finite ones. A structure is pseudofinite if it satisfies every first-order sentence that holds in all finite structures of the same language. This idea gained importance through works by Ax, who studied pseudofinite fields, and later by Hrushovski and others in the context of model-theoretic algebra. Pseudofiniteness has since played a key role in finite model theory and asymptotic classes. The article considers universal theories T, the number of isomorphism types of whose finite models is finite. It is proved that all cyclic submodels of a T-pseudofinite model of this theory are finite.

Keywords. universal axiomatizable class of L-structures, theory of a class of L-structures, pseudofinite structure, T-pseudofinite structure.

1 Introduction

The concept of pseudofinite structures emerged as part of a broader effort in model theory to understand the relationship between finite and infinite models, particularly within first-order logic. The development of this idea can be traced back to the 1960s, at the intersection of logic, algebra, and number theory, when logicians began to investigate infinite structures that could be characterized by the same first-order sentences as finite structures.

A structure is called pseudofinite if it is infinite, yet satisfies every first-order sentence that holds in all finite structures of the same language. More precisely, a model $\mathfrak{M}$ is pseudofinite if it is elementarily equivalent to an ultraproduct of finite structures. This notion allows logicians to treat certain infinite models as “limits” or idealizations of finite ones, enabling the application of model-theoretic methods to problems originally rooted in finite mathematics.

2020 Mathematics Subject Classification: 03C60, 03C52.

Funding: This work was supported by the Ministry of Science and Higher Education (agreement № 075-02-2025-1638/1 of 10.03.2025).

DOI: https://doi.org/10.70474/fxyhwj58

One of the foundational developments in the area was James Ax's work in the late 1960s, particularly his characterization of pseudofinite fields. In his landmark paper "The Elementary Theory of Finite Fields" (1968), Ax proved that the theory of finite fields is complete and that every pseudofinite field is elementarily equivalent to an ultraproduct of finite fields. He further showed that pseudofinite fields are perfect, have exactly one extension of each finite degree, and are pseudo-algebraically closed. This result brought substantial attention to the utility of ultraproducts in connecting finite and infinite model-theoretic behavior.

Throughout the 1970s and 1980s, pseudofiniteness became an increasingly important concept in various branches of mathematical logic and algebra. Researchers began to apply it not only in the study of fields but also to groups, rings, and other algebraic structures. The broader idea of interpreting "finiteness-like" behavior in infinite models proved valuable in understanding the asymptotic properties of classes of finite structures, which became central in finite model theory and descriptive complexity theory.

In the 1990s and 2000s, Ehud Hrushovski significantly expanded the theoretical framework surrounding pseudofinite structures. His work on Zariski geometries and non-standard finite fields used pseudofinite techniques to derive deep results in number theory and algebraic geometry. Hrushovski's application of model-theoretic tools to diophantine geometry and the Mordell-Lang conjecture marked a new era in the interplay between logic and classical mathematics.

Pseudofiniteness also plays a crucial role in the study of asymptotic classes, random structures, and finite model theory, particularly in the context of computer science. The work of Macpherson, Pillay, and others on simple theories and the classification of pseudofinite groups has led to further connections with permutation group theory and stability theory.

Today, the study of pseudofinite structures continues to be a vibrant area within model theory. It lies at the crossroads of logic, algebra, and combinatorics, providing a unifying framework for analyzing infinite structures through the lens of finite approximations. This duality remains a powerful conceptual and technical tool in both pure and applied model theory.

Recently, the model theory of pseudofinite structures is an actively developing area of mathematics. In [1–3] and [4], the model-theoretic properties of theories of pseudofinite fields, groups, rings, and acts over monoids are studied. Clearly, given a pseudofinite model $\mathfrak{M}$ of some theory T in a language L and a sentence true in $\mathfrak{M}$, a finite model of this sentence may not be a model of T. For example, if ${}_SA$ is a pseudofinite act over a monoid S and ${}_SA \models \Phi$, then $\mathfrak{B} \models \Phi$ for some finite structure $\mathfrak{B}$ in the language of acts over a monoid S, but $\mathfrak{B}$ may not be an act over S. So, it is natural to consider the concept of T-pseudofiniteness for a theory T of a language L, which was introduced in [7]. A model $\mathfrak{M}$ of a theory T in a language L is called T-pseudofinite if every sentence in a language L true in $\mathfrak{M}$ has a finite model, which is a model of the theory T. It is clear that T-pseudofiniteness implies pseudofiniteness, and pseudofiniteness implies T-pseudofiniteness for every finite axiomatizable theory T. In [7, 8],

T-pseudofinite acts over a monoid S are considered, where T is a theory of all acts over S.

Also, we can note the articles [5] and [6]. In [5], S. Malyshev gives the description of types of pregeometries with an algebraic closure operator for acyclic theories. In [6], N. Markhabatov and Ye. Baisalov consider acyclic graphs approximated by finite acyclic graphs.

In this work, we consider universal theories T, the number of isomorphism types of whose finite models is finite. We prove that all cyclic submodels of a T-pseudofinite model of this theory are finite.

2 Preliminaries

A structure $\mathfrak{M}$ in a language L is called *pseudofinite* if every sentence true in $\mathfrak{M}$ has a finite model. Let T be a theory of a language L. A model $\mathfrak{M}$ of a theory T in the language L is called T-*pseudofinite* if every sentence in a language L true in $\mathfrak{M}$ has a finite model, which is a model of the theory T.

Theorem 1 (A.A. Stepanova, E.L. Efremov, S.G. Chekanov [7])**.** *Let T be a theory of a language L and $\mathfrak{M}$ be a model of T. Then $\mathfrak{M}$ is a T-pseudofinite structure if and only if $\mathfrak{M}$ is elementarily equivalent to the ultraproduct of finite models of the theory T.*

Theorem 2 (A.A. Stepanova, E.L. Efremov, S.G. Chekanov [3])**.** *Every coproduct of finite S-acts is a T-pseudofinite S-act, where T is the theory of all S-acts.*

A class K of L-structures is called *axiomatizable* if there exists a set Z of sentences of the language L such that for any structure $\mathfrak{A}$,

$$\mathfrak{A} \in K \iff (\text{the language of } \mathfrak{A} \text{ is } L \text{ and } \mathfrak{A} \models \Phi \text{ for all } \Phi \in Z). \tag{1}$$

An axiomatizable class K of L-structures is called *universal axiomatizable* if there exists a set Z of $\forall$-sentences of the language L for which (1) holds.

A substructure $\mathfrak{B}$ of an L-structure $\mathfrak{A}$ is called *one-generated* or *cyclic* if there exists $b \in B$ such that the intersection of all substructures of $\mathfrak{A}$ containing b coincides with $\mathfrak{B}$. In this case, we denote the substructure $\mathfrak{B}$ by $\langle b \rangle$.

3 Main result

Theorem 3. *Let $l \in \omega$, let K be a universal axiomatizable class of L-structures such that the cardinality of any finite cyclic structure in K is less than $l+1$, and let T be a theory of K. If $\mathfrak{A}$ is a T-pseudofinite structure, then the cardinality of any cyclic substructure of $\mathfrak{A}$ is less than $l+1$.*

Proof. Let the hypotheses of the theorem be satisfied, and let $\mathfrak{A} \in K$ be a T-pseudofinite structure. By Theorem 1, $\mathfrak{A} \equiv \mathfrak{B}$, where $\mathfrak{B} = \prod_{i \in I} \mathfrak{B}_i / D$, $\mathfrak{B}_i$ are finite models of T, D is an ultrafilter on I. Since T is a theory of an axiomatizable class K, then $\mathfrak{B} \in K$ and $\mathfrak{B}_i \in K$ for all $i \in I$. We prove that the cardinalities of all cyclic substructures of $\mathfrak{B}$ are less than $l+1$. Assume the opposite, that is, there exists a cyclic substructure $\langle b/D \rangle$ of $\mathfrak{B}$ such that $|\langle b/D \rangle| > l$. Then $\mathfrak{B} \models \Phi_{t_0,\ldots,t_l}(b/D)$, where

$$\Phi_{t_0,\ldots,t_l}(x) \leftrightharpoons \exists x_0 \ldots \exists x_l \left(\bigwedge_{0 \le i < j \le l} x_i \ne x_j \wedge \bigwedge_{0 \le i \le l} x_i = t_i(x) \right),$$

$t_i(x)$ are some terms of the language L. By Los's theorem,

$$J = \{ i \in I \mid \mathfrak{B}_i \models \Phi_{t_0,\ldots,t_l}(b(i)) \} \in D,$$

that is, $|\langle b(i) \rangle| > l$ for all $i \in J$. Since K is a universal axiomatizable class, then $\langle b(i) \rangle \in K$ for all $i \in J$. But the cardinality of any finite cyclic structure in K is less than $l+1$. A contradiction. Consequently, the cardinalities of all cyclic substructures of $\mathfrak{B}$ are less than $l+1$.

Now we prove that the cardinality of any cyclic substructure of $\mathfrak{A}$ is less than $l+1$. Assume the converse, that is, there exists $a \in A$ such that $|\langle a \rangle| > l$. Then $\mathfrak{A} \models \Phi_{g_0,\ldots,g_l}(a)$ for some terms $g_0, \ldots, g_l$ of the language L. Since the structures $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent, $\mathfrak{B} \models \exists x \Phi_{g_0,\ldots,g_l}(x)$. Therefore, there exists a cyclic substructure $\mathfrak{B}$ of cardinality greater than l. This contradiction proves the theorem. □

4 Corollaries from the main result

Corollary 4. *Let S be a monoid, $l \in \omega$, K be the class of all S-acts such that the cardinality of any finite cyclic S-act is less than $l+1$, and T be a theory of K. If ${}_SA$ is a T-pseudofinite S-act, then the cardinality of any cyclic subact of ${}_SA$ is less than $l+1$.*

Proof. Since the class K is universal axiomatizable, this Corollary follows from Theorem 3. □

By Corollary 4, we obtain the following corollary.

Corollary 5 (A.A. Stepanova, E.L. Efremov, S.G. Chekanov [7]). *Let S be a monoid, the number of isomorphism types of finite cyclic S-acts be finite, and T be a theory of S-acts. If ${}_SA$ is a T-pseudofinite S-act, then every cyclic subact of ${}_SA$ is finite.*

Corollary 6. *Let G be an abelian group, the number of finite index subgroups of G be finite, and T be a theory of all G-acts. Then ${}_GA$ is a T-pseudofinite G-act if and only if ${}_GA$ is a coproduct of finite G-acts.*

Proof. Let ${}_GA$ be a T-pseudofinite G-act. It is well known that every act over a group G is a coproduct of cyclic acts, and every cyclic act over G is isomorphic to a G-act ${}_GG/H$, where H is a subgroup of G and the unary operations on G/H are defined as follows: $g(aH) = (ga)H$ for any $g, a \in G$. Then by Corollary 4 ${}_GA$ is a coproduct of finite G-acts.

If ${}_GA$ is a coproduct of finite G acts, then by Theorem 2 ${}_GA$ is T-pseudofinite. □

Corollary 7. *Let K be the class of all abelian groups such that the number of isomorphism types of finite cyclic subgroups of groups in K is finite, T be the theory of class K, and let G be a pseudofinite (T-pseudofinite) group. Then G is a periodic group.*

Proof. Since the class K is universal axiomatizable, this corollary follows from Theorem 3. □

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Степанова А. А., Ефремов Е. Л., Чеканов С.Г. УНИВЕРСАЛДЫҚ T ТЕОРИЯЛАРЫНЫҢ T-ПСЕВДО-ШЕКТІ МҮЛДЕЛЕРІНДЕ

Жалған ақырлы құрылымдар ұғымы 1960-жылдары, шексіз құрылымдарды белгілі бір мағынада ақырлы құрылымдар сияқты сипаттау мақсатында пайда болды. Құрылым жалған ақырлы деп аталады, егер ол сол сигнатурадағы барлық ақырлы құрылымдарда орындалатын бірінші реттік тұжырымдарға бағынса. Бұл идея алғаш рет псевдоақырлы өрістерді зерттеген Акс еңбектерінде маңызға ие болды, кейінірек Хрушевский және басқа зерттеушілер модельдік теориялық алгебра саласында оны әрі қарай дамытты. Содан бері жалған ақырлылық ақырлы модельдер теориясында және асимптотикалық кластарда маңызды рөл атқарады. Мақалада T әмбебап теориялары қарастырылады, олардың шекті модельдерінің изоморфизм түрлерінің саны шекті. Бұл теорияның T-псевдофинитті моделінің барлық циклдік ішкі модельдері ақырлы болатыны дәлелденді.

Түйін сөздер: L-құрылымдардың әмбебап аксиоматизацияланатын класы, L-құрылымдар класының теориясы, псевдофинитті құрылым, T-псевдофинитті құрылым.

Степанова А. А., Ефремов Е. Л., Чеканов С.Г. О T-ПСЕВДОКОНЕЧНЫХ МОДЕЛЯХ УНИВЕРСАЛЬНЫХ ТЕОРИЙ T

Понятие псевдоконечных структур возникло в 1960-х годах в рамках попыток понять бесконечные структуры, которые в определённом смысле ведут себя как конечные. Структура называется псевдоконечной, если она удовлетворяет каждому предложению первого порядка, которое выполняется во всех конечных структурах того же языка. Эта идея приобрела значение благодаря работам Акса, изучавшего псевдоконечные поля, а позже Хрушевского и других — в контексте модельно-теоретической алгебры. Псевдоконечность с тех пор играет важную роль в теории конечных моделей и асимптотических классов. В статье рассматриваются универсальные теории T, число типов изоморфизмов конечных моделей которых конечно. Доказывается, что все циклические подмодели T-псевдоконечной модели этой теории конечны.

Ключевые слова: универсально аксиоматизирумый класс L-структур, теория класса L-структур, псевдоконечная структура, T-псевдоконечная структура.