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Evaluation of solutions of one class of finite-dimensional nonlinear equations. II

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Abstract. In this article, we obtain two theorems on a priori estimates for solutions of nonlinear equations in a finite-dimensional space. These theorems are proved under certain conditions, which are borrowed from the conditions which are satisfied by finite-dimensional approximations of one class of nonlinear initial-boundary value problems. This article is a continuation of the first part with the same title. In this paper, we prove the second theorem.

Keywords. Differential operator, nonlinear equation, existence of a solution, uniqueness of a solution, a priori estimation of a solution.

1 Introduction

In many problems of mathematical physics, the law of energy conservation makes it possible to prove the existence of a solution that satisfies an energy estimate. However, when the number of spatial variables $n \geq 3$, such estimates generally do not allow the use of perturbation methods.

Solutions for which perturbation theory is not applicable—more precisely, those that do not permit linearization or refinement through small parameter expansions—are commonly referred to as “weak” solutions.

The applicability of perturbation methods plays a central role in the analysis of problems in mathematical physics. Accordingly, the theory of differential equations places a significant emphasis on establishing the existence of solutions that admit such techniques.

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Solutions that permit the application of perturbation theory are often referred to as “strong” solutions, though this terminology may depend on the specific analytical framework.

Many problems of mathematical physics can be written in “restricted notation” (in the form of an integral equation), usually of the following form

$$f(u) = u + L(u) = g, \tag{1}$$

where $L(u)$ is the nonlinear part. This equation is often studied in the metric of some Banach or Hilbert space H.

When moving to an “abbreviated notation” the energy estimate, usually performed for problems in mathematical physics, will turn into an a priori estimate of the following form

$$\|G(u)\| \leq C \cdot \|u + L(u)\| = C\|g\|, \tag{2}$$

where C is a constant number independent of $u \in H$, and G is a completely continuous operator in H.

An a priori estimate (2) usually does not allow the use of perturbation theory. Therefore, it becomes necessary to obtain an estimate of the following form

$$\|u\| \leq \varphi\,(\|f(u)\|), \tag{3}$$

where $\varphi(\cdot)$ is a continuous function on $[0, \infty)$.

The presence of an estimate of the form (3), as a rule, opens the possibility of using perturbation theory (with an appropriate choice of the space H).

A very important problem is the problem of the existence of a sequence of finite-dimensional approximations of the problem (1) (more precisely, approximations of the operation $u+L(u)$):

$$f_1(\cdot),\ f_2(\cdot),\ \dots\ ,\ f_n(\cdot),\ \dots \tag{4}$$

considered in the spaces

$$H_1,\ H_2,\ \dots\ ,\ H_n,\ \dots\ , \quad \dim H_n = n, \tag{5}$$

such that a priori estimates of the form (2) are satisfied and it is possible to obtain an estimate similar to (3).

It is implied that $H_n\,(n = 1, 2, \dots)$ is a subspace of H and the metric H_n is the metric induced from the metric of H.

The problem of describing the dynamics of an incompressible fluid, due to its theoretical and applied importance, attracts the attention of many researchers.

This work is devoted to the problem of the existence and smoothness of solutions to equations of mathematical physics [1].

The articles [2–4] provide a fairly complete analysis of the current state of the problem and a review of the available literature, and propose methods to solve the problem. The articles [5–13] are devoted to the study of the solvability in general of equations of mathematical physics, the continuous dependence of the solution of a parabolic equation, and the smoothness of the solution.

This work arose as a result of numerous attempts by the authors to solve the problem of the existence of a strong solution to an equation of mathematical physics.

In this work, we obtain two theorems on a priori estimates of solutions to nonlinear equations in a finite-dimensional Hilbert space. The work consists of four sections. The first section is devoted to the introduction and origin of the problem. The second section provides the notation used and the formulation of the main results. The third section provides a proof of Theorem 1, which in the limit gives weak solvability of many problems of mathematical physics. In the fourth paragraph, we prove Theorem 2, which in the limit allows us to establish strong solvability of some problems of mathematical physics that admit perturbation theory. The conditions of the theorems are such that they can be used in studying a certain class of initial-boundary value problems to obtain strong a priori estimates in the presence of weak a priori estimates.

This paper is the second part of the paper [14].

2. The conditions used and the formulation of the results

Let us derive uniform estimates for nonlinear problems in a finite-dimensional space. The equations under consideration are (usually) analogs of finite-dimensional approximations of equations of mathematical physics written in "abbreviated notation".

Throughout this section, H is a finite-dimensional real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$.

We will be interested in an equation of the following form

$$u + L(u) = g \in H, \tag{6}$$

where $L(\cdot)$ is a nonlinear continuous transformation, g is an element of the space H. The solution u of problem (6) is sought in H.

We are focused on such finite-dimensional equations of the form (6) that are finite-dimensional approximations of infinite-dimensional problems of the form (6) in an infinite-dimensional Hilbert space. In this case, it will turn out to be very important to obtain estimates that are independent of the approximation number and allow one to pass to the limit and obtain a priori estimates in the limit for solving the infinite-dimensional problem.

It will be very important to obtain estimates that do not depend on the number of approximations, allowing one to pass to the limit and to obtain in the limit a priori estimates for solving an infinite-dimensional problem. Infinite-dimensional problems of the form (6), on which we are focused in what follows, are, as a rule, problems of mathematical physics written in a limited form.

Here and everywhere below, $f(u)$ will mean an operation of the form

$$f(u) := u + L(u). \tag{7}$$

If $\xi \in [0, +\infty)$ is a parameter and the vector $u(\xi)$ is a vector function continuously differentiable with respect to the parameter ξ, then we will assume that the vector-function $L(u(\xi))$, is also continuously differentiable, as well as the expressions that arise from $L(u)$ and $f(u)$.

We introduce the notation L_u:

$$(L(u(\xi)))_\xi = L_{u(\xi)} u_\xi(\xi). \tag{8}$$

It is obvious that L_u (for each $u \in H$) will be a linear operator

$$L_u v = (L(u(\xi)))|_{u_\xi = v}. \tag{9}$$

We have

$$(f(u(\xi)))_\xi = u_\xi + L_u u_\xi = (E + L_u) u_\xi.$$

In what follows, if $u_0, v_0 \in H$, then the vector $L_{u_0} v_0$ is understood as follows: we take a continuously differentiable vector function $u(\xi)$ such that

$$u|_{\xi=0} = u_0, \quad u_\xi(\xi)|_{\xi=0} = v_0$$

and for $L_{u_0} v_0$ we take the vector

$$L_{u_0} v_0 = (L(u(\xi))_\xi \big|_{\xi=0}.$$

Here and everywhere in what follows, E is an identity operator.
Let us denote

$$D_u = E + L_u, \qquad D_u^* = E + L_u^*, \tag{10}$$

$$D_u^* f(u) = (E + L_u^*) f(u). \tag{11}$$

$$M_u a = \Big(D^*_{u(\xi)} f(u(\xi))\Big)_\xi \Big|_{\substack{u(\xi) = u \\ u_\xi(\xi) = a}} = M_u u_\xi\big|_{u_\xi = a} = M_u a. \tag{12}$$

We present the conditions that we are going to use.

Condition U1: *For operators $L(\cdot)$, L_u, L_u^*, D_u, D_u^* the following conditions are satisfied*

$$\begin{cases} \|M_u - M_v\|_{H\to H} + \|L(u) - L(v)\| + \|L_u - L_v\|_{H\to H} + \\ \qquad\qquad + \|L_u^* - L_v^*\|_{H\to H} \le \psi\left(\|u\|\right)\psi\left(\|v\|\right)\|u-v\|, \\ \|M_v u\| + \|D_v^* u\| + \|D_v u\| \le \psi\left(\|v\|\right)\|u\|, \end{cases} \tag{13}$$

where $\|\cdot\| = \|\cdot\|_H$, $\psi(\cdot)$ *is a non-decreasing on* $[0,\infty)$, *positive continuous function.*

Condition U2: *There exist linear invertible operators T and Q such that*

$$\|T\| \le C_T, \quad \|Q\| \le C_T, \quad \|T^{-1}\| < \infty, \quad \|Q^{-1}\| < \infty, \tag{14}$$

and for any $u \in H$ the inequalities hold

$$\langle Tu, L(u)\rangle \ge 0, \quad \langle Tu, u\rangle \ge \|Qu\|^2. \tag{15}$$

In (14) C_T *is some fixed constant number.*

In what follows, C or c (uppercase or lowercase, with or without indices) will denote constant numbers (generally speaking, different in different places), independent of the adjacent factors.

Theorem 1. [14] *Let condition U1 and condition U2 be satisfied. Then for any $g \in H$ the problem*

$$f(u) = g \tag{16}$$

has a solution $u \in H$, satisfying the estimate

$$\|Qu\|^2 \le C_T \|g\|^2, \tag{17}$$

where Q is the operator from condition U2, and C_T is the constant from condition U2.

The proof of Theorem 1 is given in the first part of the article [14].

The notation of the transformations $f(u)$, $L(u)$, the operators $L_u\cdot$, $D_u\cdot$, $M_u\cdot$ (defined for each $u \in H$, (see (6)–(11)) and their conjugates L_u^*, D_u^* and M_u will be used without reservations.

We will also introduce the following notations:

$$J(u) = \|u\|^2 \exp\left\{-\|f(u)\|^2\right\}, \tag{18}$$

$$N(u) = D_u^* f(u) - \gamma(u)\, u\,. \tag{19}$$

We often use the notations (18) and (19) without reservations, as well as the notations that arise in the formulations of conditions U1 and U2, and the notations that arise in the formulations of conditions U3 and U4 given below.

Condition U3: *There exists an invertible operator G, such that*

$$\|G\|_{H\to H} \le C_0 < \infty, \quad \|G^{-1}\| < \infty \tag{20}$$

and for any $u \in H$ the inequality

$$\|G\,u\|^2 \le d_0\ \|f(u)\|^2, \tag{21}$$

where $d_0 > 0$ is a constant.

Condition U4: *If $0 \neq u_0 \in H$, $\gamma(u) > \|u\|^{-2}$ and $N(u) = 0$, then strict inequalities are satisfied*

$$\inf_{\{a\}} \frac{\langle M_u P_u a, P_u a\rangle - \gamma(u)\|P_u a\|^2}{\|P_u a\|^2} < 0 < \sup_{\{a\}} \frac{\langle M_u P_u a, P_u a\rangle - \gamma(u)\|P_u a\|^2}{\|P_u a\|^2}, \tag{22}$$

where $P_u a$ is an orthogonal projector.

The following theorem is true.

Theorem 2. *If conditions U1, U3, and U4 are satisfied, then for any $u \in H$ the a priori estimate holds:*

$$\|u\|^2 \le C\ \exp\left\{\|f(u)\|^2\right\}. \tag{23}$$

Note that the estimate (23) is satisfied if conditions U1, U3 and the following condition U5 are satisfied.

Condition U5: *There exist constant numbers c_0, c_1, m and a self-adjoint operator T, such that if $\|u\| \ge 1$, then the inequalities are satisfied*

$$\|L(u)\| \ge c_0\ \|Tu\|^m, \qquad \|u\| \le c_1\ \|u\|^m. \tag{24}$$

Remark 1. If the conditions of Theorem 1 are satisfied, that is, conditions U1 and U2 are satisfied, then condition U3 is also satisfied.

Remark 2. Theorem 1 allows us to prove the existence of a "weak" solution to some problems of mathematical physics. To prove the existence of a "strong" solution, which allows us to use perturbation theory for some problems of mathematical physics, we need another finite-dimensional theorem, which will be proved under conditions U1, U3, and U4.

For $0 \neq u_0 \in H$, we set

$$a\,(u_0) = \sup \exp\left\{-\|u\|^2 + \|f(u)\|^{2\alpha}\right\} = \sup\ R(u), \tag{25}$$

where $\alpha > 1$ and the *supremum* is taken over all vectors $u \in H$ such that

$$J(u) \ge J(u_0)\,. \tag{26}$$

Lemma 1. *For any $c > 0$, the set*

$$M^{(c)} = \{u : J(u) \geq c\} \tag{27}$$

is compact.

Since H is finite-dimensional, it suffices to prove the boundedness of the set $M^{(c)}$.

Proof. We prove the lemma by reasoning "by contradiction". Let the set $M^{(c)}$ be unbounded.

For $u \in M^{(c)}$ we have

$$\begin{gathered} 0 < c \leq J(u) = \|u\|^2 \exp\left\{-\|f(u)\|^2\right\} \leq \\ \leq \|u\|^2 \exp\{-d_0^{-1}\|G\,u\|^2\} \leq \|u\|^2 \exp\left\{-c_1 \|u\|^2\right\}, \end{gathered} \tag{28}$$

where $c_1 > 0$ is a constant.

When deriving (28), we have used condition U3.

When $\|u\| \to \infty$, the right side of (28) tends to zero. We have obtained a contradiction. Consequently, the set $M^{(c)}$ is bounded. The lemma is proved.

The following lemma holds.

Lemma 2. *Let $0 \neq u_0 \in H$. Then there exists a vector $\widetilde{u}_0 \in H$, realizing the supremum* (25)–(26). *For $\widetilde{u}_0$ the following inequalities hold*

$$J(\widetilde{u}_0) \geq J(u_0), \quad \exp\left\{-\|\widetilde{u}_0\|^2 + \|f(\widetilde{u}_0)\|^{2\alpha}\right\} \geq \exp\left\{-\|u_0\|^2 + \|f(u_0)\|^{2\alpha}\right\}. \tag{29}$$

Proof. The functionals $J(u_0)$ and $\exp\left\{-\|u\|^2 + \|f(u)\|^{2\alpha}\right\}$ are continuous. Since *supremum* is taken over a set which is compact according to Lemma 1, we obtain the existence of $\widetilde{u}_0$. The fulfillment of the inequalities (29) follows from the definition (see (25) and (26)).

The lemma is proved.

Let $u = u(\xi)$ be a continuously differentiable vector function.
Then for the functionals

$$J(u(\xi)) = \|u\|^2 \exp\left\{-\|f(u)\|^{2\alpha}\right\}, \quad R(u(\xi)) = \exp\left\{-\|u\|^2 + \|f(u)\|^{2\alpha}\right\} \tag{30}$$

we have

$$\begin{gathered} J_\xi(\,u(\xi)) = 2J(u)\left[\left\langle \left(\frac{1}{\|u\|^2} - \gamma(u)\right) u - D_u^* f(u) + \gamma(u)u, u_\xi \right\rangle\right] = \\ 2J(u)\left[\left\langle \left(\frac{1}{\|u\|^2} - \gamma(u)\right) u - N(u), u_\xi \right\rangle\right], \end{gathered} \tag{31}$$

$$R_\xi(\,u(\xi)) = 2R(u)\left[\left\langle N(u) + \alpha\,\|f(u)\|^{2(\alpha-1)}\, D_u^* f(u), u_\xi \right\rangle\right] =$$

$$2R(u)\left[\left\langle\left(-1+\alpha\,\gamma(u)\,\|f(u)\|^{2(\alpha-1)}\right)\,u+\alpha\,\|f(u)\|^{2(\alpha-1)}N(u),u_\xi\right\rangle\right], \tag{32}$$

recall that $N(u)$ is from (19).

If we define the vector function $u=u(\xi)$ as a solution to the problem

$$\begin{cases} u_\xi = x\,u + y\,N(u), \\ u|_{\xi=0} = \widetilde{u}_0, \end{cases} \tag{33}$$

then from (31) and (32) we derive:

$$J_\xi(u(\xi)) \;=\; 2J(u)\left[\left(\frac{1}{\|u\|^2}-\gamma(u)\right)\|u\|^2x-\|N(u)\|^2y\right], \tag{34}$$

$$R_\xi(u(\xi)) \;=\; 2R(u)\left[\left(-1+\alpha\,\gamma(u)\,\|f(u)\|^{2(\alpha-1)}\right)\|u\|^2x+\right. \tag{35}$$

$$\left.+\;\alpha\,\|f(u)\|^{2(\alpha-1)}\,\|N(u)\|^2\,y\right]. \tag{36}$$

The following lemma is true.

Lemma 3. *Let $0\neq u_0\in H$ and $\widetilde{u}_0$ be a vector that realizes the supremum* (25)–(26). *Then, if $\|f(u_0)\|\geq 1$, then the following inequalities hold:*

$$\left(\frac{1}{\|\widetilde{u}_0\|^2}-\gamma(\widetilde{u}_0)\right)\cdot\left(-1+\alpha\,\gamma(\widetilde{u}_0)\,\|f(\widetilde{u}_0)\|^{2(\alpha-1)}\right)\leq 0.$$

Proof. Assume the contrary.

Let the lemma inequality not hold. In the problem (33) we choose

$$x=\left(\frac{1}{\|\widetilde{u}_0\|^2}-\gamma(\widetilde{u}_0)\right),\; y=0,$$

then from (34) and (36) we obtain that the quantities $J(u(\xi))$ and $R(u(\xi))$ do not decrease in the neighborhood of the point $\xi=0$, and $R(u(\xi))$ strictly increases. Therefore, there exists a point $\xi_0>0$ such that

$$J(u(\xi_0))\geq J(\widetilde{u}_0),\quad R(u(\xi_0))>R(\widetilde{u}_0), \tag{37}$$

and the second inequality is strict.

The fulfillment of (37) contradicts the origin of the vector $\widetilde{u}_0$. Therefore, the lemma is proved.

The following lemma holds.

Lemma 3. *Let $0\neq u_0\in H$ and $\widetilde{u}_0$ be a vector that realizes the supremum* (25)–(26). *Then, if $N(\widetilde{u}_0)\neq 0$, then the equality holds:*

$$\|\widetilde{u}_0\|^2=\alpha\,\|f(\widetilde{u}_0)\|^{2(\alpha-1)}. \tag{38}$$

Proof. Let us consider a system of linear inhomogeneous equations

$$\begin{cases} \left(\dfrac{1}{\|\widetilde{u}_0\|^2} - \gamma(\widetilde{u}_0)\right) \|\widetilde{u}_0\|^2\, x - \|N(\widetilde{u}_0)\|^2\, y = d_1, \\ \\ \left(-1 + \alpha\, \gamma(\widetilde{u}_0) \|f(\widetilde{u}_0)\|^{2(\alpha-1)}\right) \|\widetilde{u}_0\|^2\, x + \alpha\, \|f(\widetilde{u}_0)\|^{2(\alpha-1)} \|N(\widetilde{u}_0)\|^2\, y = d_2, \end{cases} \tag{39}$$

with respect to unknowns x and y.

According to a well-known theorem of linear algebra, due to $\|N(\widetilde{u}_0)\|^2 \neq 0$, we obtain that the system has a unique solution if

$$\Delta = \left(\frac{1}{\|\widetilde{u}_0\|^2} - \gamma(\widetilde{u}_0)\right) \cdot \alpha\, \|f(\widetilde{u}_0)\|^{2(\alpha-1)} + \left(-1 + \alpha\, \gamma(\widetilde{u}_0) \|f(\widetilde{u}_0)\|^{2(\alpha-1)}\right) =$$

$$= \frac{\alpha}{\|\widetilde{u}_0\|^2}\, \|f(\widetilde{u}_0)\|^{2(\alpha-1)} - 1 \neq 0.$$

Therefore, if (37) is not satisfied, choosing in (38) $d_1 = 1,\ d_2 = 1$, we obtain that the quantities $J(u(\xi))$ and $R(u(\xi))$ in the neighborhood of the point $\xi = 0$ strictly increase. Consequently, there exists $\xi_0 > 0$ such that the inequalities (37) are satisfied:

$$J(u(\xi_0)) \geq J(\widetilde{u}_0), \quad R(\widetilde{u}_0) > R(u_0).$$

The fulfillment of these inequalities contradicts the origin of the vector $\widetilde{u}_0$, realizing *supremum* (25)–(26). Therefore, (38) is fulfilled.

The lemma is proved.

The following lemma is true.

Lemma 5. *Let $0 \neq u_0 \in H$ and $\widetilde{u}_0$ be a vector that realizes the supremum* (25)–(26). *Assume that the following condition is satisfied:*

$$N(\widetilde{u}_0) = 0, \quad 1 - \gamma(\widetilde{u}_0)\, \|\widetilde{u}_0\|^2 \neq 0. \tag{40}$$

Then the equality (38) *from Lemma 4 is satisfied, i.e.*

$$\|\widetilde{u}_0\|^2 = \alpha\, \|f(\widetilde{u}_0)\|^{2(\alpha-1)}.$$

Proof. We define the vector function $u = u(\xi)$ as a solution to the problem

$$\begin{cases} u_\xi = x \cdot \xi\, u + P_u\, a, \\ u|_{\xi=0} = \widetilde{u}_0\,. \end{cases} \tag{41}$$

From (31) and (32), using (45), we derive:

$$J_\xi(u(\xi)) = 2J(u)\left[\left(\frac{1}{\|u\|^2} - \gamma(u)\right)\xi\,\|u\|^2 x - \langle N(u), a\rangle\right] =$$
$$2J(u)\left[\left(\frac{1}{\|u\|^2} - \gamma(u)\right)\xi\,\|u\|^2 x - \xi\cdot\left\langle\left(\frac{1}{\xi}\int_0^\xi N_\eta\,(u(\eta))\,d\eta\right),\,a\right\rangle\right] = \tag{42}$$
$$2J(u)\left[\left(\frac{1}{\|u\|^2} - \gamma(u)\right)\xi\,\|u\|^2 x - \xi\left\langle N_\eta\,(u(\eta))\,\big|_{\eta=0}, a\right\rangle + \xi^2\,O(1)\right];$$

$$R_\xi(u(\xi)) = 2R(u)\left[\left(-1 + \alpha\,\gamma(u)\,\|f(u)\|^{2(\alpha-1)}\right)\,\xi\,\|u\|^2\,x + \right.$$
$$\left.\alpha\,\|f(u)\|^{2(\alpha-1)}\cdot\xi\left\langle N_\eta\,(u(\eta))\,\big|_{\eta=0}, a\right\rangle + \xi^2\,O(1)\right]. \tag{43}$$

By virtue of (40) for small (but not equal to zero) ξ, we can eliminate x from (42) and (43) by setting

$$\|u\|^2\,x = \left(\frac{1}{\|u\|^2} - \gamma(u)\right)^{-1}\frac{1}{\xi}\,\left\langle\int_0^\xi N_\eta\,(u(\eta))\,d\eta,\,a\right\rangle.$$

Then we have

$$J_\xi(u(\xi)) = 0, \tag{44}$$

$$R_\xi(u(\xi)) = 2R(u)\left[\left(-1 + \alpha\,\gamma(u)\,\|f(u)\|^{2(\alpha-1)}\right)\left(\tfrac{1}{\|u\|^2} - \gamma(u)\right)^{-1} + \right.$$
$$\left.\alpha\,\|f(u)\|^{2(\alpha-1)}\right]\,\xi\,\left(\left\langle N_\eta\,(u(\eta))\,\big|_{\eta=0}, a\right\rangle + \xi^2\,O(1)\right) = \tag{45}$$
$$2R(u)\left(\tfrac{1}{\|u\|^2} - \gamma(u)\right)^{-1}\left(-1 + \tfrac{\alpha\,\|f(u)\|^{2(\alpha-1)}}{\|u\|^2}\right)\cdot$$
$$\cdot\left[\xi\left\langle N_\eta\,(u(\eta))\,\big|_{\eta=0}, a\right\rangle + \xi^2\,O(1)\right].$$

At the same time

$$\langle N_\eta\,(u(\eta))\,\big|_{\eta=0}, a\rangle = \langle M_{\widetilde{u}_0}P_{\widetilde{u}_0}a, a\rangle\,. \tag{46}$$

By virtue of condition U4, the vector a should be chosen such that $P_{\widetilde{u}_0}a = a$ and so that the strict inequality holds

$$2R(\widetilde{u}_0)\left(\frac{1}{\|\widetilde{u}_0\|^2} - \gamma(\widetilde{u}_0)\right)^{-1}\left(-1 + \frac{\alpha\,\|f(\widetilde{u}_0)\|^{2(\alpha-1)}}{\|\widetilde{u}_0\|^2}\right)\cdot\left\langle N_\eta\,(u(\eta))\,\big|_{\eta=0}, a\right\rangle > 0.$$

From this and from (44) and (45) it follows that there exists a small $\xi_0 > 0$ (not equal to zero) such that the equality and strict inequality

$$J(u(\xi_0)) = J(\widetilde{u}_0),$$

$$R(u(\xi_0)) > R(\widetilde{u}_0),$$

that contradict the origin of the vector $\widetilde{u}_0$ are satisfied. Therefore, the lemma is proved.

Lemma 6. *Let $0 \neq u_0 \in H$ and $\widetilde{u}_0$ be a vector that realizes the supremum* (25)–(26). *Then at least one of the equalities either a) or b) holds:*

$$\begin{aligned} &a)\ \|\widetilde{u}_0\|^2 = \alpha\, \|f(\widetilde{u}_0)\|^{2(\alpha-1)}; \\ &b)\ \frac{1}{\|\widetilde{u}_0\|^2} - \gamma(\widetilde{u}_0) = 0. \end{aligned} \tag{47}$$

Proof. If equality b) from (47) holds, then the lemma is proved.

If equality b) does not hold in (47), then equality a) is obtained from Lemma 3 in the case $N(\widetilde{u}_0) \neq 0$ and from Lemma 5 in the case $N(\widetilde{u}_0) = 0$.

The lemma is proved.

3. Proof of Theorem 2

Let u_0 be a vector whose norm must be estimated by the norm of the vector $f(u_0)$. Let

$$\widetilde{b} = \sup\{|D_0(u)| + |D_1(u)|\}, \tag{48}$$

where *supremum* is taken over all vectors $u \in H$ such that

$$J(u) \geq J(u_0)\, e^{-1}. \tag{49}$$

In (48) $D_0(u)$ and $D_1(u)$ are defined by the equalities:

$$\begin{cases} D_0(u) = \left(\left\langle f(u(\eta)),\, D_{u(\eta)}u(\eta)\right\rangle\right)_\eta \big|_{u_\eta = u}, \\ D_1(u) = \left(\|u(\eta)\|^2 - \alpha\|f(u(\eta))\|^{2(\alpha-1)}\, \gamma(u(\eta))\, \|u(\eta)\|^2\right)_\eta \big|_{u_\eta = u}. \end{cases} \tag{50}$$

In the case where $L(u)$ is a bilinear transformation, we obtain

$$\begin{aligned} D_0(u) &= (\langle u + L(u,u),\, u + L_u u\rangle)_\eta \big|_{u_\eta = u} = \\ &= (\langle u_\eta + L_u u_\eta,\, u + L_u u\rangle + \langle u + L(u,u),\, u_\eta + 2L_u u_\eta\rangle) \big|_{u_\eta = u} = \\ &= \langle u + 2L(u),\, u + 2L(u)\rangle + \langle u + L(u),\, u + 4L(u)\rangle = \\ &= \|2f(u) - u\|^2 + \langle f(u),\, 4f(u) - 3u\rangle\,. \end{aligned} \tag{51}$$

If the transformation $L(u)$ is continuously differentiable, then the functionals $D_0(u)$ and $D_1(u)$ will also be continuously differentiable.

Since by (49) *supremum* is taken over a compact set, there exists a vector realizing *supremum* (48)-(49).

The vector realizing *supremum* (48)–(49) will be denoted by u_b. The existence of u_b is proved in the same way as in Lemma 2.

Let us choose a sequence of positive numbers

$$\delta_0,\ \delta_1,\ \ \dots\ , \delta_n,\ \dots \tag{52}$$

such that

$$0 < \delta_j < \frac{1}{4}, \qquad \sum_{j=0}^{\infty} \delta_j^2 = \frac{1}{4}, \qquad \sum_{j=0}^{\infty} \delta_j = \infty. \tag{53}$$

From the vector u_0 we construct the vector $\widetilde{u}_0$ that realizes *supremum* (25)-(26). According to Lemma 2, the inequalities are satisfied.

$$J(\widetilde{u}_0) \geq J(u_0), \quad R(\widetilde{u}_0) \geq R(u_0). \tag{54}$$

Recall that $J(\cdot)$ and $R(\cdot)$ are from (18).

According to Lemma 6, the vector $\widetilde{u}_0$ satisfies at least one of the conditions either a) or b) from (47) of Lemma 6.

If a) from (47) is satisfied, then we obtain the estimate

$$J(\widetilde{u}_0) = \|\widetilde{u}_0\|^2 \exp\left\{-\|f(\widetilde{u}_0)\|^{2\alpha}\right\} = \alpha \|f(\widetilde{u}_0)\|^{2(\alpha-1)} \exp\left\{-\|f(\widetilde{u}_0)\|^{2\alpha}\right\} \leq$$

$$\leq \alpha \sup_{x>0} x^{\alpha-1} e^{-x} = \alpha\, (\alpha-1)^{\alpha-1} e^{-\alpha+1} < \alpha^{\alpha}.$$

But $J(u_0) \leq J(\widetilde{u}_0)$ (see Lemma 2).

Therefore

$$\|u_0\|^2 \leq \alpha^{\alpha} \exp\left\{\|f(u_0)\|^2\right\}. \tag{55}$$

Therefore, if equality a) from (47) holds, Theorem 2 will be proved.

If b) from (47) holds, i.e. if the equality

$$\frac{1}{\|\widetilde{u}_0\|^2} - \gamma(\widetilde{u}_0) = 0, \tag{56}$$

then we take $\{u_0, \widetilde{u}_0\}$ as the initial pair and continue the construction.

Let the pairs be built

$$\{u_0, \widetilde{u}_0\}\,,\ \dots, \{u_n, \widetilde{u}_n\}\,,\ n \geq 0\,.$$

Moreover, $\widetilde{u}_n$ implements *supremum* (25)–(26), in which u_0 is taken as u_n.

For $\widetilde{u}_n$, according to Lemma 6, at least one of the following conditions, either a), or b, is satisfied:

$$
\begin{aligned}
&a)\ \|\widetilde{u}_n\|^2 = \alpha\, \|f(\widetilde{u}_n)\|^{2(\alpha-1)};\\
&b)\ \frac{1}{\|\widetilde{u}_n\|^2} - \gamma(\widetilde{u}_n) = 0.
\end{aligned}
\tag{57}
$$

If condition a) from (57) is satisfied, then we interrupt the process of constructing pairs. If equality b) from (57) is satisfied, then we proceed to constructing the pair $\{u_{n+1}, \widetilde{u}_{n+1}\}$.

We define the vector function $u = u(\xi)$ as a solution to the problem

$$
\begin{cases}
u_\xi = -u,\\
u|_{\xi=0} = \widetilde{u}_n.
\end{cases}
\tag{58}
$$

For $J(u(\xi))$ and $R(u(\xi))$ we have:

$$
\begin{aligned}
J_\xi(u(\xi)) &= 2J(u) \left\langle \left(\frac{1}{\|u\|^2} - \gamma \right) u, u_\xi \right\rangle = 2J(u)\, \big(\gamma(u)\|u\|^2 - 1\big) = \\
&= 2J(u) \left(\int_0^\xi \left(\gamma\left(u(\eta)\right)\, \|u(\eta)\|^2\right)_\eta d\eta \right) = \\
&= 2J(u) \int_0^\xi \left(\langle f(u(\eta)), D_{u(\eta)} u(\eta)\rangle\right)_\eta d\eta\,;
\end{aligned}
\tag{59}
$$

$$
\begin{aligned}
R_\xi(u(\xi)) &= 2R(u) \left[\left(-1 + \alpha\, \gamma(u)\, \|f(u)\|^{2(\alpha-1)} \right) \cdot \left(-\|u\|^2 \right) \right] = \\
&= 2R(u) \left[\left(\|\widetilde{u}_n\|^2 - \alpha\, \|f(\widetilde{u}_n)\|^{2(\alpha-1)} \right) + \right. \\
&\left. + \int_0^\xi \left(\|u\|^2 - \alpha\, \gamma(u)\, \|u\|^2\, \|f(u)\|^{2(\alpha-1)} \right)_\eta d\eta \right].
\end{aligned}
\tag{60}
$$

When deriving (59) and (60), it was taken into account that equality b) from (57) is satisfied.

Let us choose the number ξ_n from the condition

$$
\xi_n = \frac{1}{8\sqrt{\widetilde{b}}}\, \delta_n.
\tag{61}
$$

where δ_0 is from (52)–(53) and $\widetilde{b}$ is from (48).

We integrate (59) from 0 to $\xi \leq \xi_n$:

$$J(u(\xi)) = J(\widetilde{u}_n)\,\exp\left\{2\int_0^{\xi}\left(\int_0^{\eta}\left(\left\langle f(u(\tau)), D_{u(\tau)}u(\tau)\right\rangle\right)_{\tau} d\tau\right) d\eta\right\} \geq$$
$$\geq J(\widetilde{u}_n)\,\exp\left\{-2\int_0^{\xi}\left(\int_0^{\eta}\left|D_0(u(\tau))\right| d\tau\right) d\eta\right\}. \tag{62}$$

If

$$J(\widetilde{u}_n) \geq J(u_0)\, e^{-\frac{9}{10}}, \tag{63}$$

then by virtue of (62) for small $\xi > 0$ we have

$$u(\xi) \in \left\{u :\ J(u(\xi)) \geq J(u_0)\, e^{-1}\right\}. \tag{64}$$

Therefore, from (62) we obtain

$$J(u(\xi)) \geq J(\widetilde{u}_n)\,\exp\left\{-\widetilde{b}\,\xi^2\right\}.$$

It follows that (64) holds for all $\xi \in [0, \xi_n]$, where ξ_n is from (61).

From (62) we deduce

$$J(u(\xi)) \geq J(\widetilde{u}_n)\,\exp\left\{-\widetilde{b}\,\xi^2\right\} \geq J(\widetilde{u}_n)\,\exp\left\{-\frac{1}{8}\,\delta_n^2\right\}. \tag{65}$$

Inequality valid for all $\xi \in [0, \xi_n]$.

Integration (60) performed for $\xi \in [0, \xi_n]$, taking into account the inclusion (64) leads to the inequality:

$$R(u(\xi)) \geq 2R(\widetilde{u}_n)\left[\left(\xi\,\|\widetilde{u}_n\|^2 - \alpha\,\|f(\widetilde{u}_n)\|^2\right) - \frac{1}{4}\,\delta_n^2\right]. \tag{66}$$

If

$$\|\widetilde{u}_n\|^2 \leq 4\alpha\,\|f(\widetilde{u}_n)\|^2, \tag{67}$$

then we terminate the process.

If (67) is not satisfied, then instead of inequality (66) we have the inequality

$$R(u(\xi)) \geq R(\widetilde{u}_n)\left[\exp\left\{\frac{\delta_n}{\sqrt{\widetilde{b}}}\cdot\frac{3}{16}\,\|\widetilde{u}_n\|^2 - \frac{1}{4}\,\delta_n^2\right\}\right], \tag{68}$$

true for all $\xi \in [0, \xi_n]$.

Now for the left element of the pair $\{u_{n+1}, \widetilde{u}_{n+1}\}$ we take $u(\xi_n)$, and for the right element we take the vector implementing *supremum* (25)–(26), in which instead of u_0 we take that $u_{n+1} = u(\xi_n)$.

From (65), (68) and Lemma 2 we obtain:

$$J(\widetilde{u}_{n+1}) \geq J(u_{n+1}) \geq J(\widetilde{u}_n)\, \exp\left\{-\frac{1}{4}\,\delta_n^2\right\}, \tag{69}$$

$$R(\widetilde{u}_{n+1}) \geq R(u_{n+1}) \geq R(\widetilde{u}_n)\, \exp\left\{\frac{3\,\delta_n}{16\sqrt{\widetilde{\widetilde{b}}}} - \frac{1}{4}\,\delta_n^2\right\}. \tag{70}$$

The relations (64)–(70) will be true if we prove that the inequality (63) is satisfied.

From (69) for $n \geq 1$ we have

$$\begin{gathered} J(\widetilde{u}_n) \geq J(\widetilde{u}_{n-1})\, \exp\left\{-\frac{1}{4}\,\delta_{n-1}^2\right\} \geq J(\widetilde{u}_0)\, \exp\left\{-\frac{1}{4}\sum_{j=0}^{n-1}\delta_j^2\right\} \geq \\ \geq J(\widetilde{u}_0)\, \exp\left\{-\frac{1}{4}\sum_{j=0}^{\infty}\delta_j^2\right\} = J(\widetilde{u}_0)\, \exp\left\{-\frac{1}{16}\right\} \geq J(u_0)\, e^{-1}. \end{gathered} \tag{71}$$

When deriving (71) in the last transition, Lemma 1 was used.

From the calculations (64)–(69) and (71) it follows that if inequality (63) is satisfied for $j = 0, 1, \ldots, n-1$, then it is also true for n. Therefore, since for small n the fulfillment of (63) is obvious, then (63) is fulfilled for all n until the process terminates.

If the process terminates at $n = 0$, then Theorem 2 follows from the estimate (55).

If the process terminates at some finite $n \geq 1$, then condition a) from (57) is satisfied or the inequality (67) is satisfied.

If condition a) from (57) or the inequality (67) is satisfied, then we have

$$J(\widetilde{u}_n) = \|\widetilde{u}_n\|^2\, \exp\left\{-\|f(\widetilde{u}_n)\|^2\right\} \leq 4\,\alpha \sup_{x>0} x\, e^{-x} = 4\,\alpha.$$

From here and from (71) we obtain

$$J(u_0) \leq 4\,\alpha\, e \leq \ 12\,\alpha.$$

From this inequality we obtain

$$\|u_0\|^2 \leq 12\,\alpha\, \exp\left\{\|f(u_0)\|^2\right\},$$

from which the statement of Theorem 2 follows.

It remains to consider the case when the process does not terminate for any finite n.

In this case, due to the choice of $\{\delta_n\}$ (see (53)) from (70) we obtain

$$\lim_{n\to\infty} R(\widetilde{u}_{n+1}) = \infty. \tag{72}$$

Since by (71) and Lemma 1 the vectors $\widetilde{u}_0,\ \widetilde{u}_1,\ \ldots, \widetilde{u}_n,\ \widetilde{u}_{n+1},\ \ldots$ lie in a compact set, and the functional $R(\cdot)$ is continuous, we obtain that (72) cannot be satisfied. Therefore, Theorem 2 is proved.

Remark 3. By changing the choice of functionals $J(\cdot)$ and $R(\cdot)$, we can obtain other theorems.

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Қошанов Б.Д., Өтелбаев М., Шыныбеков А.Н. АҚЫРЛЫ ӨЛШЕМДІ СЫЗЫҚТЫ ЕМЕС ТЕҢДЕУЛЕРДІҢ БІР КЛАСЫНЫҢ ШЕШІМДЕРІН БАҒАЛАУ. II

Осы мақалада шекті өлшемді кеңістіктегі сызықты емес теңдеулердің шешімдеріне арналған априорлық бағалаулар туралы екі теорема келтіріледі. Бұл теоремалар сызықты емес теңдеулердің шешімдерінің бір класының бастапқы-шеттік есептердің шекті өлшемді жуықтаулары қанағаттандыратын шарттардан алынған белгілі бір шарттарға негізделіп дәлелденген. Мақала осы атаумен шыққан бірінші бөлімнің жалғасы болып табылады. Осы мақалада екінші теорема дәлелденеді.

Түйін сөздер: дифференциалдық оператор, сызықтық емес теңдеу, шешімнің бар болуы, шешімнің жалғыздығы, шешімнің априорлық бағалауы.

Кошанов Б.Д., Отелбаев М., Шыныбеков А.Н. ОЦЕНКА РЕШЕНИЙ ОДНОГО КЛАССА КОНЕЧНОМЕРНЫХ НЕЛИНЕЙНЫХ УРАВНЕНИЙ. II

В данной статье получены две теоремы об априорных оценках решений нелинейных уравнений в конечномерном пространстве. Эти теоремы доказаны при выполнении определённых условий, заимствованных из условий, которым удовлетворяют конечномерные аппроксимации одного класса нелинейных краевых задач с начальным условием. Статья является продолжением первой части с тем же названием. В данной статье доказывается вторая теорема.

Ключевые слова: дифференциальный оператор, нелинейное уравнение, существование решения, единственность решения, априорная оценка решения.