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Inverse initial problem for fractional wave equation with the Hadamard fractional derivative

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Abstract. This paper investigates an inverse initial problem for a time-fractional wave equation involving the Hadamard fractional derivative. Unlike the more widely studied Caputo and Riemann–Liouville derivatives, the Hadamard derivative is defined via a logarithmic kernel and exhibits distinct analytical features, making it suitable for modeling processes with slow memory decay and multiplicative structures. Building on prior work concerning the extremum principle and solvability of boundary value problems with Hadamard-type operators, we establish sufficient conditions for the unique solvability of the inverse problem. The analysis is carried out in terms of eigenfunction expansions and leverages properties of the two-parameter Mittag–Leffler function. The findings contribute to the theory of inverse problems for fractional wave equations and highlight the role of Hadamard derivatives in capturing complex temporal dynamics in mathematical models.

Keywords. Fractional wave equation, inverse initial problem, Hadamard fractional derivative, Mittag-Leffler function.

1 Introduction

Inverse problems for fractional partial differential equations have gained considerable attention in recent years due to their broad applicability to modeling complex phenomena in physics, engineering, and other scientific fields. In particular, fractional-wave equations, which incorporate memory effects and nonlocal behavior, provide a more accurate representation of various real-world processes than their classical counterparts [1].

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This paper addresses an inverse initial problem for a time-fractional wave equation where the fractional derivative is understood in the sense of Hadamard. The Hadamard fractional derivative, characterized by its logarithmic kernel, introduces unique analytical challenges and properties distinct from those of the more commonly studied Riemann–Liouville and Caputo derivatives. More properties of this derivative can be found in [2], [3].

In [4], an extremum principle for Hadamard fractional derivatives was considered. The authors established new estimates for the Hadamard fractional derivatives at extreme points of functions. This extremum principle was instrumental in proving the uniqueness and continuous dependence of solutions for initial boundary value problems related to linear and nonlinear time-fractional diffusion equations.

Sequential differential equations with the Hadamard fractional derivative were the subject of [5]. The Ulam–Hyers stability of Caputo-Hadamard fractional stochastic differential equations was studied in [6]. Variable-order Caputo-Hadamard fractional derivative was considered in [7].

We note works [8] and [9], where sub-diffusion and fractional diffusion-wave equations involving the Hadamard fractional derivative were analyzed. In [10], a problem with the terminal integral condition for a nonlinear fractional-differential equation with the bi-ordinal Hilfer-Hadamard derivative was targeted for the unique solvability.

We investigate the well-posedness of this problem under specific assumptions on the given data. By carefully analyzing the structure of the equation and utilizing appropriate functional analytic tools, we establish conditions that guarantee the existence and uniqueness of a solution. The results presented contribute to a broader understanding of inverse problems associated with fractional wave equations and highlight the potential of the Hadamard derivative in modeling and analysis.

2 Direct problem

Consider an initial boundary value problem for the time-fractional wave equation with the Hadamard fractional derivative in a rectangular domain. Let us consider an equation

$$ {}_H D_{1t}^{\alpha} u(t,x) - u_{xx}(t,x) = f(t,x) \tag{1} $$

in a rectangular domain $\Omega = \{(t,x) : 0 < x < 1, 1 < t < T\}$. Here $f(t,x)$ is a given function, $T > 1$ is a positive real number, and

$$ {}_H D_{1t}^{\alpha} g(t) = \left(t\frac{d}{dt}\right)^n \frac{1}{\Gamma(n-\alpha)} \int\limits_1^t \left(\log\frac{t}{s}\right)^{n-\alpha+1} \frac{g(s)}{s} ds \ \ (t > 1) $$

represents the Hadamard fractional derivative of order $\alpha(1 < \alpha \le 2$, $\log(..) = \ln(..)$ [1].

Let us formulate a direct problem for equation (1).

Direct problem. To find a function $u(t,x)$ satisfying

- the equation (1) in Ω;
- regularity conditions $u(\cdot,x)\in C^{\alpha}_{\gamma,\log}[1,T]$, $u(t,\cdot)\in C^1[0,1]\cap C^2(0,1)$;
- boundary conditions

$$u(t,0)=u(t,1)=0,\ 1\le t\le T; \tag{2}$$

- initial conditions

$${}_HI^{2-\alpha}_{1t}u(t,x)\big|_{t=1+}=\varphi(x),\ 0\le x\le 1,\ {}_HD^{\alpha-1}_{1t}u(t,x)\big|_{t=1+}=\psi(x),\ 0<x<1. \tag{3}$$

Here $\varphi(x)$ and $\psi(x)$ are given functions, ${}_HI^{\beta}_{1t}$ represents the Hadamard fractional integral of order $\beta>0$ given

$${}_HI^{\beta}_{1t}g(t)=\frac{1}{\Gamma(\beta)}\int\limits_1^t\left(\log\frac{t}{s}\right)^{\beta-1}\frac{g(s)}{s}ds,\ \ t>1,$$

the class of functions $C^{\alpha}_{\delta,\gamma}(..)$ with $0<\gamma\le 1$ is given by (see [1])

$$\begin{aligned}C^n_{\delta,\gamma}[a,b]&=\left\{g:\ \|g\|_{C^n_{\delta,\gamma}}=\sum_{k=0}^{n-1}\|\delta^k g\|_C+\|\delta^n g\|_{C_{\gamma,\log}}\right\},\\ C^0_{\delta,\gamma}[a,b]&=C_{\gamma,\log}[a,b],\ \delta=t\frac{d}{dt}.\end{aligned}$$

We search a solution to the direct problem as follows:

$$u(t,x)=\sum_{k=1}^{\infty}U_k(t)\sin k\pi x. \tag{4}$$

Substituting (4) into (1) at $f(t,x)\equiv 0$ we obtain

$${}_HD^{\alpha}_{1t}U_k(t)+(k\pi)^2U_k(t)=f_k(t), \tag{5}$$

where $f_k(t)=2\int\limits_0^1 f(t,x)\sin k\pi x dx$ are Fourier coefficients of the function $f(t,x)$ represented by Fourier-Sine series, i.e.

$$f(t,x)=\sum_{k=1}^{\infty}f_k(t)\sin k\pi x.$$

Initial conditions (2) give us

$${}_HI^{2-\alpha}_{1t}U_k(t)\big|_{t=1+}=\varphi_k,\ 0\le x\le 1,\ {}_HD^{\alpha-1}_{1t}U_k(t)\big|_{t=1+}=\psi_k,\ 0<x<1. \tag{6}$$

Here, φ_k and ψ_k are Fourier coefficients of functions $\varphi(x)$ and $\psi(x)$, respectively.

The solution of the Cauchy-type problem (5)-(6) can be represented as [1]

$$U_k(t) = \varphi_k(t-1)^{\alpha-1}E_{\alpha,\alpha}\left[-(k\pi)^2(\log t)^\alpha\right] + \psi_k(t-1)^{\alpha-2}E_{\alpha,\alpha-1}\left[-(k\pi)^2(\log t)^\alpha\right] + \\ + \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left[-(k\pi)^2\left(\log\frac{t}{s}\right)^\alpha\right] f_k(s)\frac{ds}{s}, \quad (7)$$

where $E_{a,b}(z) = \sum\limits_{n=0}^{\infty} \frac{z^n}{\Gamma(an+b)}$, with $a > 0$ and $b \in \mathbb{R}$, represents two-parameter Mittag-Leffler function [1].

It is easy to prove the following statement.

Lemma 1. *If $g(x) \in C^2[0,1]$ is such that $g(0) = g(1) = 0$, $g''(0) = g''(1) = 0$, and $g'''(x) \in L_2(0,1)$, then*

$$\sum_{k=1}^{\infty} |g_k|(k\pi)^2 \leq \sum_{k=1}^{\infty} \frac{1}{(k\pi)^2} + \left\|g'''(x)\right\|_2^2.$$

The proof of this lemma can be done using integration by parts, considering Bessel's inequality and Parseval's identity.

The convergence of the infinite series corresponding to the functions $u(t,x)$ and $u_{xx}(t,x)$ can be proved using Lemma 1 and the well-known estimate of the two-parameter Mittag-Leffler function $E_{a,b}(-z) \leq \frac{C}{1+|z|}$ for $z > 0$ [1].

Regarding the solvability of the direct problem, we can state the following.

Theorem 2. *If the functions $\varphi(x)$, $\psi(x)$, and $f(t,x)$ (with respect to the variable x) satisfy the condition of Lemma 1 and $f(\cdot,x) \in C_{\gamma,\log}[1,T]$, then a solution of the direct problem does exist, moreover, it is unique and is represented by Formula (4), where $U_k(t)$ will be found using (7).*

3 Inverse initial problem

In this section, we consider an inverse problem of finding an initial condition using the additional data at a fixed time.

Inverse initial problem. To find a pair of functions $\{u(t,x);\psi(x)\}$ satisfying

- the equation (1) in Ω;
- regularity conditions $u(\cdot,x) \in C^\alpha_{\gamma,\log}[1,T]$, $u(t,\cdot) \in C^1[0,1] \cap C^2(0,1)$, $\psi(x) \in C[0,1]$;
- boundary conditions (2);

- the first initial condition of (3);
- over-determination condition $u(\xi, x) = \zeta(x)$, $0 \leq x \leq 1$ for fixed $\xi \in (1, T]$.

Here, $\varphi(x)$ and $\zeta(x)$ are given functions.

Inverse initial problems for differential equations were considered in many works. For example, see [11]–[14]. Namely, in [11], authors investigated the unique solvability of the inverse initial problem for the heat equation with the Bessel operator. Then in [12], the result was generalized for the time-fractional heat equation with the same operator in the space variable. In [13], a similar inverse problem was targeted at the sub-diffusion equation with a variable coefficient involving a more general fractional derivative. The work [14] is devoted to the unique solvability of the inverse initial problem for the fractional wave equation.

The following statement holds:

Theorem 3. *Let $1 < \alpha \leq 4/3$. Then if the functions $\varphi(x)$, $\zeta(x)$, and $f(t,x)$ (concerning the variable x) satisfy the condition of Lemma 1 and $f(\cdot, x) \in C_{\gamma,\log}[1,T]$, then a solution of the inverse initial problem does exist, moreover, it is unique and represented by the following formula:*

$$u(t,x) = \sum_{k=1}^{\infty} \Bigg[\varphi_k (t-1)^{\alpha-1} E_{\alpha,\alpha}[-(k\pi)^2 (\log t)^\alpha] + \psi_k (t-1)^{\alpha-2} E_{\alpha,\alpha-1}\left[-(k\pi)^2 (\log t)^\alpha\right] + \\ + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left[-(k\pi)^2 \left(\log\frac{t}{s}\right)^\alpha\right] f_k(s) \frac{ds}{s} \Bigg] \sin k\pi x, \quad (8)$$

$$\psi(x) = \sum_{k=1}^{\infty} \frac{1}{(\xi-1)^{\alpha-2} E_{\alpha,\alpha-1}\left[-(k\pi)^2 (\log \xi)^\alpha\right]} \Bigg\{ \zeta_k - \varphi_k (\xi-1)^{\alpha-1} E_{\alpha,\alpha}[-(k\pi)^2 (\log \xi)^\alpha] - \\ - \int_1^\xi \left(\log \frac{\xi}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left[-(k\pi)^2 \left(\log\frac{\xi}{s}\right)^\alpha\right] f_k(s) \frac{ds}{s} \Bigg\} \sin k\pi x; \quad (9)$$

Proof. We assume that function $\psi(x)$ is given and then use the solution of the direct problem, given by

$$u(t,x) = \sum_{k=1}^{\infty} \sin k\pi x \Bigg\{ \varphi_k (t-1)^{\alpha-1} E_{\alpha,\alpha}\left[-(k\pi)^2 (\log t)^\alpha\right] + \psi_k (t-1)^{\alpha-2} \times \\ \times E_{\alpha,\alpha-1}\left[-(k\pi)^2 (\log t)^\alpha\right] + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left[-(k\pi)^2 \left(\log\frac{t}{s}\right)^\alpha\right] f_k(s) \frac{ds}{s} \Bigg\}. \quad (10)$$

Substituting (10) into the over-determination condition, one will get

$$\zeta(x)=\sum_{k=1}^{\infty}\sin k\pi x\Bigg\{\varphi_k(\xi-1)^{\alpha-1}E_{\alpha,\alpha}\left[-(k\pi)^2(\log\xi)^\alpha\right]+\psi_k(\xi-1)^{\alpha-2}\times$$

$$\times E_{\alpha,\alpha-1}\left[-(k\pi)^2(\log\xi)^\alpha\right]+\int\limits_1^{\xi}\left(\log\frac{\xi}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left[-(k\pi)^2\left(\log\frac{\xi}{s}\right)^{\alpha}\right]f_k(s)\frac{ds}{s}\Bigg\}. \quad (11)$$

In [15], it was shown that the Mittag-Leffler function $E_{a,b}(z)$ does not have zeros for $1<a\le 4/3$, $z,\ b\in\mathbb{R}$. You can also see [14]. Therefore, dividing the coefficient of the function $\psi(x)$ in (11), one can easily get (9).

The convergence of infinite series representing the solution can be proved using Lemma 1. Namely, using (8) and considering the estimate $|zE_{a,b}(-z)|\le C$ for $z>0$, we get

$$|u(t,x)|\le\sum_{k=1}^{\infty}\left[C_1|\varphi_k|+C_2|\psi_k|+C_3\int\limits_1^{t}|f_k(s)|\frac{ds}{s}\right].$$

Here $C_i\ (i=\overline{1,3})$ are positive real numbers. Further, since they do not have principal importance, we denote them as C. Integration by parts and the well-known inequality $2ab\le a^2+b^2$ yield

$$|u(t,x)|\le C\sum_{k=1}^{\infty}\frac{1}{(k\pi)^2}\left[|\varphi_k^{(1)}|^2+|\psi_k^{(1)}|^2+\int\limits_1^{t}|f_k^{(1)}(s)|^2\frac{ds}{s}\right],$$

where $\varphi_k^{(1)}=\int\limits_0^1\varphi'(x)\cos k\pi x dx$, $\psi_k^{(1)}=\int\limits_0^1\psi'(x)\cos k\pi x dx$, $f_k^{(1)}(t)=\int\limits_0^1 f_x(t,x)\cos k\pi x dx$. Using Parseval's identity, one can easily get

$$|u(t,x)|\le C\left(\sum_{k=1}^{\infty}\frac{2}{(k\pi)^2}+\left\|\varphi'(x)\right\|_2^2+\left\|\psi'(x)\right\|_2^2+\int\limits_1^{t}\|f_x(s,\cdot)\|_2^2\frac{ds}{s}\right).$$

Here $\|\cdot\|_2$ presents the $L_2(0,1)$-norm. Similarly, we will get

$$|\psi(x)|\le C\left(\sum_{k=1}^{\infty}\frac{1}{(k\pi)^2}+\left\|\zeta'(x)\right\|_2^2+\left\|\varphi'(x)\right\|_2^2+\int\limits_1^{\xi}\|f_x(s,\cdot)\|_2^2\frac{ds}{s}\right).$$

Note that to get this, we have imposed the following conditions on the given functions:

$$\varphi(x),\ \zeta(x),\ f(t,x)\in C[0,1],\ \varphi(0)=\varphi(1)=0,\ \zeta(0)=\zeta(1)=0,\ f(t,0)=f(t,1)=0,$$

$$\varphi'(x),\ \zeta'(x),\ f_x(t,\cdot) \in L_2(0,1).$$

To prove the uniform convergence of infinite series corresponding to $u_{xx}(t,x)$, we will impose more conditions on the given functions as it was present in Lemma 1.

The uniqueness of the solution to the inverse initial problem follows from the completeness of the system $\{\sin k\pi x\}_{k=1}^{\infty}$. Namely, assuming that the problem has two different set of solutions $\{u_1(t,x),\ \psi_1(x)\}$, $\{u_2(t,x),\ \psi_2(x)\}$, and denoting

$$u(t,x) = u_1(t,x) - u_2(t,x), \quad \psi(x) = \psi_1(x) - \psi_2(x),$$

we will get the corresponding homogeneous problem. Then we multiply both sides of (4) by $\sin m\pi x$, and integrate along $[0,1]$:

$$\int_0^1 u(t,x)\sin m\pi x dx = \int_0^1 \sum_{k=1}^{\infty} U_k(t)\sin(k\pi x)\sin(m\pi x)dx.$$

Based on orthogonality of the system $\{\sin k\pi x\}_{k=1}^{\infty}$, one can easily get

$$U_k(t) = 2\int_0^1 u(t,x)\sin k\pi x dx. \tag{12}$$

The solution of the homogeneous case of Problem (5)–(6), from (7) easy to deduce that $U_k(t) \equiv 0$. Hence, due to (12), considering that the system $\{\sin k\pi x\}_{k=1}^{\infty}$ is complete, we obtain $u(t,x) \equiv 0$, which proves the uniqueness of the solution to the considered inverse initial problem.

Theorem 3 has been proved. □

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Алимов Зухридин, Кербал Себти, АДАМАР БӨЛШЕК ТУЫНДЫСЫ ҚАТЫСҚАН БӨЛШЕК РЕТТІ ТОЛҚЫН ТЕҢДЕУІ ҮШІН КЕРІ БАСТАПҚЫ ЕСЕП

Бұл мақалада уақыт айнымалысы бойынша Адамар бөлшек туындысы қатысқан бөлшек ретті толқын теңдеуі үшін кері бастапқы есеп қарастырылады. Көп зерттелетін Риман–Лиувилль мен Капуто туындыларынан айырмашылығы, Адамар туындысы логарифмдік ядро арқылы анықталып, баяу жад әсерлері мен мультипликативті құрылымдарды сипаттауға мүмкіндік береді. Авторлар Адамар типті операторлармен байланысты шекаралық есептердің шешілуі және экстремум принципі жөніндегі алдыңғы жұмыстарға сүйене отырып, кері есептің жалғыз шешімі үшін жеткілікті шарттарды дәлелдейді. Зерттеу Фурье қатарлары мен екі параметрлі Миттаг–Леффлер функциясының қасиеттеріне негізделген. Алынған нәтижелер бөлшек толқындық теңдеулерге арналған кері есептер теориясын толықтырады және Адамар туындыларының күрделі уақытша динамикаларды сипаттаудағы маңызын көрсетеді.

Түйін сөздер: Бөлшек ретті толқын теңдеуі, кері бастапқы есеп, Адамар бөлшек туындысы, Миттаг–Леффлер функциясы.

Алимов Зухридин, Кербал Себти, ОБРАТНАЯ НАЧАЛЬНАЯ ЗАДАЧА ДЛЯ ДРОБНОГО ВОЛНОВОГО УРАВНЕНИЯ С ДРОБНОЙ ПРОИЗВОДНОЙ АДАМАРА

В данной статье рассматривается обратная начальная задача для дробного волнового уравнения с дробной производной Адамара по времени. В отличие от более известных производных Римана–Лиувилля и Капуто, производная Адамара определяется с помощью логарифмического ядра и обладает особыми аналитическими свойствами, что делает её подходящей для моделирования процессов с медленным затуханием памяти и мультипликативной структурой. Основываясь на ранее полученных результатах по принципу экстремума и разрешимости краевых задач с производными Хадамара, авторы устанавливают достаточные условия единственности решения. Метод основан на разложении по собственным функциям и использовании свойств двухпараметрической функции Миттага–Леффлера. Полученные результаты вносят вклад в развитие теории обратных задач для дробных волновых уравнений и подчёркивают роль производных Адамара в моделировании сложной временной динамики.

Ключевые слова: Дробное волновое уравнение, обратная начальная задача, дробная производная Адамара, функция Миттаг–Леффлера.