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On the solvability of the main inverse problem for a system of the first-order stochastic differential equations with degenerate diffusion

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Abstract. The problem of constructing a system of first-order Itô stochastic differential equations by the given properties of motion is considered. Necessary and sufficient conditions for the existence of a given integral manifold of the constructed equation are obtained in terms of the equation's coefficients. These conditions are derived separately for two cases: when the manifold depends on all independent variables and when it depends only on a part of the variables.

Keywords. stochastic differential equations, integral manifold, inverse problem, degenerate diffusion.

1 Introduction

The inverse problems for systems of Itô stochastic differential equations (SDEs) occupy a significant place in the theory of stochastic processes, lying at the intersection of differential geometry and stochastic analysis. Classical stochastic calculus, initiated by the foundational work of K. Itô, traditionally focuses on the direct problem: analyzing the behavior of solutions when the equations are given. In contrast, the construction of stochastic systems from prescribed geometric or dynamical properties, such as the existence of an invariant manifold, presents more intricate challenges and remains relatively less developed.

One of the central tasks in this context is to determine under what conditions a system of first-order Itô SDEs admits a given integral manifold. This problem naturally generalizes

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concepts from the deterministic theory of differential equations to the stochastic setting. The presence of both drift and diffusion components introduces additional constraints, requiring the manifold to remain invariant not only under deterministic flow but also under stochastic perturbations.

Recent progress in this direction has led to the derivation of necessary and sufficient conditions for the existence of such manifolds, formulated explicitly in terms of the coefficients of the system. The analysis typically distinguishes between two cases: when the manifold depends on all independent variables and when it depends only on a part of the independent variables. This distinction reflects the geometric flexibility of the manifold structure and allows a finer classification of integrability conditions.

These developments contribute to the broader effort of extending classical geometric and analytic techniques to stochastic systems. They also have potential applications in stochastic control, filtering theory, and mathematical modeling in various applied fields where invariant structures play a crucial role. As such, they provide a rigorous foundation for the systematic construction of stochastic models with prescribed qualitative behavior.

2 Case 1. The given properties of motion depend on all variables

Let a set

$$\Lambda(t) : \lambda(x, y, t) = 0, \;\; \lambda \in R^m, \;\; \lambda = \lambda(x, y, t) \in C^{221}_{xyt} \tag{1}$$

be given. It is required to construct the equations of a motion for the class of the first-order Itô stochastic differential equations with degenerate diffusion

$$\begin{cases} \dot{x} = f_1(x, y, t) \\ \dot{y} = f_2(x, y, t) + \sigma(x, y, t)\dot{\xi} \end{cases} \tag{2}$$

such that the set (1) is the integral manifold of System (2).

Here $x \in R^{n_1}$, $y \in R^{n_2}$, $n_1 + n_2 = n$, $\xi \in R^k$, and $\sigma(x, y, t)$ is a $(n \times k)$-dimension matrix; $\{\xi_1(t, \omega), ..., \xi_k(t, \omega)\}$ is a system of independent Wiener processes [1] defined on some probability space (Ω, U, P).

Suppose that the vector functions $f_1(x, y, t)$, $f_2(x, y, t)$ and the matrix $\sigma(x, y, t)$ satisfy the following conditions:

(i) $f_1(x, y, t)$, $f_2(x, y, t)$, and $\sigma(x, y, t)$ are continuous in t and Lipschitz continuous in x, y in the whole space $R^n \ni z = (x^T, y^T)^T$,

(ii) $f_1(x, y, t)$, $f_2(x, y, t)$, and $\sigma(x, y, t)$ satisfy the condition of linear growth

$$\|f_1(z, t)\|^2 + \|f_2(z, t)\|^2 + \|\sigma(z, t)\|^2 \leq L(1 + \|z\|^2)$$

which ensures the existence and uniqueness up to stochastic equivalence of the solution $z(t)$ of Equation (2) in R^n with the initial condition $z(t_0) = z_0$ that is a strictly Markov process continuous with probability one [1]. We say $p(x, y, t)$ belongs to the class K ($p \in K$) if it satisfies conditions (i) and (ii).

A particular case of the statement of this stochastic problem is the problem of constructing a second-order equation

$$\ddot{x} = g(x, \dot{x}, t) + \sigma(x, \dot{x}, t)\dot{\xi}.$$

By substituting $\dot{x} = y$ it is reduced to the first-order equations system

$$\begin{cases} \dot{x} = y \\ \dot{y} = g(x, y, t) + \sigma(x, y, t)\dot{\xi} \end{cases}$$

with degenerate diffusion in half of the independent variables. Previously, this problem was studied in [2, 3, 4].

The problem of constructing the system (2) by the given set (1) was studied in [5, 6, 7, 8] in the absence of stochastic perturbations ($\sigma \equiv 0$). In [5, 6], Galiullin proposed a formulation, classification of the inverse problems of dynamics, and methods for solving them in the class of ordinary differential equations (ODEs). One of the general methods for solving the inverse problems of dynamics for ODE (the quasi-inversion method) was proposed in [8].

We generalize the quasi-inversion method to the class of first-order Itô stochastic differential equations with degenerate diffusion.

To solve the posed problem, we apply the Ito stochastic differentiation rule and construct the equation of perturbed motion

$$\dot{\lambda} = \frac{\partial \lambda}{\partial t} + \frac{\partial \lambda}{\partial x} f_1 + \frac{\partial \lambda}{\partial y} f_2 + S + \frac{\partial \lambda}{\partial x}\sigma\dot{\xi}, \tag{3}$$

where $S = \dfrac{1}{2}\left(\dfrac{\partial^2 \lambda}{\partial y^2} : \sigma\sigma^T\right)$, and by $\dfrac{\partial^2 \lambda}{\partial y^2} : D$, following [1], we mean the vector, elements of which are traces of the products of matrices of the second derivatives of the elements $\lambda_\mu(x, y, t)$ of the vector $\lambda(x, \dot{x}, t)$ with respect to $\dot{x}$ and the matrix D, $D = \sigma\sigma^T$.

We introduce arbitrary Yerugin functions [7]: A and B such that $A(0, x, \dot{x}, t) \equiv 0$, $B(0, x, \dot{x}, t) \equiv 0$, and

$$\dot{\lambda} = A(\lambda, x, \dot{x}, t) + B(\lambda, x, \dot{x}, t)\dot{\xi}. \tag{4}$$

By comparing equations (3) and (4), we get the relations

$$\frac{\partial \lambda}{\partial y} f_2 = A - \frac{\partial \lambda}{\partial t} - \frac{\partial \lambda}{\partial x} f_1 - S, \quad \frac{\partial \lambda}{\partial y}\sigma_i = B_i, \ i = \overline{1, k}, \tag{5}$$

from which we need to determine the vector-functions f_1, f_2 and the matrix σ.

The following lemma is required to solve the problem.

Lemma 1 (I.A. Mukhametzyanov, R.G. Mukharlyamov [8], p. 12–13)**.** *The set of all solutions of the linear system*

$$H\vartheta = g,\ H = (h_{ij}),\ \vartheta = (\vartheta_i),\ g = (g_i),\ i = \overline{1,k},\ j = \overline{1,n},\ k \le n,$$

is determined by the expression $\vartheta = s[HC] + H^{+}g$, where the matrix H has rank m.

Here s is an arbitrary scalar value, $[HC] = [h_1 ... h_i c_{i+1} ... c_{n-1}]$ is a vector product of vectors $h_i = (h_{ik})$ and arbitrary vectors $c_\rho = (c_{\rho k})$, $\rho = \overline{m+1, n-1}$; $H^{+} = H^T(HH^T)^{-1}$, H^T is a matrix transposed to H.

According to Lemma 1, from Relations (5) we define the vector function f_2 and the columns σ_i, $(i = \overline{1,k})$ of the matrix σ.

$$f_2 = s_1 \left[\frac{\partial\lambda}{\partial y} C\right] + \left(\frac{\partial\lambda}{\partial y}\right)^{+} \left(A - \frac{\partial\lambda}{\partial t} - \frac{\partial\lambda}{\partial x} f_1 - S\right), \tag{6}$$

$$\sigma_i = s_2 \left[\frac{\partial\lambda}{\partial y} C\right] + \left(\frac{\partial\lambda}{\partial y}\right)^{+} B_i,\ i = \overline{1,k}, \tag{7}$$

where $\sigma_i = (\sigma_{1i}, \sigma_{2i}, ..., \sigma_{n_2 i})^T$ is i-th column of the matrix σ, $B_i = (B_{1i}, B_{2i}, ..., B_{mi})^T$ is i-th column of the matrix B, s_i, k are arbitrary scalar values.

Thus, the following statement holds.

Theorem 2. *For the system of first-order Itô differential equations* (2) *to have a given integral manifold* (1)*, it is necessary and sufficient for arbitrarily given vector functions $f_1 \in K$ the vector function f_2 and the columns σ_i of the matrix σ of System* (2) *are of the form* (6) *and* (7)*, respectively.*

3 Case 2. The given properties of motion depend on a subset of variables

Let a set

$$\Lambda(t) : \lambda(x,t) = 0,\ \ \lambda \in R^m,\ \lambda = \lambda(x,t) \in C^{22}_{xt} \tag{8}$$

be given. It is required to construct the equations of motion for the class of first-order Itô stochastic differential equations with degenerate diffusion

$$\begin{cases} \dot{x} = g_1(x,y,t) \\ \dot{y} = g_2(x,y,t) + \sigma(x,y,t)\dot{\xi} \end{cases} \tag{9}$$

such that the set (8) is the integral manifold of system (9).

Suppose that the vector functions $g_1(x,y,t)$, $g_2(x,y,t)$ and the matrix $\sigma(x,y,t)$ satisfy the following conditions:

(iii) $g_1(x,y,t)$, $g_2(x,y,t)$, and $\sigma(x,y,t)$ are continuous in t and Lipschitz continuous in x in the whole space $R^n \ni z = (x^T, y^T)^T$,

(iiii) $g_1(x,y,t)$, $g_2(x,y,t)$, and $\sigma(x,y,t)$ satisfy the condition of linear growth

$$\|g_1(z,t)\|^2 + \|g_2(z,t)\|^2 + \|\sigma(z,t)\|^2 \leq L(1 + \|z\|^2)$$

which ensures the existence and uniqueness up to stochastic equivalence of the solution $z(t)$ of equation (9) in R^n with the initial condition $z(t_0) = z_0$ that is a strictly Markov process continuous with probability one [1], as in Case 1.

Let us assume that $g_1 \in C^{121}_{xyt}$. Using the Ito stochastic differentiation rule, we have

$$\dot{\lambda} = \frac{\partial \lambda}{\partial t} + \frac{\partial \lambda}{\partial x} g_1(x,y,t),$$

$$\ddot{\lambda} = \frac{\partial^2 \lambda}{\partial t^2} + \frac{\partial^2 \lambda}{\partial t \partial x} g_1 + \frac{d}{dt}\left(\frac{\partial \lambda}{\partial x}\right) g_1 + \frac{\partial \lambda}{\partial x}\dot{g}_1 =$$

$$= \frac{\partial^2 \lambda}{\partial t^2} + 2\frac{\partial^2 \lambda}{\partial t \partial x} g_1 + g_1^T \frac{\partial^2 \lambda}{\partial x^2} g_1 + \frac{\partial \lambda}{\partial x} g_{1t} + \frac{\partial \lambda}{\partial x} g_{1x}^T g_1 + \lambda_x g_{1y}^T g_2 + \frac{1}{2}\lambda_x g_{1yy} : \sigma\sigma^T + \lambda_x g_{1y}^T \sigma \dot{\xi}. \quad (10)$$

Let us introduce arbitrary Yerugin functions [7]: A_1 and B_1 such that $A_1(0,0,x,y,t) \equiv 0$, $B(0,0,x,y,t) \equiv 0$ and

$$\ddot{\lambda} = A_1(\lambda, \dot{\lambda}, x, y, t) + B(\lambda, \dot{\lambda}, x, y, t)\dot{\xi}. \quad (11)$$

Based on equations (10) and (11) we obtain the relations

$$\begin{cases} G_1 g_2 = A_1 - G_2, \\ G_1 \sigma_i = B_{1i}, \ i = \overline{1,k}, \end{cases} \quad (12)$$

where

$$G_1 = \lambda_x g_{1y}^T,$$

$$G_2 = \frac{\partial^2 \lambda}{\partial t^2} + 2\frac{\partial^2 \lambda}{\partial t \partial x} g_1 + g_1^T \frac{\partial^2 \lambda}{\partial x^2} g_1 + \frac{\partial \lambda}{\partial x} g_{1t} + \frac{1}{2}\lambda_x g_{1yy} : \sigma\sigma^T,$$

from which we need to determine g_1, g_2 and σ.

Using Lemma 1, from relations (12) we define the function g_2 and matrix σ

$$g_2 = \widetilde{s}_1[G_1 C] + (G_1)^+(A_1 - G_2), \quad (13)$$

$$\sigma_i = \widetilde{s}_2[G_1 C] + (G_1)^+ B_{1i}. \quad (14)$$

Theorem 3. *For the system of first-order Itô differential equations* (9) *to have a given integral manifold* (8), *it is necessary and sufficient that for arbitrarily given vector functions* $g_1 \in K$, *the vector function* g_2 *and the columns* σ_i *of the matrix* σ *of System* (9) *are of the form* (13) *and* (14), *respectively.*

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Тілеубергенов М.Ы., Василина Г.Қ., Медетбеков М.М. АЗЫНҒАН ДИФФУЗИЯЛЫ БІРІНШІ РЕТТІ СТОХАСТИКАЛЫҚ ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕР ЖҮЙЕСІНІҢ НЕГІЗГІ КЕРІ ЕСЕБІНІҢ ШЕШІМДІЛІГІ ТУРАЛЫ

Қозғалыстың берілген қасиеттері бойынша бірінші ретті Ito стохастикалық дифференциалдық теңдеулер жүйесін құру есебі қарастырылды. Теңдеудің коэффициенттері бойынша, құрастырылған теңдеу үшін берілген интегралдық көпбейненің бар болуының қажетті және жеткілікті шарттары барлық тәуелсіз айнымалылардан тәуелді және айнымалылардың бір бөлігінен ғана тәуелді болған кезде екі жағдай бөлек алынды.

Кілттік сөздер: стохастикалық дифференциалдық теңдеулер, интегралдық көпбейне, кері есеп, азынған диффузия.

Тлеубергенов М.И., Василина Г.К., Медетбеков М.М. О РАЗРЕШИМОСТИ ОСНОВНОЙ ОБРАТНОЙ ЗАДАЧИ СИСТЕМЫ СТОХАСТИЧЕСКИХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ПЕРВОГО ПОРЯДКА С ВЫРОЖДАЮЩЕЙСЯ ДИФФУЗИЕЙ

Рассматривается задача построения системы стохастических дифференциальных уравнений Ито первого порядка по заданным свойствам движения. В терминах коэффициентов уравнения получены необходимые и достаточные условия существования у построенного уравнения заданного интегрального многообразия отдельно в двух случаях, когда оно зависит от всех независимых переменных и когда оно зависит лишь от части переменных.

Ключевые слова: стохастические дифференциальные уравнения, интегральное многообразие, обратная задача, вырождающаяся диффузия.