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On the function depth in an o-stable ordered group of a finite convexity rank

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Abstract. We investigate the monotonicity properties of unary functions definable in ordered groups whose elementary theories are o-stable and have finite convexity rank. The notion of o-stability, combining o-minimality and stability, ensures tameness of types around cuts. Prior work established piecewise or local monotonicity of definable functions in weakly o-minimal structures, with key contributions by Pillay, Steinhorn, Wencel, and others. We build on these results by focusing on local monotonicity, n-tidiness, and the depth of definable functions. In particular, we show that any such function is piecewise n-tidy for some finite n, extending the theory of monotonicity beyond weakly o-minimal structures to a broader o-stable context.

Keywords. O-minimal theory, NIP theory, piecewise monotonicity, local monotonicity, o-stable theory, the convexity rank.

1 Preliminaries

The notion of o-stability combines both notions of o-minimality and stability. Roughly speaking, a linearly ordered structure is o-stable if, for any cut, there exist a few complete one-types that are consistent with this cut. B. Baizhanov and V. Verbovskiy showed in [1] that a weakly o-minimal theory is o-stable. A sharper result follows from the description of weakly o-minimal structures by B. Kulpeshov, that a linearly ordered structure is weakly o-minimal if and only if any cut has at most two extensions up to complete one-types over this structure and the sets of all realizations of these one-types are convex [5]. So, we can say that any weakly o-minimal theory has Morley o-rank 1 and Morley o-degree at most 2, so, these theories are o-ω-stable.

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A. Pillay and C. Steinhorn started the investigation of the piecewise monotonicity of definable unary functions in linearly ordered structures in [9]. R. Wencel extended their result to the class of non-valuational weakly o-minimal ordered groups [14]. D. Macpherson, D. Marker, and C. Steinhorn introduced the notion of a local monotonicity and a tidy function and proved that any unary function that is definable in a structure of a weakly o-minimal theory is tidy. V. Verbovskiy introduced the notion of the depth of a function and proved that any function definable in a structure of a weakly o-minimal theory has finite depth and is piecewise n-tidy for some finite natural n [10]. Also, the question of monotonicity of unary functions has been studied in many other articles for different classes of theories. V. Verbovskiy and A. Dauletiyarova proved piecewise monotonicity of a unary function definable in an ordered non-valuational group with an o-stable theory [13]. Here, we aim to consider local monotonicity and the notions of n-tidy and the depth of a unary function definable in an ordered group with an o-stable theory of a finite convexity rank.

In the following section, we provide some standard definitions and notations.

Let $\mathcal{M} = (M, <, \dots)$ be a totally ordered structure, a be an element of M, and let A and B be subsets of M. As usual, we write

$$\begin{aligned} a &< B, \text{ if } a < b \text{ for any } b \in B, \\ A &< B, \text{ if } a < b \text{ for any } a \in A \text{ and } b \in B. \end{aligned}$$

A partition $\langle C, D\rangle$ of M is called a *cut* if $C < D$. Given a cut $\langle C, D\rangle$, one can construct a partial type $\{c < x < d : c \in C,\ d \in D\}$, which we also call a cut and use the same notation $\langle C, D\rangle$. If the set C is definable, then the cut is called *quasirational*; if in addition $\sup C \in M$, then the cut $\langle C, D\rangle$ is called *rational*. A non-definable cut is called *irrational*. If $C = (-\infty, c)$ we denote this cut by c^-, and if $C = (-\infty, c]$ we denote it by c^+. If $C = M$, we denote this cut $+\infty$. The notation $\sup A$ stands for such a cut $\langle C, D\rangle$, that $C = \{c \in M : c < \sup A\}$. If the set C is definable we sometimes distinguish cuts defined by $\sup C$ and $\inf D$ as: $\sup C$ stands for $\langle C, D\rangle \cup \big\{C(x)\big\}$ and $\inf D$ stands for $\langle C, D\rangle \cup \big\{\neg C(x)\big\}$. A cut $\langle C, D\rangle$ in an ordered group is called *non-valuational* [7, 14] if $d - c$ converges to 0 whenever c and d converge to $\sup C$ and $\inf D$, respectively. A cut, which is not non-valuational, is called *valuational*. Observe that for a valuational cut $\langle C, D\rangle$, there is a convex non-trivial subgroup H such that $\sup C = \sup(a + H)$ for some a, and this cut is definable iff the subgroup H is definable. An ordered group G is said to be of *non-valuational type*, if any quasirational cut is non-valuational. Note that G is of non-valuational type if and only if there is no definable non-trivial convex subgroup in G.

The set of all cuts $\langle C, D\rangle$ that are definable in $\mathcal{M}$ and such that the set D has no smallest element will be denoted by $\overline{M}$. The set M can be regarded as a subset of $\overline{M}$ by identifying an element $a \in M$ with the cut $\big\langle(-\infty, a], (a, +\infty)\big\rangle$. After such identification, $\overline{M}$ is naturally equipped with a linear ordering extending $(M, <)$: $\langle C_1, D_1\rangle \le \langle C_2, D_2\rangle$ if and only if $C_1 \subseteq C_2$. Clearly, $(M, <)$ is a dense substructure of $(\overline{M}, <)$.

A subset A of a totally ordered set M is called *convex* if for any a and $b \in A$ the interval $[a, b]$ is a subset of A. The *length* of a convex set A is defined as $\sup\{a-b : a, b \in A\}$. A *convex component* of a set A is a maximal convex subset of A. The *convex hull* A^c of A is defined as

$$A^c = \big\{b \in M : \exists a_1, a_2 \in A\,(a_1 \leq b \leq a_2)\big\},$$

that is, it is the least convex set containing the set A.

2 Introduction

The aim of this paper is to investigate the properties of unary functions that are definable in an o-stable ordered group whose convexity rank is finite, say, n.

Let $\mathcal{M} = (M, <, \dots)$ be a totally ordered structure. Recall that a function can be defined from its graph and for each function, it is easy to construct its graph. So, we consider an arbitrary formula: $\Phi(x, \bar{y})$. Let B be the set of all such $\bar{b}$, that $\Phi(\mathcal{M}, \bar{b}) \neq \emptyset$. Given an element $\bar{b}$ we consider the definable set $\Phi(\mathcal{M}, \bar{b})$. Then we can consider $\sup \Phi(\mathcal{M}, \bar{b})$ as an element of $\overline{M}$. So, the set

$$\{(\bar{b}, \sup \Phi(\mathcal{M}, \bar{b})) : \bar{b} \in B\}$$

defines the graph of some function f from B to $\overline{M}$. The main property we consider here is the monotonicity of a function. So, below, we define $f(\bar{b}_1) \geq f(\bar{b}_2)$ in terms of Φ. We prefer to work with a formula $\Phi(x; \bar{y})$ rather than with a function $f(\bar{y})$.

So, let $\Phi(x, \bar{y})$ be an M-definable formula. We write

$$\Phi(\mathcal{M}, \bar{y}_1) \geq \Phi(\mathcal{M}, \bar{y}_2), \text{ if } \mathcal{M} \models \forall x_2 \exists x_1 [\bigwedge_{i=1}^{2} \Phi(x_i, \bar{y}_i) \to x_2 \leq x_1]$$

it means that $\sup \Phi(\mathcal{M}, \bar{y}_1) \geq \sup \Phi(\mathcal{M}, \bar{y}_2)$. Then

$$\Phi(\mathcal{M}, \bar{y}_1) = \Phi(\mathcal{M}, \bar{y}_2) \Leftrightarrow \Phi(\mathcal{M}, \bar{y}_1) \leq \Phi(\mathcal{M}, \bar{y}_2) \wedge \Phi(\mathcal{M}, \bar{y}_1) \geq \Phi(\mathcal{M}, \bar{y}_2)$$

$$\Phi(\mathcal{M}, \bar{y}_1) < \Phi(\mathcal{M}, \bar{y}_2) \Leftrightarrow \Phi(\mathcal{M}, \bar{y}_1) \leq \Phi(\mathcal{M}, \bar{y}_2) \wedge \Phi(\mathcal{M}, \bar{y}_1) \neq \Phi(\mathcal{M}, \bar{y}_2)$$

Now we consider the case where the length of the tuple $\bar{y}$ is 1, that is, y is a variable.

We say that $\Phi(x, y)$ is strictly increasing on a set I, if

$$\forall y \forall z [y, z \in I \wedge y < z \to \Phi(\mathcal{M}, y) < \Phi(\mathcal{M}, z)].$$

If $E(y, z)$ is an equivalence relation with convex classes on a set I, then $\Phi(\mathcal{M}, y)$ is strictly increasing on the quotient set $I/_E$, if

$$\forall y \forall z [y, z \in I \wedge \neg E(y, z) \wedge y < z \to \Phi(\mathcal{M}, y) < \Phi(\mathcal{M}, z)].$$

We define strictly decreasing and constant behavior in a similar way to strictly increasing behavior.

We assume that $\operatorname{dom}\Phi(\mathcal{M},y)=\{y\in M:\mathcal{M}\models(\exists x)\Phi(x,y)\}$.

Definition 1 (M. Dickmann; D. Macpherson, D. Marker, C. Steinhorn)**.**

- A weakly o-minimal structure is a totally ordered structure $\mathcal{M}=(M,<,\dots)$ such that any definable subset of M is a finite union of convex disjoint sets under the ordering $<$.
- A theory is weakly o-minimal if all of its models are.

Definition 2 (D. Macpherson, D. Marker, C. Steinhorn,[7])**.** If $\mathcal{M}$ is a totally ordered structure, $\Phi(x,y)$ is an M-definable formula, $I\subset\operatorname{dom}(\Phi(\mathcal{M},y))$, then we say that Φ is tidy on I, if one of the following holds:

1. $\forall x\in I$ there is an infinite interval $J\subset I$ such that $x\in J$ and Φ is strictly increasing on J (we say that Φ is locally increasing on I).
2. $\forall x\in I$ there is an infinite interval $J\subset I$ such that $x\in J$ and Φ is strictly decreasing on J (we say that Φ is locally decreasing on I).
3. $\forall x\in I$ there is an infinite interval $J\subset I$ such that $x\in J$ and Φ is constant on J (we say that Φ is locally constant on I).

and, if for some $x\in I$ the set $\{y\in I\mid\Phi(\mathcal{M},t)$ is strictly monotonic on $(x,y)\cup(y,x)\}$ and has a maximum or a minimum, then $\Phi(\mathcal{M},t)$ is strictly monotonic on I.

Definition 3. If Φ and I are like in Definition 2, then we say that Φ is n-tidy on I if the following holds:

- $\forall z\forall y\forall t\;[\Phi(\mathcal{M},z)=\Phi(\mathcal{M},y)\wedge z<t<y\to\Phi(\mathcal{M},z)=\Phi(\mathcal{M},t)]$
- $\Phi^{(n)}$ is tidy on I/E_{n-1}, where $\Phi^{(n)}(x,y):=\exists z[E_{n-1}(y,z)\wedge\Phi(x,z)]$
- $(\forall y\in I)\;E_n(I,y)/E_{n-1}$ has no minimum and maximum.
- $|I/E_n|\geq\omega$.

Where E_n is an equivalence relation on I such that

$$\begin{array}{rcl}E_n(z,y) & \Leftrightarrow & E_{n-1}(z,y)\vee\\ & \vee & [[z<y\wedge\neg E_{n-1}(z,y)\to\Phi^{(n)}\restriction[z,y]/E_{n-1}\text{ is strictly monotonic}]\wedge\\ & \wedge & [y<z\wedge\neg E_{n-1}(z,y)\to\Phi^{(n)}\restriction[y,z]/E_{n-1}\text{ is strictly monotonic}]]\end{array}$$

Here 0-tidy is tidy, $\Phi^{(0)}=\Phi,\ E_0(z,y)\Leftrightarrow z=y$.

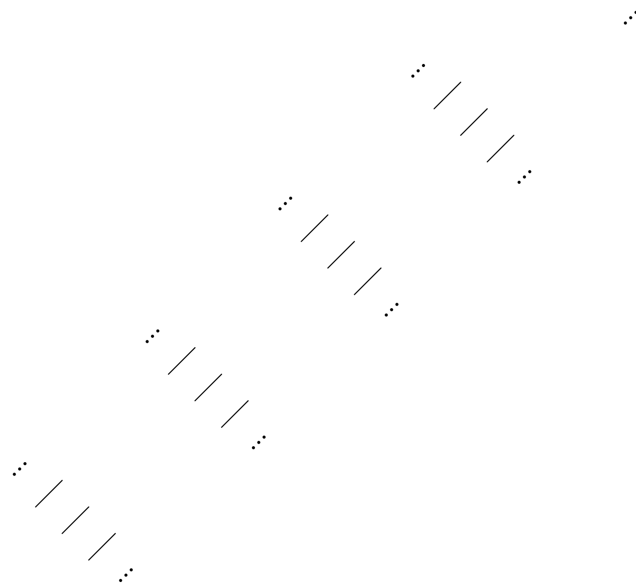

Figure 1: The example of the graph of a function of depth 3

Definition 4. If Φ and I like in Definition 2, then we say that Φ is strongly tidy on I if there exists $n \in N$ such that Φ is $(n-1)$-tidy on I and $\Phi^{(n)}$ is strictly monotonic on I/E_{n-1}. So we say that the depth of Φ on I equals n.

Definition 5 (D. Macpherson, D. Marker, C. Steinhorn, [7])**.** Let $\mathcal{M}$ be a weakly o-minimal structure. We say that $\mathcal{M}$ has monotonicity if the following holds: whenever $\Phi(x, y, \bar{a})$ is a formula with $\bar{a} \in M$, there is $m \in N$ and a partition of $\mathrm{dom}(\Phi(M, y, \bar{a}))$ into definable sets $X, I_1, \ldots, I_m$ such that X is finite, each I_i is convex and on each I_i the formula $\Phi(x, y, \bar{a})$ is tidy.

Definition 6 (V. Verbovskiy, [10])**.** Let $\mathcal{M}$ be a weakly o-minimal structure. We say that $\mathcal{M}$ has strong monotonicity (monotonicity [7]), if the following holds: whenever $\Phi(x, y, \bar{a})$ is a formula with $\bar{a} \in M$, there is $m \in N$ and a partition of $\mathrm{dom}(\Phi(M, y, \bar{a}))$ into definable sets X, I_1, $\ldots$, I_m such that X is finite, each I_i is convex and on each I_i the formula $\Phi(x, y, \bar{a})$ is strongly tidy.

Definition 7. Let $\mathcal{M}$ be a weakly o-minimal structure. Then we say that $\mathcal{M}$ has finite depth, if the following holds: whenever $\Phi(x, y, \bar{z})$ is a formula, there exists $n \in N$ such that for any $\bar{a} \in M$ and for any convex set $I \subset \mathrm{dom}(\Phi(\mathcal{M}, y, \bar{a}))$, on which $\Phi(\mathcal{M}, y, \bar{a})$ is strongly tidy, the depth of $\Phi(\mathcal{M}, y, \bar{a})$ on I is less than n.

Theorem 8 (D. Macpherson, D. Marker, C. Steinhorn, [7]). *If all models of $Th(\mathcal{M})$ are weakly o-minimal, then $\mathcal{M}$ has monotonicity.*

Theorem 9 (V. Verbovskiy, [10]). *If all models of $Th(M)$ are weakly o-minimal, then M has strong monotonicity and finite depth.*

Definition 10 (B. Baizhanov, V. Verbovskiy, [1], [11]).

1. An ordered structure $\mathcal{M}$ is *o-stable in* λ if for any $A \subseteq M$ with $|A| \leq \lambda$ and for any cut $\langle C, D\rangle$ in $\mathcal{M}$ there are at most λ 1-types over A which are consistent with the cut $\langle C, D\rangle$, i.e.
$$\left|S^1_{\langle C,D\rangle}(A)\right| \leq \lambda.$$
2. A theory T is *o-stable in* λ if every model of T is. Sometimes, we write T is o-λ-stable.
3. A theory T is *o-stable* if there exists an infinite cardinal λ in which T is o-stable.

In [11], V. Verbovskiy proved that any ordered group whose elementary theory is o-stable is Abelian.

Lemma 11 (V. Verbovskiy, [11]). *Let G be an ordered group of non-valuational type whose elementary theory is o-stable. Then any equivalence relation in G has at most finitely many infinite convex classes.*

Let $\Gamma = \{(x, f(x)) : x \in \mathrm{dom}(f)\}$ be the graph of a function f.

We denote by $\lim_{x\to a+0} f$ the set of all elements $b \in \overline{G}$ such that (a, b) is a limit point of the set $\{(x, f(x)) : x \in \mathrm{dom}(f), x > a\}$. In other words, $\lim_{x\to a+0} f$ is the set of all the right-hand limit points of the function f at the point a.

Similarly, we define $\lim_{x\to a-0} f$ as the set of all such b that (a, b) is a limit point of the set $\{(x, f(x)) : x \in \mathrm{dom}(f), x < a\}$.

Furthermore, we define
$$\lim_{x\to a} f \triangleq \lim_{x\to a-0} f \cup \lim_{x\to a+0} f.$$

Fact 12 (J. Goodrick, [4]). *For any densely ordered structure A and any function $f : A \to \overline{A}$, for any $a \in A$, the set $\lim_{x\to a} f(x)$ is nonempty.*

Lemma 13 (V. Verbovskiy, A. Dauletiyarova, [13]). *Let an ordered group $(G, <, +, f, 0, \dots)$, whose order is dense, have an o-stable theory. Then there exists a natural number k such that for any element $a \in \mathrm{dom}\, f$, the set $\lim_{x\to a} f(x)$ has at most k elements.*

Due to Lemma 13, we can define k functions $f_1, \dots, f_k$, where k is taken from Lemma 13, as follows: $f_i(x)$ is the i-th element of $\lim_{x\to a} f(x)$. So, we can define finitely many functions which have at most one limit $\lim_{x\to a+0} f$ and at most one limit $\lim_{x\to a-0} f$. So, without loss of generality, we may assume that a function under consideration has at most one left-hand limit and at most one right-hand limit.

The definition of the convexity rank of a formula with one free variable was introduced in [5] and extended on an arbitrary set in [6] by B. Kulpeshov:

Definition 14 (B. Kulpeshov, [5, 6]). Let T be a weakly o-minimal theory, $\mathcal{M} \models T$, $A \subseteq M$. *The rank of convexity of the set* A ($RC(A)$) is defined as follows:

1. $RC(A) = -1$ if $A = \emptyset$.
2. $RC(A) = 0$ if A is finite and non-empty.
3. $RC(A) \geq 1$ if A is infinite.
4. $RC(A) \geq \alpha + 1$ if there exists a parametrically definable equivalence relation $E(x, y)$ and an infinite sequence of elements $b_i \in A, i \in \omega$, such that:

- For every $i, j \in \omega$ whenever $i \neq j$ we have $M \models \neg E(b_i, b_j)$;
- For every $i \in \omega$, $RC(E(x, b_i)) \geq \alpha$ and $E(M, b_i)$ is a convex subset of A.

5. $RC(A) \geq \delta$ if $RC(A) \geq \alpha$ for all $\alpha < \delta$, where δ is a limit ordinal.

If $RC(A) = \alpha$ for some ordinal α, we say that $RC(A)$ is defined. Otherwise (that is, if $RC(A)) \geq \alpha$ for all α), we put $RC(A) = \infty$.

The rank of convexity of a formula $\phi(x, \bar{a})$, where $\bar{a} \in M$, is defined as the rank of convexity of the set $\phi(M, \bar{a})$, that is, $RC(\phi(x, \bar{a})) \triangleq RC(\phi(M, \bar{a}))$.

The convexity rank of a 1-type p is defined as the rank of convexity of the set $p(M)$, that is, $RC(p) \triangleq RC(p(M))$.

Obviously, a theory that extends the theory of a linear order has the convexity rank 1 if there are no parametrically definable equivalence relations with infinitely many infinite convex classes.

3 Main result

Let A be a definable convex set. We define the following convex subgroups:

$$H_A^+ = \{g \in G : a + |g| \in A \text{ for any } a \in A\},$$

$$H_A^- = \{g \in G : a - |g| \in A \text{ for any } a \in A\}.$$

Following [8], we say that the right shore of A is long if H_A^+ is trivial; similarly, the left shore of A is long if H_A^- is trivial.

Lemma 15. *Let $\mathcal{G}$ be an ordered o-stable group and E a definable equivalence relation with convex classes. Then the number of infinite E-classes with a long shore is finite.*

Proof. Let E be a definable equivalence relation with infinitely many convex classes. Assume to the contrary that there exist infinitely many infinite E-classes with a long shore. By Dirichlet's principle, without loss of generality, we may assume that there are infinitely many infinite E-classes with the right long shore. Moreover, without loss of generality, we may assume that there exists an infinite increasing sequence $\langle a_i : i < \omega\rangle$ of representatives of infinite E-classes with a long right shore, such that $\neg E(a_i, a_j)$ for each $i < j < \omega$. Since we can consider a sufficiently saturated model, we may suppose that there exists a positive element $b \in G$ such that $E(a_i, a_i + b)$ holds for each $i < \omega$.

Let $\varphi(x; b)$ say that x belongs to an E-class whose length is at least b and whose right shore is long and the distance between x and the right shore of $[x]_E$ is less than b. Since the right shore is long, this set is not empty.

Let $C = \{g \in G : g < a_i \text{ for some } i\}$ and $D = G \setminus C$. Since for each $c \in C$ and each b_1 and b_2 with $0 < b_1 < b_2 < b$ it holds that

$$\varphi(\mathcal{G}, b_1) \cap (c, \sup C) \subset \varphi(\mathcal{G}, b_2) \cap (c, \sup C)$$

we obtain that $\mathcal{G}$ has the strict order property inside the cut $\sup C$. Since an expansion of a model of an o-stable theory by a convex unary predicate preserves o-stability [11], we may add a convex predicate P which names C. So, we obtain the strict order property inside the cut

$$\{c < x < d : c \in C,\ d \in D\} \cup \{P(x)\},$$

that contradicts o-stability, [1]. □

Theorem 16. *Let $\mathcal{G} = (G, <, +, \dots)$ be an ordered group with an o-stable theory. Let $\mathcal{G}$ have finitely many, say k, non-trivial proper definable convex subgroups. Then $RC(G) = k$. And vice versa, if $RC(G)$ is finite, say k, then the number of non-trivial proper definable convex subgroups is equal to k.*

Proof. Let $\{0\} < H_1 < \cdots < H_k < G$ be a chain of all definable convex subgroups of $\mathcal{G}$. Then $E_k(x, y) \triangleq x - y \in H_k$ is an equivalence relation with convex classes, and the chain $E_1, \dots, E_k$ of equivalence relations demonstrates that $RC(G)$ is at least k.

Assume that $RC(G) = k$ and a chain of equivalence relations $E_1, \dots, E_k$ witnesses it, where E_i refines E_{i+1} for each positive $i < k$. By Lemma 15, without loss of generality, after removing finitely many equivalence classes, we may assume that each shore of each class of each equivalence relation is short, that is, not long, so, it defines a non-trivial subgroup. By the first paragraph of the proof of this theorem, the number of definable convex non-trivial proper subgroups is at most k, say, n. Let

$$\{0\} < H_1 < \cdots < H_n < G$$

be the sequence of all definable convex non-trivial proper subgroups of $\mathcal{G}$.

First, we consider E_1. Since H_1 is the least non-trivial definable convex subgroup, and each shore of each infinite E_1-class defines a subgroup, each infinite E_1-class consists of cosets of H_1.

Assume that there are infinitely many cosets of H_2 such that each contains an E_1-class as a proper subset and this E_1-class consists of infinitely many cosets of H_1. Then we consider $\mathcal{G}/H_1$ with the full induced structure. Its elementary theory is o-stable [11]. We obtain an equivalence relation in $\mathcal{G}/H_1$ with infinitely many infinite convex classes, but these classes are proper subsets of cosets of H_2/H_1. So, there exists a definable convex subgroup $H'/H_1 < H_2/H_1$. Let H' be the pre-image of H'/H_1 in G. It is definable and convex. We obtain a contradiction with the fact that there are exactly n definable convex non-trivial proper subgroups. So, either an E_1-class consists of finitely many cosets of H_1 or consists of cosets of H_2. Note that if an E_1-class consists of finitely many cosets of H_1, then the order in G/H_1 is discrete. Moreover, there is a positive integer m_1 such that if a E_1-class consists of finitely many cosets of H_1, then the number of these cosets is at most m_1. Indeed, otherwise by compactness there are infinite E_1-classes and we obtain a contradiction as above.

Considering G/H_i we can conclude by the similar reasons that either an E_{i+1}-class consists of finitely many cosets of H_{i+1} or consists of cosets of H_{i+2}.

These imply that $k \leq n$. So, we obtain the equality $k = n$. □

Theorem 17. *Let G be an order-stable ordered group of non-valuational type with a dense order, let A be a sort in $\overline{G}$, and let $f : D \subset G \to A$ be a definable and continuous function. Then the function f is piecewise monotonic; that is, there exists some $m \in \mathbb{N}$ and a finite partition of the domain of the function $D = \operatorname{dom} f$ into definable sets $X, I_1, \ldots, I_m$ such that X is finite, each I_i is convex for $i < m$, and $f \restriction I_i$ is monotonic.*

Theorem 18. *Let $\mathcal{G}$ be an ordered group with an o-stable theory and let f be a definable continuous unary function. Let $RC(G) = n$. Let $\{0\} < H_1 < \cdots < H_n < G$ be the chain of all its definable convex subgroups and let G/H_i be dense for each i.*

Then f is strongly tidy and its depth is at most n. In other words, an ordered group with an o-stable theory and a finite convexity rank has strong monotonicity and finite depth bounded by its convexity rank.

Proof. As we mentioned it below Lemma 13, we may assume that the graph f has at most one limit point from the left and at most one limit point from the right for each element. By Theorem 17 the restriction of f to any coset of H_1 is piecewise monotone. Let E_1 be an equivalence relation on $\operatorname{dom} f$ with convex classes, on which f is monotone. Then, as we know, the E_1-classes are cosets of H_1. Now we consider G/H_1 with the complete induced structure and we find that f/H_1 is monotone on the cosets of H_2/H_1. Here,

$$f/H_1(a) \triangleq \sup\{f(x) : x \in a + H_1\}.$$

Proceeding by induction, we conclude that the statements of the theorem hold. The bound for the depth of f follows from Theorem 16. □

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Вербовский В.В. ШЕКТЕУЛІ ДӨҢЕСТІК РАНГІ БАР РЕТТЕЛГЕН-ТҰРАҚТЫ ТОПТАҒЫ ФУНКЦИЯНЫҢ ТЕРЕҢДІГІ ТУРАЛЫ

Біз элементар теориялары о-тұрақты және шектеулі дөңестіктік рангі бар реттелген топтарда анықталатын унар функциялардың монотондық қасиеттерін зерттейміз. о-тұрақтылық ұғымы о-минималдылық пен тұрақтылықты біріктіріп, үзілістер маңындағы типтердің тәртіптілігін қамтамасыз етеді. Бұған дейінгі еңбектерде Пиллэй, Стайнхорн, Венцел және басқалардың маңызды үлесімен әлсіз о-минимал құрылымдарда анықталатын функциялардың кесінділік немесе локальді монотондығы көрсетілген. Біз бұл нәтижелерді жалғастырып, локальді монотондыққа, n-тәртіптілікке және анықталатын функциялардың тереңдігіне назар аударамыз. Атап айтқанда, мұндай функцияның кейбір шекті натурал n үшін кесінділік n-тәртіпті болатынын көрсетеміз. Бұл монотондық теориясын әлсіз о-минимал құрылымдардан кеңірек о-тұрақты контекске дейін жалғастырады.

Түйін сөздер: реттелген минимал теория, NIP теориясы, кесінділік монотондық, локальді монотондық, реттелген тұрақты теория, дөңестік ранг.

Вербовский В.В. О ГЛУБИНЕ ФУНКЦИИ В О-СТАБИЛЬНОЙ УПОРЯДОЧЕННОЙ ГРУППЕ С КОНЕЧНЫМ РАНГОМ ВЫПУКЛОСТИ

Мы исследуем свойства монотонности унарных функций, определимых в упорядоченных группах, элементарные теории которых являются о-стабильными и имеют конечный ранг выпуклости. Понятие о-стабильности, объединяющее о-минимальность и стабильность, обеспечивает упорядоченность типов в окрестности сечений. В предыдущих работах, в частности благодаря вкладу Пиллея, Стайнхорна, Венцеля и других, была установлена кусочная или локальная монотонность определимых функций в слабо о-минимальных структурах. Мы развиваем эти результаты, сосредотачиваясь на локальной монотонности, n-упорядоченности и глубине определимых функций. В частности, мы показываем, что любая такая функция является кусочно n-упорядоченной для некоторого конечного n, расширяя теорию монотонности за пределы слабо о-минимальных структур к более общему о-стабильному контексту.

Ключевые слова: о-минимальная теория, теория с NIP, кусочная монотонность, локальная монотонность, упорядоченно-стабильная теория, ранг выпуклости.