Kazakh Mathematical Journal ISSN 1682–0525

25:3 (2025) 6–12

Cuts of a totally ordered field of rational functions over an Archimedean field

Nataliya Yu. Galanova

Tomsk State University, Tomsk, Russia

galanova@math.tsu.ru

Communicated by: Viktor V. Verbovskiy

Received: 11.05.2025 ⋆ Accepted/Published Online: 01.07.2025 ⋆ Final Version: 11.05.2025

Abstract. We study the classification of Dedekind cuts in the field $K(\alpha)$, where $K \subseteq \mathbb{R}$ and α is a positive infinitesimal. The cuts are analyzed according to three criteria: fundamentality, symmetry, and algebraicity. We prove that every non-principal cut in $K(\alpha)$ that is both non-fundamental and asymmetric must be algebraic. For such cuts, we construct a sign-changing polynomial whose root realizes the cut. Furthermore, we investigate the properties of these polynomials and their dependence on the structure of the base field K. The results contribute to the broader understanding of algebraic and order-theoretic properties in non-Archimedean extensions of real fields, particularly in the context of model theory and real algebraic geometry. The classification developed here provides a constructive approach to identifying algebraic cuts and offers insights into the interaction between infinitesimal elements and the topological structure of real closed fields.

Keywords. cut (gap), totally ordered field, quotient field, formal power series.

1 Introduction

The monograph [5] and the review article [4] present established approaches to the study of totally ordered fields, which appeared in the works of Artin and Schreier [1] and developed together with a non-standard analysis, an analysis of non-Archimedean valued fields, and a model theory. One of the directions for the investigation of ordered fields is connected with the cut (gap) theory. The theory of cuts dates back to Dedekind's work and has now received significant development [11]. The present paper continuous this theme.

Let $\langle F, \cdot, +, < \rangle$ be a totally ordered field (by ordered, we will always mean "totally ordered"). A pair of non-empty subsets $A, B \subset F$ is called a *cut*, if $A < B$ and $A \cup B = F$. Let, as in [12], (A, B) be a cut in F, the set A is called a *short shore*, if there exists $a_0 \in A$ such

2020 Mathematics Subject Classification: 03E04, 12J15, 12J25.

DOI: https://doi.org/10.70474/trtm5125

that for all $a \in A$ we have $a + (a - a_0) \in A$, the element a_0 is called *close* to the shore B. If a shore is not short, then it is called a *long shore*. If both A and B are long shores, then the cut (A, B) is called *symmetric* [12]. If one of the shores is long and the other one is short, then such a cut (A, B) is called *asymmetric* [12] (or a *ball* cut [11]). Let F_1 be an ordered extension of a field F and $x \in F_1$. We say that x *realizes* a cut (A, B) [11, 14], if

$$\forall a \in A \ \forall b \in B \ (a \leqslant x \leqslant b).$$

We say that a cut (A, B) is *algebraic* [12, 14], if in some F_1 extending F, some $x \in F_1 \backslash F$ realizes (A, B) and is algebraic over F.

A cut (A, B) is called *principal*, if A has a largest element or B has a smallest element; otherwise (A, B) is called *non-principal* (or a *gap* [11], or an *irrational* cut [2]). A cut (A, B) is called *fundamental* [12], if for all positive $\varepsilon \in F$ there exist $x \in A$ and $y \in B$ such that $y - x < \varepsilon$. A fundamental non-principal cut is also called a Scott cut [14]. From the definitions, it is easy to see that every non-principal fundamental cut is symmetric; every principal fundamental cut is asymmetric and non-algebraic.

Let L be a totally ordered set. A subset $H \subset L$ is said to be cofinal to L, if

$$\forall l \in L \ \exists h \in H \ (l \leq h).$$

A subset $H \subset L$ is said to be coinitial to L, if

$$\forall l \in L \ \exists h \in H \ (l \geq h).$$

The least cardinality of a set among all sets that are cofinal (coinitial) to L is called cofinality (coinitiality) of the set L and is denoted $cf(L)$ $(coi(L))$.

Throughout this paper $\mathbb{N}$ is the set of all natural numbers, $\mathbb{Q}$ is the field of all rational numbers, $\mathbb{R}$ is the field of all real numbers.

We write F^+ for $\{x \in F \mid x > 0\}$. For $x, y \in F^+$, let $x \sim y$ if there exists $n \in \mathbb{N}$ such that

$$x \leq ny \text{ and } y \leq nx.$$

Let G_F be the set of equivalence classes of F mod $\sim$. We denote the $\sim$-class of an element x by $\widehat{x}$, which is an element of G_F. Note that

$$\widehat{x} = \{y \in F^+ \mid \ (\exists\, m, n \in \mathbb{N}) \ \frac{1}{n}x \leq y \leq mx\} = \{y \in F^+ \mid \ (\exists\, n \in \mathbb{N}) \ \frac{1}{n}x \leq y \leq nx\}.$$

We write $x \ll y$ if $nx < y$ for any $n \in \mathbb{N}$. Clearly, $\widehat{x} < \widehat{y} \Leftrightarrow x \ll y$. We put $\widehat{x} \cdot \widehat{y} = \widehat{x \cdot y}$. So, we obtain that $(G_F, \cdot, <)$ is a totally ordered group. If an ordered group is isomorphic to G_F, then this group is called *a group of Archimedean classes of* F (*an Archimedean group of* F). There exists an ordered embedding of the group $(G_F, \cdot, <)$ in the field F, so we may assume that $G_F \subset F$ [5, 10, 9, 4, 12].

An ordered field F is *real-closed* if it does not have a proper algebraic extension to an ordered field, or equivalently, if every positive element in F is a square and every polynomial over F of an odd degree has a root in F. The Artin-Schreier theorem asserts that every ordered field F has an algebraic extension to a real-closed field $\overline{F}$ whose order is an extension of the order on F, and that $\overline{F}$ is unique up to an isomorphism that leaves all elements of F fixed [5].

2 Cuts of a totally ordered field of rational functions over an Archimedean field

Let K be a totally ordered subfield of $\mathbb{R}$, $\mathbb{Q} \subseteq K \subseteq \mathbb{R}$; we have $G_K = \{1\}$, thus the field K is Archimedean. Let $K(\alpha)$ be a field of rational functions over K, where α is a transcendental element over K. A transcendental extension of K of transcendence degree 1 is obtained by forming the polynomial ring $K[\alpha]$ with indeterminate α and taking its quotient field. We define the order relation on the set $K(\alpha)$ so that α is an infinitesimal: for a polynomial with coefficients from K, we put

$$r_k\alpha^{n-k} + r_{k-1}\alpha^{n-k+1} + ... + r_0\alpha^n > 0,$$

if the coefficient r_k at the lowest degree α is greater than zero; the fraction is assumed to be greater than zero if the numerator and the denominator of the fraction are of the same sign [5, 3]. For example, $(3\alpha^5 - 2\alpha^7 - 0,4\alpha^2) < 0$, because $-0,4 < 0$. Note that the field $K(\alpha)$ is non-Archimedean and its group of Archimedean classes is identified with $G_{K(\alpha)} = \{\alpha^k \mid k \in \mathbb{Z}\}$. By this order, for all $n \in \mathbb{N}$ the following hold

$$0 < \cdots < \alpha^3 < \alpha^2 < n \cdot \alpha < \frac{1}{n} = \frac{1}{n}\alpha^0 < \alpha^{-1} < \ldots$$

We have the following chains of extensions [4]:

$$K \subsetneq K(\alpha) \subsetneq K((\alpha)) \subsetneq K((\mathbb{Q})) = \big\{ \sum_{\gamma\in\Gamma} r_\gamma\alpha^\gamma \mid \gamma \in \mathbb{Q},\ r_\gamma \in K,\ \Gamma \subset \mathbb{Q},\ \Gamma \text{ is well-ordered}\big\},$$

the last field is called a Hahn field. Aslo we have the following:

$$\mathbb{Q}(\alpha) \subseteq K(\alpha) \subseteq \overline{K}(\alpha) \subseteq \mathbb{R}(\alpha) \subsetneq \mathbb{R}((\alpha)) = \big\{ \sum_{n=m}^{\infty} r_n\alpha^n \mid m \in \mathbb{Z},\ r_n \in \mathbb{R}\big\},$$

where the last field is a field of formal Laurent series over $\mathbb{R}$.

It is known that

$$\begin{aligned} \sqrt{1+\alpha} &= \sum_{k\in\mathbf{N}\cup\{0\}} \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cdot .. \cdot(\frac{1}{2}-k+1)}{k!}\alpha^k \in K((\alpha)) \setminus K(\alpha) \\ e^\alpha &= 1 + \alpha + \frac{\alpha^2}{2!} + \cdots \in K((\alpha)) \setminus K(\alpha). \end{aligned}$$

We consider the field $K(\alpha)$ from the point of view of classifying cuts by type:

fundamental/non-fundamental, symmetric/asymmetric, algebraic/non-algebraic.

To define the cut (A, B), it suffices for us to know A, then $B = K(\alpha) \setminus A$.
For $\mathbb{Q} \subseteq K \subsetneq \overline{K} \subsetneq \mathbb{R}$; there are the following examples of cuts:

	fund.	sym.	alg.	example
1	1	1	1	$A = \{x \in K(\alpha) \mid x < \sqrt{1+\alpha}\}$
2	1	1	0	$A = \{x \in K(\alpha) \mid x < e^{\alpha}\}$
3	1	0	1	this case is not possible for any field
4	1	0	0	$A = \{x \in K(\alpha) \mid x \leqslant \alpha\}$
5	0	1	1	$A = \{x \in K(\alpha) \mid x < x_0, x_0 \in \overline{K} \setminus K\},\ if\ \overline{K} \setminus K \neq \emptyset$
6	0	1	0	$A = \{x \in K(\alpha) \mid x < x_0, x_0 \in \mathbb{R} \setminus \overline{K}\},\ if\ \mathbb{R} \setminus \overline{K} \neq \emptyset\}$
7	0	0	1	$A = \{x \in K(\alpha) \mid \exists n \in \mathbf{N}\ x < n\alpha\}$
8	0	0	0	this case is not possible; we prove it in Theorem 1

Details of the analysis of the given examples of cuts (1)–(7) for $K(\alpha)$ with $K = \mathbb{Q}$ can be found in [13], for the examples that are in the table above, the reasoning is similar.

For a more general case, we prove the following.

Theorem 1. *Let K be a totally ordered field, $\mathbb{Q} \subseteq K \subseteq \mathbb{R}$. There is no non-principal cut in the field $K(\alpha)$, which is non-fundamental, asymmetric, and non-algebraic.*

Proof. We consider a non-principal, non-fundamental, asymmetric cut (A, B) of the field $K(\alpha)$, where the set A is short, and the set B is long. The case where A is long and B is short is treated similarly. We prove that the cut (A, B) is algebraic. To do this, it suffices to show that there is a polynomial with coefficients from the field $K(\alpha)$, the root of which is between the sets A and B.

1) From the non-fundamentality condition, we first prove that there exists $k_0 \in \mathbb{Z}$ such that $k_0 = min\{k \in \mathbb{Z} \mid (\forall x \in A)(\forall y \in B)\ y - x > \alpha^k\}$. Indeed, the set $\{\alpha^k \mid k \in \mathbf{Z}\}$ is coinitional and it is also confinal to the set $(0, +\infty)_{K(\alpha)}$. By the condition there exists $\varepsilon \in K(\alpha)^+$ such that $(\forall x \in A)(\forall y \in B)\ y - x \geqslant \varepsilon$. Then there is $k \in \mathbf{Z}$ such that $\alpha^k < \varepsilon \leqslant \alpha^{k-1}$, so the set $M = \{k \in \mathbb{Z} \mid (\forall x \in A)(\forall y \in B)\ y - x > \alpha^k\}$ is not empty; in addition, if k belongs to M, then all integers larger than k belongs to M. Similarly, there exist $m \in \mathbb{Z}$, $x \in A$, and $\exists y \in B$ such that $y - x < \alpha^m$. Then $m \notin M$ and m is a lower bound of the set M. Therefore, the set M has the smallest element, which we denote by k_0.

2) Since (A, B) is non-principal, we obtain that for all $x \in A$ and $m \in \mathbf{N}$ it holds that $(x + m\alpha^{k_0} \in A)$. Indeed, let $x \in A$, $m \in \mathbf{N}$. Assume $x_1 = x + \alpha^{k_0} \in B$. Then $x_1 - x = \alpha^{k_0}$, but it should be $x_1 - x > \alpha^{k_0}$, for a contradiction. So $x_1 = x + \alpha^{k_0} \in A$, further by induction we obtain that $x_2 = x_1 + \alpha^{k_0} \in A$, ..., $x_m = x_{m-1} + \alpha^{k_0} = x_0 + m\alpha^{k_0} \in A$.

3) Note that there are no other Archimedean classes in $G_{K(\alpha)}$ between the Archimedean classes $\widehat{\alpha^{k_0}} = \{x \in K(\alpha) \mid \exists\, m, n \in \mathbb{N}\ \frac{1}{n}\alpha^{k_0} \leq x \leq m\alpha^{k_0}\}$ and $\widehat{\alpha^{k_0-1}}$.

4) Let x_0 be close to B, $x_1 \in A$, and $x_0 < x_1$. Then for any $x \in A$ with $x_1 < x$, we have that $x_1 + (x_1 - x) < x + (x - x_0) \in A$ and $x_1 + (x_1 - x) \in A$, which by definition means that x_1 is also close to B.

5) We prove there exists $x_0 \in A$ such that x_0 is close to B and $x_0 + \alpha^{k_0-1} \in B$. According to 1), from the minimality of k_0 there exist $x_1 \in A$ and $y_1 \in B$ such that $y_1 - x_1 \leqslant \alpha^{k_0-1}$. If x_1 is close to B, then $x_0 = x_1$. If this is not the case, then due to the asymmetry of the cut, there exists $x_0 \in A$ that is close to B and larger than x_1. So, $y_1 - x_0 \leqslant y_1 - x_1 \leqslant \alpha^{k_0-1}$, $y_1 \leqslant x_0 + \alpha^{k_0-1}$, and $x_0 + \alpha^{k_0-1} \in B$.

Next, in the proof of the theorem, we assume that x_0 as in 5).

6) We show that $x_0 + \frac{1}{n}\alpha^{k_0-1} \in B$ for all $n \in \mathbf{N}$. Assume the contrary: there exists $n_0 = max\{n \in \mathbf{N} \mid x_0 + \frac{1}{n}\alpha^{k_0-1} \in B\}$. Then $x_0 + \frac{1}{n_0}\alpha^{k_0-1} \in B$, $x_0 + \frac{1}{n_0+1}\alpha^{k_0-1} \in A$. But since x_0 is close to B, so

$$\left(x_0 + \frac{1}{n_0+1}\alpha^{k_0-1}\right) + \frac{m}{n_0+1}\alpha^{k_0-1} \in A$$

for all $m \in \mathbf{N}$. For $m = n_0$ we obtain that $x_0 + \alpha^{k_0-1} \in B$, that contradicts the choice of x_0, see 5).

7) We prove that there are no elements from $K(\alpha)$ between the sets $\{x_0 + m\alpha^{k_0}\}_{m\in\mathbf{N}}$ and $\{x_0 + \frac{1}{n}\alpha^{k_0-1}\}_{n\in\mathbf{N}}$. Assume the contrary; let there exist $x \in K(\alpha)$ such that

$$x_0 + m\alpha^{k_0} < x < x_0 + \frac{1}{n}\alpha^{k_0-1}$$

for all $m, n \in \mathbf{N}$. We have $x \in A \sqcup B$. If $x \in A$ then $\forall m \in \mathbf{N}\ m\alpha^{k_0} < x - x_0 < \alpha^{k_0-1}$, then according to 3), $x - x_0 \in \widehat{\alpha^{k_0-1}}$ and there exists $n \in \mathbf{N}$ such that $\frac{1}{n}\alpha^{k_0-1} \leqslant x - x_0$, which implies that $x \in B$. If $x \in B$ then $\forall n \in \mathbf{N}\ \ x - x_0 < \frac{1}{n}\alpha^{k_0-1}$, and $x - x_0 \in \widehat{\alpha^{k_0}}$, there exists $m \in \mathbf{N}$ such that $x - x_0 < m\alpha^{k_0}$ and $x \in A$.

8) From 7) it follows that $\{x_0 + m\alpha^{k_0}\}_{m\in\mathbf{N}}$ is confinal to A, and $\{x_0 + \frac{1}{n}\alpha^{k_0-1}\}_{n\in\mathbf{N}}$ is coinitional to B.

9) It remains to insert the root of the polynomial with coefficients from $K(\alpha)$ between the sets A and B. To do this, take, for example, the element $\alpha^{k_0-1/2}$, it does not belong to $K(\alpha)$ and it is in the extension between the elements of the Archimedean classes $\widehat{\alpha^{k_0}}$ and $\widehat{\alpha^{k_0-1}}$, so $A < x_0 + \alpha^{k_0-1/2} < B$. We select a polynomial with coefficients from $K(\alpha)$ with the root $x_0 + \alpha^{k_0-1/2}$, say, $t = x_0 + \alpha^{k_0-1/2}$. Then an example of the desired sign-changing polynomial in the cut (A, B) is the following: $f(t) = (t - x_0)^2 - \alpha^{2k_0-1}$. $\square$

The proof of the theorem defines an algorithm for searching for a sign-changing polynomial on its asymmetric, non-fundamental cut in $K(\alpha)$ for $\mathbb{Q} \subseteq K \subseteq \mathbb{R}$.

Examples (1)–(8) give a complete classification of cuts in $K(\alpha)$, for $\mathbb{Q} \subseteq K \subsetneq \overline{K} \subsetneq \mathbb{R}$ by the type of fundamentality, symmetry, and algebraicity.

Note that there exist fields of formal power series with non-principal, non-fundamental, asymmetric, non-algebraic cuts [8].

References

[1] Artin E., Schreier O. *Algebraische Konstruktion Reeller Körper.* Abh. Math. Sem. Hamb. Univ. 5 (1925), 85–99.

[2] Baizhanov B.S., Verbovsky V.V. *Orderly stable theories*, Algebra and Logic. 50:3 (2011), 303–325.

[3] Bourbaki N. *Elemente de Mathematique algebre*, chapitre III Hermann Paris, 1948, chapitre VI Hermann Paris, 1952.

[4] Comicheo A.B., Shamseddine K. *Summary on non-Archimedean valued fields*, Contemporary Mathematics. 704 (2018), 1–36. http://dx.doi.org/10.1090/conm/704/14158.

[5] Dales H.J., Woodin H. *Super-real fields.* Oxford: Clarendon Press, 1996.

[6] Galanova N. *Symmetric gaps of fields of bounded formal power series.* (Rus.). J. Algebra and Logic. 42:1 (2003), 26–36.

[7] Galanova N.Yu. *On symmetric cuts of a real-closed field.* Vestnik Tomskogo gosudarstvennogo universiteta. Matematika i mekhanika.[Tomsk State University Journal of Mathematics and Mechanics]. 53 (2018), 5–15. DOI: 10.17223/19988621/53/1.

[8] Galanova N. *On cuts of the quotient field of a ring of formal power series*, Vestnik Tomskogo gosudarstvennogo universiteta. Matematika i mekhanika.[Tomsk State University Journal of Mathematics and Mechanics]. 83 (2023), 5–16. doi:10.17223/19988621/83/1.

[9] Fuchs L. *Partially ordered algebraic systems.* Pergamen Press, 1963.

[10] Knight J. F., Lange K. *Lengths of developments in $K((G))$.* Selecta Mathematica. 25:14 (2019), 2–36. https:// doi.org/10.1007/s00029-019-0448-0.

[11] Kuhlmann F.-V. *Selected methods for the classification of cuts and their applications*, Proceedings of the 5th Joint Conferences on Algebra, Logic and Number Theory, June 24-29, 2018, Bedlewo. Banach Center Publications. 121 (2020), 85–106. DOI: 10.4064/bc121-9.

[12] Pestov G. *To the gap theory of ordered fields* (Rus.). Sib. Math. J. 42:6 (2001), 1350-1360.

[13] Podkorytov M. V. *O nekotoryh metodah issledovaniya linejno uporyadochennyh polej formal'nyh stepennyh ryadov* [On some research methods for totally ordered fields of formal power series]: vypusknaya bakalavrskaya rabota po napravleniyu podgotovki: 01.03.01 — Matematika / Podkorytov, Maksim Vadimovich — Tomsk:[b.i.] (2024) https://vital.lib.tsu.ru/vital/access/manager/Repository/vital:21192.

[14] Shelah S. *Quite Complete Real Closed fields*, Israel Journal of Mathematics. 142 (2004), 261–272.

Галанова Н.Ю. АРХИМЕДТІК ӨРІС ҮСТІНДЕГІ РАЦИОНАЛ ФУНКЦИЯЛАР ӨРІСІНІҢ СЫЗЫҚТЫ РЕТТЕЛУІНДЕГІ КЕСІНДІЛЕР

Бұл мақалада $K \subseteq \mathbb{R}$ және α — оң шексіз кіші элемент болған жағдайда, $K(\alpha)$ өрісіндегі Дедекинд кесінділерінің фундаменталдық, симметриялық және алгебралық түрлері бойынша жіктелуі қарастырылады. $K(\alpha)$ өрісіндегі әрбір принципті емес, әрі фундаменталды емес және симметриялы емес кесінді міндетті түрде алгебралық болатыны дәлелденеді. Мұндай кесінділер үшін осы кесіндіні жүзеге асыратын, таңбасын өзгертетін көпмүше құру әдісі ұсынылады. Сонымен қатар, бұл көпмүшелердің қасиеттері мен олардың базалық өріс K-тің құрылымына тәуелділігі зерттеледі. Алынған нәтижелер нақты сандар өрісінің Архимедке жатпайтын кеңейтулерінің алгебралық және тәртіптік қасиеттерін тереңірек түсінуге мүмкіндік береді, әсіресе модельдік теория мен нақты алгебралық геометрия тұрғысынан. Ұсынылған жіктеу алгебралық кесінділерді конструктивті түрде сипаттауға жағдай жасап, шексіз кіші элементтердің нақты жабық өрістердің топологиялық құрылымымен байланысын айқындайды.

Түйін сөздер: кесінді (бос орын), сызықты реттелген өріс, бөлінді өріс, формалды дәрежелік қатар.

Галанова Н.Ю. СЕЧЕНИЯ В ЛИНЕЙНО УПОРЯДОЧЕННОМ ПОЛЕ РАЦИОНАЛЬНЫХ ФУНКЦИЙ НАД АРХИМЕДОВЫМ ПОЛЕМ

В данной работе исследуется классификация (Дедекиндовых) сечений в поле $K(\alpha)$, где $K \subseteq \mathbb{R}$, а α — положительная бесконечно малая величина. Сечения рассматриваются с точки зрения фундаментальности, симметричности и алгебраичности. Доказано, что каждое собственное сечение в $K(\alpha)$, которое одновременно нефундаментально и несимметрично, является алгебраическим. Для таких сечений построен меняющий знак многочлен, корень которого порождает соответствующее сечение. Полученные результаты способствуют более глубокому пониманию алгебраических и порядковых свойств неархимедовых расширений вещественных полей, особенно в контексте теории моделей и вещественной алгебраической геометрии. Классификация сечений обеспечивает конструктивный подход к распознаванию алгебраических сечений и выявляет взаимосвязь между бесконечно малыми элементами и топологической структурой вещественно замкнутых полей.

Ключевые слова: сечение, линейно упорядоченное поле, поле частных, формально степенной ряд.