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Complexity estimates for theories of some classes of prime models

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Abstract. In this paper, the problems of algorithmic characterization in subclasses of the class of prime models of a finite rich signature are studied. Models with algebraic elements and models with first-order definable elements are considered. Conditions for the existence of strong constructivizations of models of different dimensions in these classes are formulated. Estimates of the algorithmic complexity of elementary theories of some subclasses of the class of prime strongly constructivizable models are given.

Keywords: Algorithmic complexity estimates, semantic class of models, prime model, model with algebraic elements, model with first-order definable elements.

An important stage in the development of modern model theory is the results of the work [21], which presents an existence criterion of a prime model for a countable complete theory. The methods developed in this article were applied to characterize wider classes of atomic, countable saturated, universal, and homogeneous models, as well as to study various dependencies between these classes. In the works [3], [5], some criteria of strong constructivizability of prime models of complete decidable theories are studied. As demonstrations, these works consider some natural subclasses of the class of prime models consisting of models with a computable family of atoms, models with algebraic elements, models with first-order definable elements, models with the finite basis property, and some others. In this paper, we specify conditions for the existence of strong constructivizations of models of different algorithmic dimensions in classes of prime models with algebraic and first-order definable elements, and establish exact algorithmic complexity estimates for elementary theories of some

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subclasses of the class of prime models.

In Section 1, we specify preliminary notions, while Section 2 contains some common facts concerning strong constructivizability in subclasses of the class of prime models. In Section 3, we formulate Theorems 3.1 and Theorem 3.2 that present the main results of the work. Section 3 also exposes initial parts of the proofs of these theorems, while essential parts of the proofs are given in Sections 4 and 5. In Section 6, we give some final notes.

1 Preliminaries

We consider theories in first-order predicate logic *with equality* and use general concepts of model theory, computability theory, Boolean algebras, and constructive models found in [6], [19], [20], [2], [4], and [16]. We consider signatures admitting Gödel's numberings of the formulas. Generally, *incomplete theories* are considered.

A finite signature is called *rich*, if it contains at least one n-ary predicate or function symbol for $n > 1$, or two unary function symbols. A theory F is called *finitely axiomatizable* if it is defined by a finite set of axioms and its signature is finite. The following notations are used: $FL(\sigma)$ is the set of all formulas of the signature σ, $FL_n(\sigma)$ is the set of all formulas of signature σ with n free variables $x_0, x_1, ..., x_{n-1}$, $SL(\sigma)$ is the set of all sentences (i.e., closed formulas) of signature σ. In the work, we use a finite rich signature σ, and consider a fixed Gödel numbering Φ_i, $i \in \mathbb{N}$, of the set $SL(\sigma)$, while $\varphi_i(\bar{x}_i)$, $i \in \mathbb{N}$, is a Gödel numbering of the set $FL(\sigma)$. We use the notation $\mathrm{Card}(A)$ for the cardinality of the set A. If b is an element in a Boolean algebra and $\alpha \in \{0, 1\}$, then b^α means b for $\alpha = 1$ and $-b$ for $\alpha = 0$. Similarly, if Φ is a formula of a theory and $\alpha \in \{0, 1\}$, then Φ^α means Φ for $\alpha = 1$ and $\neg\Phi$ for $\alpha = 0$.

Following Rogers [19, Sec. 5.2] we use the notation W_n for the nth computably enumerable set in Post's numbering of the family of all c.e. sets. Moreover, W_n^t is a finite part of the set W_n that can be computed in t steps (for definiteness, we count that $W_n^0 = \varnothing$). For $A, B \subseteq \mathbb{N}$, we denote by $A \approx B$ the fact that A and B are computably isomorphic, that is, there is a bijection $\pi \colon \mathbb{N} \to \mathbb{N}$ that is a computable function satisfying $\pi(A) = B$. For $A \subseteq \mathbb{N}$, we define $A \approx \Sigma_n^0 \Leftrightarrow_{dfn} A \in \Sigma_n^0 \;\wedge\; (\forall X \in \Sigma_n^0)(X \leqslant_m A)$.

A general terminology of decidable theories is applied and the concept of a *strongly constructivizable* (*s.c.*) model is used together with a characteristic called the *algorithmic dimension* of a s.c. model whose value may be either 1 or ω; for details, cf. [16]. A s.c. model $\mathfrak{N}$ having dimension $\dim_{s.c.}(\mathfrak{N}) = 1$ is called *autostable.*

We denote by $\mathcal{L}(T)$ the *Tarski-Lindenbaum algebra* of theory T (also called the *sentence algebra* of the theory), while $\mathcal{L}_n(T)$, $n \in \mathbb{N}$, means an analogous notion with respect to the set of formulas with free variables $x_0, ..., x_{n-1}$. A formula $\varphi(\bar{x}) \in FL_n(\sigma)$ is called an *atom* of theory T in n free variables if $\varphi(\bar{x})$ represents a class in the quotient algebra $\mathcal{L}_n(T)$ that is an atom in this Boolean algebra. Notice that the same formula $\varphi(\bar{x})$ belongs to $FL_m(\sigma)$ for any $m > n$, but in general, $\varphi(\bar{x})$ will not be an atom in the Boolean algebra

$\mathcal{L}_m(T)$. An expression the *formula* $\varphi(\bar{x})$ *is an atom* in the theory T is allowed, but the index n of the algebra $\mathcal{L}_n(T)$ relative to which the formula $\varphi(\bar{x})$ is checked whether it is an atom must be known by the context. In the case where the set $\mathcal{A} = \{(\varphi(\bar{x}), n) \mid \varphi \in FL_n(\sigma)$ and $\varphi(\bar{x})$ is an atom in $\mathcal{L}_n(T)\}$ is computable, we say that the *set of atoms* of theory T is *computable*, or we use an alternative term the *set* $\mathcal{A}$ *is uniformly computable in the parameter* n. If the set $\mathcal{A}$ is computably enumerable, we say that the *set of atoms* of theory T is *computably enumerable.*

Lemma 1.1. *Let T be a complete decidable theory. The set $\mathcal{A}'$ of formulas of T that are not atoms of T is computably enumerable.*

Proof. Immediately. □

Lemma 1.2. *Let T be a complete decidable theory. The set $\mathcal{A}$ of formulas that are atoms of T is computable if and only if the set $\mathcal{A}$ is computably enumerable.*

Proof. Immediately. □

To solve the problem of characterization for theories of semantic classes of models, we use the universal construction of finitely axiomatizable theories that can control the isomorphism type of the Tarski-Lindenbaum algebra, preserving a large layer MQL of model-theoretic properties called the *quasiexact* (or *infinitary*) layer.

Theorem 1.3. [The Universal construction] *Effectively in a c.e. index of a computably axiomatizable theory T and a Gödel number of a finite rich signature σ, one can construct a finitely axiomatizable theory $F = \mathbb{F}\mathbb{U}(T, \sigma)$ of the signature σ together with a computable isomorphism $\mu : \mathcal{L}(T) \to \mathcal{L}(F)$ between the Tarski-Lindenbaum algebras preserving the quasiexact layer MQL of model-theoretic properties; cf. listing of the layer MQL in either* [*10, Th. 4.1*], *or* [*11, Th. 0.6.1*]. *In particular, MQL contains the following model-theoretic properties*:

- (a) *existence of a prime model*,
- (b) *existence of a model with algebraic elements*,
- (c) *existence of a model with first-order definable elements*,
- (d) *existence of a strongly constructivizable prime model*,
- (e) *existence of a strongly constructivizable prime model $\mathfrak{N}$ with* $\dim_{s.c.}(\mathfrak{N}) = 1$,
- (f) *existence of a strongly constructivizable prime model $\mathfrak{N}$ with* $\dim_{s.c.}(\mathfrak{N}) = \omega$.

Proof. See [10, Th. 4.1], or [11, Th. 0.6.1]. □

Finally, we mention the following reducibilities taking place for an arbitrary class M of models of a given finite rich signature σ, where Φ and Ψ are sentences if signature σ :

$$\begin{aligned} &\texttt{(a)}\ \Psi \in \mathrm{Th}(M) \ \Leftrightarrow\ \neg\Psi \notin \{\Phi \mid \Phi \textit{ has a } M\textit{-model}\}, \\ &\texttt{(b)}\ \Psi \in \{\Phi \mid \Phi \textit{ has a } M\textit{-model}\} \ \Leftrightarrow\ \neg\Psi \notin \mathrm{Th}(M). \end{aligned} \tag{1.1}$$

The following property is useful in our studies.

Theorem 1.4. *Any constructive model of a computably axiomatizable model complete theory is a strongly constructive model.*

Proof. See [2, Prop. 2.2.4]. □

2 Strong constructivizations in the class of prime models

A model $\mathfrak{N}$ is said to be *prime* if it can be elementarily embedded in any other model of theory $T = \mathrm{Th}(\mathfrak{N})$. Vaught's criterion [21] states that a countable complete theory T has a prime model if and only if the family of principal types in this theory is dense in the family of all types (equivalently, if all Tarski-Lindenbaum algebras $\mathcal{L}_n(T)$, $n < \omega$, are atomic), while the model $\mathfrak{N}$ of theory T is prime if and only if any finite sequence of elements $\bar{a} \in |\mathfrak{N}|$ realizes a principal type of this theory.

Theorem 2.1. *Let T be a complete decidable theory with a prime model $\mathfrak{N}$. The model $\mathfrak{N}$ is strongly constructivizable if and only if the set of principal types of the theory T is computable.*

Proof. See [3, Th. 1] or [5, Sec. 1]. □

Theorem 2.2. *Let T be a complete decidable theory with a strongly constructivizable prime model $\mathfrak{N}$. The following claims are equivalent:*

(a) *the set $\mathcal{A}$ of formulas presenting atoms in the Tarski-Lindenbaum algebras $\mathcal{L}_n(T)$, $n \in \mathbb{N}$, is computably enumerable,*

(b) *the set $\mathcal{A}$ of formulas presenting atoms in the Tarski-Lindenbaum algebras $\mathcal{L}_n(T)$, $n \in \mathbb{N}$, is computable uniformly in n,*

(c) *the model $\mathfrak{N}$ is autostable with respect to strong constructivizations; that is, it holds that* $\dim_{s.c.}(\mathfrak{N}) = 1$.

Proof. See Theorem 3.5 in [16]. □

Notice that, the set of Gödel's numbers of all atoms in a complete decidable theory T is computably enumerable if and only if there is a c.e. set of representatives for these atoms in the sentence algebras; i.e., there is a computably enumerable set R, such that, the set of formulas $\big\{\psi(\bar{x}) \mid (\exists i \in R)\big[\,\bar{x} = \bar{x}_i \textit{ and } T \vdash \varphi_i(\bar{x}_i) \leftrightarrow \psi(\bar{x})\big]\big\}$ coincides with the set of all atoms of the theory T.

We consider criteria of strong constructivizability and of having a given algorithmic dimension for the two subclasses of the class of prime models.

Theorem 2.3. *Let T be a complete decidable theory, and let $\mathfrak{N}$ be a model of T with first-order definable elements. Then $\mathfrak{N}$ is a prime model; moreover, $\mathfrak{N}$ is strongly constructivizable with* $\dim_{s.c.}(\mathfrak{N}) = 1$.

Proof. The Vaught's criterion is satisfied, thus, $\mathfrak{N}$ is a prime model. Both Theorem 2.1 and Theorem 2.2(a,c) are obviously applicable; therefore, the model $\mathfrak{N}$ is strongly constructivizable with $\dim_{s.c.}(\mathfrak{N}) = 1$. □

Theorem 2.4. *Let T be a complete decidable theory, and let $\mathfrak{N}$ be a model of T with algebraic elements. Then $\mathfrak{N}$ is a prime model; moreover, the model $\mathfrak{N}$ is strongly constructivizable.*

Proof. The Vaught's criterion [21] is satisfied, therefore, $\mathfrak{N}$ is a prime model. Theorem 2.1 is obviously applicable, thus, the model $\mathfrak{N}$ is strongly constructivizable. □

3 Main statements

We fix a finite rich signature σ and use a fixed Gödel numbering Φ_i, $i \in \mathbb{N}$, for the set of sentences of signature σ, and a Gödel numbering $\varphi_i(\bar{x}_i)$, $i \in \mathbb{N}$, for the set of formulas of signature σ.

We denote by $\mathring{P}^1_{s.c.}{=}\mathring{P}^1_{s.c.}(\sigma)$ the class of models of the signature σ including all prime strongly constructivizable models of dimension 1 with algebraic elements, and by $\dot{P}^1_{s.c.}{=}\dot{P}^1_{s.c.}$ (σ) the class consisting of all prime strongly constructivizable models of dimension 1 of with first-order definable elements.

Theorem 3.1. *The following complexity estimates take place*:

(a) $\{n \mid \Phi_n$ *has a* $\mathring{P}^1_{s.c.}$*-model*$\} \approx \Sigma^0_3$,

(b) $\mathrm{Th}(\mathring{P}^1_{s.c.}) \approx \Pi^0_3$.

Theorem 3.2. *The following complexity estimates take place*:

(a) $\{n \mid \Phi_n$ *has a* $\dot{P}^1_{s.c.}$*-model*$\} \approx \Sigma^0_3$,

(b) $\mathrm{Th}(\dot{P}^1_{s.c}) \approx \Pi^0_3$.

Proof. By virtue of relations (1.1), Part (b) of Theorem 3.1 is a corollary of the estimate in Part (a) of this theorem. The same concerns Theorem 3.2. Thereby, it suffices for us to prove the claims in parts (a) of these two theorems.

First, we find an upper estimate for the set posed in Theorem 3.1(a).

A sentence Ψ of signature σ has a strongly constructivizable prime model of dimension 1 with algebraic elements if and only if there is a complete decidable theory T with $\Psi \in T$, such that the set of atoms of T is computably enumerable; moreover, for any consistent in T formula $\varphi(x)$, there is $k \in \mathbb{N} \setminus \{0\}$ and a formula $\theta(x)$ in $FL(\sigma)$ satisfying $T \vdash (\exists^k x)\,\theta(x)$ and $T \vdash (\forall z)\big(\theta(z) \to \varphi(z)\big)$. Formally, the sentence Ψ has a model in the class $\mathring{P}^1_{s.c.}$ iff there are integers m, n, and q satisfying the following properties (where $\exists^{<\omega}\bar{x}$ means $\exists^t\bar{x}$ for some t with $0 < t < \omega$):

1. $W_m \cap W_n = \varnothing \wedge W_m \cup W_n = \mathbb{N}$, $\quad\forall \wedge \forall\exists$
2. $T = \{\Phi_i \mid i \in W_m\}$ is a complete theory, $\quad\forall\exists$
3. $\Psi \in T$, $\quad\exists$
4. $\forall\varphi(\bar{x})\,\exists\theta(\bar{x})\,\big[\,T \ \vdash \neg(\exists\bar{x})\,\varphi(\bar{x})$, or $T \ \vdash (\exists^{<\omega}\bar{x})\,\theta(\bar{x}) \ \wedge \ (\forall\bar{z})\,(\theta(\bar{z}) \to \varphi(\bar{z}))\,\big]$, $\quad\forall\exists$
5. $(\forall\, i \in W_q)\,\big(\varphi_i(\bar{x}_i)$ is atom$\big) \wedge (\forall j \notin W_q)\big(\varphi_j(\bar{x}_j)$ is not an atom$\big)$. $\quad\forall \wedge \forall\exists$

Thus, a prefix of the form $\exists\forall\exists$ is obtained that is exactly what is required in Theorem 3.1(a).

By changing the above quantifier $(\exists^{<\omega}...)$ to $(\exists^{1}...)$, we obtain the same upper estimate $\exists\forall\exists$ also for the set mentioned in Part (a) of Theorem 3.2.

To finish proofs of both Theorem 3.1 and Theorem 3.2, it remains to establish exact lower bounds for complexity of the sets posed in Theorem 3.1(a) and Theorem 3.2(a).

4 A lower bound for the set in Theorem 3.1(a)

Now, we turn to the lower bound of the set in Part (a) of Theorem 3.1.

We use the following set

$$E_3 = \{n \mid W_n \textit{ is computable}\}, \tag{4.1}$$

which is Σ^0_3-complete, cf. [19, Sec. 14.8, Th. XVI].

First, we describe a computably axiomatizable theory H of an enumerable signature

$$\eta = \{M^1, \lessdot^2, M^1_0, M^1_1, U^1, \lhd^2, E^2, C^1, G^1_n, D^1_{m,n,t} \ \mid \ m, n, t \in \mathbb{N}\}, \tag{4.2}$$

called the *basic theory* for our construction.

The following set of sentences is a system of axioms for H.

1°. $M(x)$ and $U(x)$ form a partition of the universe into two nonempty domains.

2°. $\lessdot$ is a successor relation without cycles in the domain $M(x)$; $\lessdot$ is trivial outside $M(x)$; $\lessdot$ has an initial element that is distinguished by $M_0(x)$; $\lessdot$ has an ending element that is distinguished by $M_1(x)$.

3°. $E(x,y)$ is an equivalent relation in the domain $U(x)$; E is defined trivially outside $U(x)$; each E-class $[a]_E$, $a \in U$, consist of two elements.

4°. $\lhd$ is a successor relation on the E-classes without cycles, $C(x)$ distinguishes exactly one E-class, E-class $C(x)$ does not have a $\lhd$-predecessor, each E-class excepting $C(x)$ has a $\lhd$-predecessor, each E-class has a $\lhd$-successor.

5°_n. $G_n(x) \Leftrightarrow (\exists z_0, z_1, ..., z_n)\,\big[\,C(z_0) \wedge z_0 \lhd z_1 \lhd ... \lhd z_n \wedge E(z_n, x)\,\big]$, $n \in \mathbb{N}$.

$6^\circ_{m,n,t}$. If $\mathrm{Card}(M) = m{+}1$ and $n \in W^{t+1}_m \smallsetminus W^t_m$, then predicate $D_{m,n,t}(x)$ distinguishes a unique element z, moreover, $G_n(z)$ is satisfied; otherwise, if $\mathrm{Card}(M) \neq m{+}1$ or $\big(\mathrm{Card}(M) = m{+}1$ but $n \notin W^{t+1}_m \smallsetminus W^t_m\,\big)$, the predicate $D_{m,n,t}(x)$ is defined trivially anywhere, $m, n, t \in \mathbb{N}$.

System of axioms of H is complete.

We start to describe models of the theory H.

The domain $X = M(\mathfrak{M})$ of an arbitrary model $\mathfrak{M}$ of theory H may be either finite or infinite. In the former case, X represents a finite $\vartriangleleft$-succession starting from an element distinguished with $M_0(x)$, while its ending element is distinguished with $M_1(x)$. In the latter case, X represents an infinite $\vartriangleleft$-succession consisting of a chain of type ω starting from an element distinguished with $M_0(x)$, and another chain of type ω^* whose ending element is distinguished with $M_1(x)$; an arbitrary extra collection of infinite chains of type $\omega^* + \omega$ may also exist in the domain $M(\mathfrak{M})$.

Now we turn to describe possibilities for E-classes in the domain $U(x)$.

First, we consider two simple cases with so-called non-standard fragments. If the domain $M(\mathfrak{M})$ in a model $\mathfrak{M}$ is infinite, according to axioms, all predicates $D_{k,j,i}(x)$ in such a model are defined trivially in each class $[a]_E$, $a \in U(\mathfrak{M})$. Furthermore, if the class $[a]_E$, $a \in U(\mathfrak{M})$, in a model $\mathfrak{M}$ of H is different from all special E-classes $G_n(x)$, $n \in \mathbb{N}$, by axiom 6°, all predicates $D_{k,j,i}(x)$ are defined trivially in this class. In both cases, it is possible to find an automorphism of the model with a permutation within the pair of elements forming the E-class.

Now, we concern the "standard" part in a model. Consider an E-class of the form $G_n(x)$, $n \in \mathbb{N}$, in a model $\mathfrak{M}$ with $\mathrm{Card}\big(M(\mathfrak{M})\big) = m + 1$ for some $m \in \mathbb{N}$. If $n \in W_m$, then there is a unique value t satisfying $n \in W_m^{t+1} \smallsetminus W_m^t$. For this t, Axiom 6° forces the predicate $D_{m,n,t}(x)$ to distinguish one of the two elements in the E-class $G_n(\mathfrak{M})$, thus making both elements in this E-class to be first-order definable in $\mathfrak{M}$. As for the other predicates $D_{k,j,i}$, they are defined trivially in the class $G_n(\mathfrak{M})$. In the other case, when $\mathrm{Card}\big(M(\mathfrak{M})\big) = m+1$ but $n \notin W_m^{t+1} \smallsetminus W_m^t$, for all $t \in \mathbb{N}$, by Axiom 6°, any predicate $D_{m,n,t}$ is defined trivially in the E-class $G_n(\mathfrak{M})$, thereby admitting automorphisms of the model $\mathfrak{M}$ with a permutation of the pair of elements forming the class $G_n(\mathfrak{M})$. In particular, elements in this class are not first-order definable in $\mathfrak{M}$.

It follows from the description above that, for an arbitrary model $\mathfrak{M}$ of theory H, the following property (presenting the key idea of the construction) is satisfied:

$$\begin{gathered} G_n(x) \textit{ is an atom in the theory } \mathrm{Th}(\mathfrak{M}) \ \Leftrightarrow \\ \mathrm{Card}\big(M(\mathfrak{M})\big) \geqslant \omega \ \vee \ (\exists\, m \in \mathbb{N})\,\big(\,\mathrm{Card}\big(M(\mathfrak{M})\big) = m+1 \ \wedge \ n \notin W_m\,\big). \end{gathered} \tag{4.3}$$

Having described models, we turn to studying properties of the theory H.

Lemma 4.1. *Theory H is model complete.*

Proof. Apply Robinson's criterion [18, Th. 4.2.1] stating that, a theory is model complete iff for any pair of its models $\mathfrak{N}$ and $\mathfrak{M}$ with $\mathfrak{N} \subseteq \mathfrak{M}$, any primitive formula $(\exists \bar{z})\varphi(\bar{z}, \bar{c})$ with constants $\bar{c} \in \mathfrak{N}$ that is true in $\mathfrak{M}$ is also true in the submodel $\mathfrak{N}$. For this, it is possible to use Ehrenfeucht-Fraissé games, [1]. □

A model $\mathfrak{N}$ of theory H is said to be *primitive*, if its universum $|\mathfrak{N}|$ consists of the domain $M(\mathfrak{N})$ with either a finite number of elements, or a countable set of elements forming a pair of $\lhd$-chains of types ω and ω^* (as described above); moreover, the domain $U(\mathfrak{N})$ is limited with just E-classes $G_n(x)$, $n \in \mathbb{N}$. It follows from axioms and the definition that any primitive model is countable.

Let H_m, $m \in \mathbb{N}$, be an extension $H + \{\mathrm{Card}(M) = m{+}1\}$ of the theory H.

Lemma 4.2. *The following assertions hold.*

(a) *For any model $\mathfrak{M} \in \mathrm{Mod}(H)$, there is a submodel $\mathfrak{N} \subseteq \mathfrak{M}$ that is a primitive model. In particular, for any $m \in \mathbb{N}$, there is a primitive model $\mathfrak{N} \in \mathrm{Mod}(H_m)$.*

(b) *A primitive model $\mathfrak{N}$ of theory H_m, $m \in \mathbb{N}$, is embeddable in all other models of the theory H_m.*

Proof. Directly, based on the description of models of the theories H and H_m. □

Lemma 4.3. *The operator $m \mapsto H_m$, $m \in \mathbb{N}$, sends integers to computably axiomatizable theories; moreover, this operator is computable and satisfies the following properties*:

(a) *in the case when the set W_m is computable, H_m is a complete decidable theory having a strongly constructivizable (prime) model $\mathfrak{N}$ with algebraic elements and with dimension* $\dim_{s.c.}(\mathfrak{N}) = 1$,

(b) *in the case when the set W_m is not computable, H_m is a complete decidable theory having a strongly constructivizable (prime) model $\mathfrak{N}$ with algebraic elements and with dimension* $\dim_{s.c.}(\mathfrak{N}) = \omega$.

Proof. Consider an arbitrary integer $m \in \mathbb{N}$.

By construction, the operator $m \mapsto H_m$ is computable. In particular, each theory H_m, $m \in \mathbb{N}$, is computably axiomatizable. By Lemma 4.1, the theory H is model complete, thus the theory H_m extending H is also model complete. By Lemma 4.2, the theory H_m has a primitive model $\mathfrak{N}$ which is embeddable into any other model of this theory. By Robinson's theorem [18, Th. 4.2.3], the theory H_m is complete. The primitive model $\mathfrak{N}$ is actually a prime model since all embeddings of models in a model complete theory are elementary. By Janiczak's theorem [7], the theory H_m is decidable since it is complete and computably axiomatizable. From the description of models we have that $\mathfrak{N}$ is a model with algebraic elements. By Theorem 2.4, the prime model $\mathfrak{N}$ of theory H_m is strongly constructivizable.

Based on the description of models, it is possible to characterize atoms of the theory H_m. Namely, any formula $\varphi(\bar{x})$ that is an atom of the theory H_m must be presented via a Boolean expression constructed from formulas of the form $G_n(x_i)$, $D_{m,n,t}(x_i)$, $G_n(x_i) \wedge \neg D_{m,n,t}(x_i)$, $x_i = x_j$, and $x_i \neq x_j$ for different variables x_i, x_j occurred in the tuple $\bar{x}$. In the case when the set W_m is computable, this together with (4.3) allows us to point out a computable enumerable sequence of formulas $\theta_i(\bar{x}_i)$, $i \in \mathbb{N}$, presenting all atoms of the Tarski-Lindenbaum algebras of H_m. By Theorem 2.2(a,c), the prime model $\mathfrak{N}$ of theory H_m has algorithmic dimension 1.

In the other case when the set W_m is not computable, the relation (4.3) ensures that the set of atoms of theory H_m is not computably enumerable. Applying Theorem 2.2(a,c) again, we obtain that the prime model $\mathfrak{N}$ of theory H_m has an infinite algorithmic dimension. □

Now, we apply the universal construction $\mathbb{F}\mathbb{U}$, cf. Theorem 1.3, in order to transform the computably axiomatizable theory H_m into a finitely axiomatizable theory $F_m = \mathbb{F}\mathbb{U}(H_m, \sigma)$ of the pre-assigned signature σ. By Theorem 1.3, there is a computable isomorphism $\mu : \mathcal{L}(H_m) \to \mathcal{L}(F_m)$ preserving all properties involved in the definition of the class $\mathring{P}^1_{s.c.}$. Thereby, by Lemma 4.3, the following relation takes place for all $m \in \mathbb{N}$:

$$\begin{aligned} m \in E_3 &\Leftrightarrow \ \textit{theory } H_m \textit{ has a model in the class } \ \mathring{P}^1_{s.c.}(\eta) \\ &\Leftrightarrow \ \textit{theory } F_m \textit{ has a model in the class } \ \mathring{P}^1_{s.c.}(\sigma). \end{aligned} \tag{4.4}$$

By construction, the transformation $m \mapsto H_m \mapsto F_m$ is effective. Thus, the relation (4.4) establishes the required lower estimate for the set posed in Part (a) of Theorem 3.1.

Theorem 3.1 is proved. □

5 A lower bound for the set in Theorem 3.2(a)

We use the set E_3 defined in (4.1) that is mentioned to be Σ^0_3-complete.

We describe a computably axiomatizable theory H' of an enumerable signature

$$\eta' = \{M^1, \lessdot^2, M^1_0, M^1_1, U^1, \lhd^2, E^2, C^1, V^1, I^0_k, J^0_k, G^1_n, D^1_{m,n,t} \mid m, n, t, k \in \mathbb{N}\}, \tag{5.1}$$

called the *basic theory* for our construction.

The following set of sentences is a system of axioms for H'.

1°. $M(x)$ and $U(x)$ form a partition of the universe into two nonempty domains.

2°. $\lessdot$ is a successor relation without cycles in the domain $M(x)$; $\lessdot$ is trivial outside $M(x)$; $\lessdot$ has an initial element that is distinguished by $M_0(x)$; $\lessdot$ has an ending element that is distinguished by $M_1(x)$.

3°. $E(x, y)$ is an equivalence relation in the domain $U(x)$; E is defined trivially outside $U(x)$; each E-class $[a]_E$, $a \in U$, consist of *at most* two elements; *moreover*,

$$(\forall x)\left[\, V(x) \leftrightarrow U(x) \ \wedge \ (\exists z)(E(x, z) \wedge x \neq z)\,\right],$$

$$I_k \leftrightarrow (\exists^{\geqslant k} z)\left[\, U(z) \wedge \neg V(z)\,\right], \text{ and } J_k \leftrightarrow (\exists^{\geqslant 2k} z)\left[\, U(z) \wedge V(z)\,\right]$$

(notice that J_k states that $\exists^{\geqslant k}$ two-element E-classes), $k \in \mathbb{N}$.

4°. $\lhd$ is a successor relation on the E-classes without cycles, $C(x)$ distinguishes exactly one E-class, E-class $C(x)$ does not have a $\lhd$-predecessor, each E-class excepting $C(x)$ has a $\lhd$-predecessor, each E-class has a $\lhd$-successor.

5°_n. $G_n(x) \;\Leftrightarrow\; (\exists z_0, z_1, ..., z_n)\left[\,C(z_0) \wedge z_0 \lhd z_1 \lhd ... \lhd z_n \wedge E(z_n, x)\,\right]$, $n \in \mathbb{N}$.

$6^\circ_{m,n,t}$. If $\mathrm{Card}(M) = m{+}1$ and $n \in W_m^{t+1} \smallsetminus W_m^t$, then predicate $D_{m,n,t}(x)$ distinguishes a unique element z, moreover, $G_n(z) \wedge V(z)$ is satisfied (*i.e., E-class $G_n(z)$ must consist of two elements in this situation*); otherwise, if $\mathrm{Card}(M) \neq m+1$ or $\big(\mathrm{Card}(M) = m{+}1$ but $n \notin W_m^{t+1} \smallsetminus W_m^t\big)$, the predicate $D_{m,n,t}(x)$ is defined trivially anywhere, $m, n, t \in \mathbb{N}$.

System of axioms of H' is complete.

System of axioms for the theory H' is obtained by a modification of axioms of the theory H with using an additional unary predicate $V(x)$ and two series I_k, J_k, $k \in \mathbb{N}$, of nullary predicates, i.e., propositional variables (all changes are highlighted within the axioms of theory H'). An idea of the new theory H' is that E-classes can now be either two-element or one-element. Predicate $V(x)$ indicates two-element E-classes, predicates I_k, $k \in \mathbb{N}$, fix the number of one-element E-classes, while predicates J_k, $k \in \mathbb{N}$, fix the number of two-element E-classes. It is required that, any E-class where axiom 6° initiates a predicate $D_{m,n,t}(x)$ must contain two elements. It is possible to extend the description of models of theory H given in Section 4 to a description of models of the theory H'. Notice that, in some sense, the class of models of theory H' is an extension of the class $\mathrm{Mod}(H)$. Namely, the previous theory H coincides with an extension $H^* = H' + (\forall z)[U(z) \to V(z)]$ of the theory H' modulo ignoring additional first-order definable predicates I_k, J_k, $k \in \mathbb{N}$, as well as the unary predicate $V(x)$ satisfying $V(z) \leftrightarrow U(z)$ in the theory H^*).

Similarly to the properties of the theory H presented in Section 4, we can describe possible actions of automorphisms in a model $\mathfrak{M}$ of the theory H' within its E-classes. Based on this, we obtain that

$$\begin{gathered}\textit{for any model } \mathfrak{M} \in \mathrm{Mod}(H'),\ G_n(x) \textit{ is an atom in } \mathrm{Th}(\mathfrak{M}) \;\Leftrightarrow \\ \mathrm{Card}\big(M(\mathfrak{M})\big) \geqslant \omega \;\vee\; (\exists\, m \in \mathbb{N})\,\big(\,\mathrm{Card}\big(M(\mathfrak{M})\big) = m{+}1 \;\wedge\; n \notin W_m\,\big).\end{gathered} \tag{5.2}$$

Moreover, the following properties take place for the model $\mathfrak{M}$:

$$\mathrm{Card}(G_n(\mathfrak{M})) = \begin{cases} 1, & \textit{if } G_n(\mathfrak{M}) \nsubseteq V(\mathfrak{M}), \\ 2, & \textit{if } G_n(\mathfrak{M}) \subseteq V(\mathfrak{M}), \end{cases} \tag{5.3}$$
$$\textit{whenever } G_n(x) \textit{ is an atom in the theory } \mathrm{Th}(\mathfrak{M}).$$

Given an integer $m \in \mathbb{N}$. We denote by H'_m an extension $H' + \{\mathrm{Card}(M) = m{+}1\}$ of the theory H'. Furthermore, we introduce notations

$$\begin{aligned} &\texttt{(a)}\ K_m = \mathbb{N} \smallsetminus W_m, \\ &\texttt{(b)}\ \mathfrak{A}_m = 2^{K_m} = \big\{\alpha \mid \alpha \textit{ is a mapping from } K_m \textit{ to } \{0,1\}\big\}, \\ &\texttt{(c)}\ H'_m[\alpha] = H'_m + \{(\forall z)\,(\,G_n(z) \to (V(z))^{\alpha(n)}\,) \mid n \in K_m\,\},\ \alpha \in \mathfrak{A}_m. \end{aligned} \tag{5.4}$$

Notice that in the case $W_m = \mathbb{N}$, we have $\mathfrak{A}_m = \varnothing$. In this particular case, the record $H'_m[\alpha]$ (with dummy α) may be used counting that $H'_m[\alpha] = H'_m$.

It follows from the description of models that, for an arbitrary mapping $\alpha \in \mathfrak{A}_m$, theory $H'_m[\alpha]$ is a consistent extension of the theory H'_m. Theories in (5.4)(c) differ from each other for different α. Therefore, in the case $W_m \neq \mathbb{N}$, the family (5.4)(c) contains more than one theory; thus, H'_m in this case is an incomplete theory. For the models at a whole, we have for an arbitrary $m \in \mathbb{N}$:

$$\mathrm{Mod}(H'_m) \;=\; \bigcup \left\{ \,\mathrm{Mod}\big(H'_m[\alpha]\big) \;\mid\; \alpha \in \mathfrak{A}_m \,\right\}. \tag{5.5}$$

Moreover, all theories in the right-hand side of equality (5.5) are incompatible with each other.

Lemma 5.1. *Theory H' is model complete.*

Proof. Axioms of theory H' restrict possible cases of embeddings $\mathfrak{N} \subseteq \mathfrak{M}$ for arbitrary models $\mathfrak{N}$ and $\mathfrak{M}$ of this theory. Unary predicate $V(x)$ prevents any one-element E-class of the model $\mathfrak{N}$ to become a two-element E-class in the model $\mathfrak{M}$. Nullary predicates I_k, $k \in \mathbb{N}$, prevent appearance of new one-element E-classes in (non-standard fragments) of the model $\mathfrak{M}$ in the case when the number of such classes in the submodel $\mathfrak{N}$ is finite, while the nullary predicates J_k, $k \in \mathbb{N}$, similarly fix the number of two-element E-classes. Based on these properties with using the description of models of H', it is possible to establish model completeness of the theory H' by Robinson's criterion [18, Th. 4.2.1] using Ehrenfeucht-Fraissé games [1]. □

A model $\mathfrak{N}$ of theory H' is said to be *primitive*, if its universum $|\mathfrak{N}|$ consists of the domain $M(\mathfrak{N})$ with either a finite number of elements, or a countable set of elements forming a pair of $\lhd$-chains of types ω and ω^* (as described in Section 4); moreover, the domain $U(\mathfrak{N})$ is limited with just E-classes $G_n(x)$, $n \in \mathbb{N}$. It follows from axioms and the definition that any primitive model is countable.

Lemma 5.2. *The following assertions hold.*

(a) *For an arbitrary primitive model $\mathfrak{N}$ of the theory H' with $\mathrm{Card}(M(\mathfrak{N})) = m + 1$, $m \in \mathbb{N}$, there is a mapping $\alpha \in \mathfrak{A}_m$ such that $\mathfrak{N} \in \mathrm{Mod}(\,H'_m[\alpha]\,)$.*

(b) *For any $m \in \mathbb{N}$ and an arbitrary mapping $\alpha \in \mathfrak{A}_m$, there is a primitive model $\mathfrak{N}$ of the theory H' such that $\mathfrak{N} \in \mathrm{Mod}(H'_m[\alpha])$; moreover, such a model is defined uniquely, up to an isomorphism.*

(c) *For any $m \in \mathbb{N}$ and an arbitrary mapping $\alpha \in \mathfrak{A}_m$, a primitive model $\mathfrak{N}$ of theory $H'_m[\alpha]$ is embeddable in any other model $\mathfrak{M}$ of the theory $H'_m[\alpha]$.*

Proof. (a) Given a primitive model $\mathfrak{N}$ of the theory H' with $\mathrm{Card}(M(\mathfrak{N})) = m + 1$. Let us define a mapping α from K_m to $\{0, 1\}$ as follows. For $n \in K_m$, we put $\alpha(n) = 1$ whenever $\mathfrak{N} \models (\exists z) \big[\, G_n(z) \wedge V(z) \,\big]$, and put $\alpha(n) = 0$ otherwise. Based on description of models of the theory H', it is possible to check that $\mathfrak{N}$ is a model of theory $\mathrm{Mod}(H'_m[\alpha])$.

(b), (c) From description of models of theory H'_m and definition of a primitive model. $\square$

Lemma 5.3. *Let $\mathfrak{N}$ be a primitive model of the theory $H'_m[\alpha]$, $m \in \mathbb{N}$, $\alpha \in \mathfrak{A}_m$. The following assertions hold*:

(a) *$\mathfrak{N}$ is a model with algebraic elements,*

(b) *$\mathfrak{N}$ is a model with first-order definable elements if and only if either $W_m = \mathbb{N}$ or* $(\forall x \in K_m)\,(\,\alpha(x) = 0\,)$,

(c) *if the primitive model $\mathfrak{N}$ is a model with first-order definable elements, the following relation takes place*:

$$n \notin W_m \;\Leftrightarrow\; \mathrm{Card}(G_n(\mathfrak{N})) = 1. \tag{5.6}$$

Proof. Based on description of models of the theory H'_m and the definition of a primitive model together with claims (5.2) and (5.3). $\square$

Lemma 5.4. *The operator $m \mapsto H'_m$, $m \in \mathbb{N}$, sends integers to computably axiomatizable theories; moreover, this operator is computable and satisfies the following properties*:

(a) *in the case when W_m is computable, the theory H'_m has a prime model $\mathfrak{N}$ with first-order definable elements; moreover, the model $\mathfrak{N}$ is strongly constructivizable with dimension* $\dim_{s.c.}(\mathfrak{N}) = 1$,

(b) *in the case when W_m is not computable, the theory H'_m does not have a strongly constructivizable model with first-order definable elements.*

Proof. By construction, the passage $m \mapsto W_m \mapsto H'_m$ is computable.

Consider a function $\alpha : K_m \to \{0, 1\}$ that is identically equal to zero (the argumentation below also applies to the case $W_m = \mathbb{N}$). By Lemma 5.1, theory H' is model complete. Therefore, its extension $H'_m[a]$ is also a model complete theory; moreover, this theory has a primitive model by Lemma 5.2. By Robinson's theorem [18, Th. 4.2.3], the theory $H'_m[\alpha]$ is complete because it has a model $\mathfrak{N}$ embeddable in the other models of the theory. Actually, the primitive model $\mathfrak{N}$ of theory $H'_m[\alpha]$ is prime because all embeddings of models in a model complete theory are elementary. By Lemma 5.3(a), $\mathfrak{N}$ is a model with algebraic elements. Moreover, by choice of the mapping α together with Lemma 5.3(b) we obtain that $\mathfrak{N}$ is a model with first-order definable elements. Since $H'_m[\alpha]$ is an extension of H'_m, we obtain $\mathfrak{N} \in \mathrm{Mod}(H'_m)$. In view of Theorem 5.2 and Theorem 5.3(b) together with (5.5), the theory H'_m cannot have other models with first-order definable elements. Thus, we obtain that the theory H'_m has a unique (up to an isomorphism) model $\mathfrak{N}$ with first-order definable elements.

Suppose that the set W_m is computable. In this case, an extension $H'_m[\alpha]$ of the theory H'_m with the earlier chosen mapping α has a computable system of axioms (5.4)(c). Therefore, the theory $H'_m[\alpha]$ is decidable because it is complete. Applying Theorem 2.3, we obtain that the model $\mathfrak{N}$ is strongly constructivizable and $\dim_{s.c.}(\mathfrak{N}) = 1$.

Now, we consider the case when the set W_m is not computable. By virtue of relation

(5.6), the theory $\mathrm{Th}(\mathfrak{N})$ is undecidable. In particular, the model $\mathfrak{N}$ cannot be strongly constructivizable. $\square$

Apply the universal construction in order to transform the computably axiomatizable theory H'_m into a finitely axiomatizable theory $F'_m = \mathbb{F}\mathbb{U}(H'_m, \sigma)$ of signature σ. By Theorem 1.3, there is a computable isomorphism $\mu : \mathcal{L}(H'_m) \to \mathcal{L}(F'_m)$ between the sentence algebras of these theories preserving properties involved in the definition of the class $\dot{P}^1_{s.c.}$. Thereby, by Lemma 5.4, the following relation takes place for all $m \in \mathbb{N}$:

$$\begin{aligned} m \in E_3 \Leftrightarrow\ & \textit{theory } H'_m \textit{ has a model in the class } \dot{P}^1_{s.c.}(\eta') \\ \Leftrightarrow\ & \textit{theory } F'_m \textit{ has a model in the class } \dot{P}^1_{s.c.}(\sigma). \end{aligned} \tag{5.7}$$

By construction, the transformation $m \mapsto H'_m \mapsto F'_m$ is effective. Thus, the relation (5.7) establishes the required lower estimate for the set posed in Part (a) of Theorem 3.2.

Theorem 3.2 is proved. $\square$

We are going to expand the result of Theorem 3.2 to the class $\dot{P}_{s.c.}{=}\dot{P}_{s.c.}(\sigma)$ consisting of all strongly constructivizable prime models of signature σ with first-order definable elements.

Lemma 5.5. $\dot{P}^1_{s.c.}\ (\sigma)\ = \dot{P}_{s.c.}(\sigma)$.

Proof. An inclusion $\subseteq$ is obvious, while an inverse inclusion $\supseteq$ between the classes is provided by Theorem 2.3. $\square$

Theorem 5.6. *For an arbitrary finite rich signature σ, the following complexity estimates take place for the class of models $\dot{P}_{s.c} = \dot{P}_{s.c}\ (\sigma)$:*

(a) $\{\, n \mid \Phi_n$ *has a* $\dot{P}_{s.c.}$*-model*$\} \approx \Sigma^0_3$,

(b) $\mathrm{Th}(\dot{P}_{s.c.}) \approx \Pi^0_3$.

Proof. By Theorem 3.2 together with Lemma 5.5. $\square$

6 Conclusion

In [8], [9], [11], and other works, a number of results on the complexity estimates for elementary theories of semantic classes of models were obtained. Later, the focus of research shifted to the more complex problem of characterizing the types of computable isomorphism of algebras of sentences of semantic classes of models; for example, see [12], [13], [14], [15], [16], [17]. At the same time, a number of simpler questions on complexity estimates of theories of semantic classes of models remain unexplored. The paper gives answers to several questions of this type. The results obtained allow us to better understand the difficulties that arise in more complex studies on characterization of sentence algebras of various semantic classes of models.

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Перетятькин М.Г. ЖАЙ МОДЕЛЬДЕРДІҢ КЕЙБІР КЛАССТАРЫНЫҢ ТЕОРИЯЛАРЫ ҮШІН КҮРДЕЛІЛІКТІ БАҒАЛАУ

Ақырғы бай таңбаның жай модельдер класының ішкі сыныптарында алгоритмдік сипаттау мәселелері зерттеледі. Алгебралық элементтері бар модельдер және бірінші ретті анықталатын элементтері бар модельдер қарастырылады. Осы кластардағы әртүрлі өлшемді модельдердің күшті конструктивтілігінің болуының шарттары тұжырымдалған. Күшті конструктивті модельдер класының кейбір ішкі сыныптарының элементар теорияларының алгоритмдік күрделілігінің бағалары келтірілген.

Түйін сөздер: Алгоритмдік күрделілікті бағалау, модельдердің семантикалық класы, жай модель, алгебралық элементтері бар модель, бірінші ретті анықталатын элементтері бар модель, ақырлы модель.

Перетятькин М.Г. ОЦЕНКИ СЛОЖНОСТИ ТЕОРИЙ НЕКОТОРЫХ КЛАССОВ ПРОСТЫХ МОДЕЛЕЙ.

Исследуются вопросы алгоритмической характеризации в подклассах класса простых моделей конечной богатой сигнатуры. Рассматриваются модели с алгебраическими элементами и модели с формульно определимыми элементами. Формулируются условия существования сильных конструктивизаций моделей разных алгоритмических размерностей в этих классах. Приводятся оценки алгоритмической сложности элементарных теорий некоторых подклассов класса простых сильно конструктивизируемых моделей.

Ключевые слова: Оценки алгоритмической сложности, семантический класс моделей, простая модель, модель с алгебраическими элементами, модель с формульно определимыми элементами.