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Non-trivial expansions of 1-transitive ordered theories

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Abstract. Here we study non-trivial expansions of 1-transitive ordered theories preserving 1-transitivity. In particular, expansions of weakly o-minimal linear orderings by equivalence relations, unary functions and arbitrary binary relations were investigated. On the base of the obtained results values of ranks for various families of weakly o-minimal theories were found.

Keywords. Weak o-minimality, 1-transitivity, convexity rank, expansion of a theory, ordered theory, rank for family of theories.

1 Preliminaries

Let L be a first-order countable language. Throughout this paper we consider L-structures and assume that L contains a binary relation symbol $<$, which is interpreted as the linear ordering in these structures.

We will say that a linearly ordered structure $M := \langle M, <, \ldots \rangle$ is 1-*is transitive* if $tp(a/\emptyset) = tp(b/\emptyset)$ for any $a, b \in M$.

Example 1. Let $M := \langle \omega, < \rangle$, where ω is the ordering on the set of natural numbers. Clearly, M is not 1-transitive, since for any $a, b \in M$ with condition $a \neq b$ we have that $tp(a/\emptyset) \neq tp(b/\emptyset)$.

Example 2. Let $M := \langle \mathbb{Z}, < \rangle$, where $\mathbb{Z}$ is the set of integers. Then it's obvious that M is 1-transitive.

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Fact 1. *Let M be a 1-transitive linearly ordered structure. Then*
(1) M has neither the smallest nor the largest elements.
(2) Any $\emptyset$-definable subset of the structure M coincides with the universe of the structure.

An *open interval* I in a structure M is a parametrically definable subset of the structure M of the form

$$I = \{c \in M : M \models a < c < b\}$$

for some $a, b \in M \cup \{-\infty, \infty\}$ where $a < b$. Similarly, we can we can define *closed, semi-open-semi-closed*, etc. intervals in M so that, for example, an arbitrary point of the structure M is itself a (trivial) closed interval.

A subset A of a linearly ordered structure M is called *convex* if for any $a, b \in A$ and $c \in M$ whenever $a < c < b$ we have $c \in A$. A *weakly o-minimal structure* ([1]) is a linearly ordered structure $M = \langle M, =, <, \ldots \rangle$ such that any definable (with parameters) subset of the structure M is a union of finitely many convex sets in M. Recall that such a structure M is called *o-minimal* if every definable (with parameters) subset of the structure M is a union of of finitely many intervals and points in M. Real closed fields with a proper convex valuation ring provide an important example of weakly o-minimal (not o-minimal) structures.

Proposition 1. *Let M be an 1-transitive o-minimal structure. Then either $\langle M, < \rangle$ is a dense linear ordering without endpoints, or $\langle M, < \rangle$ is a discrete linear ordering without endpoints.*

Proof of Proposition 1. Suppose the contrary: $\langle M, < \rangle$ is neither a dense linear ordering without endpoints nor a discrete linear ordering without endpoints. If $\langle M, < \rangle$ has any endpoint, it will contradict Fact 1. Therefore, assume that $\langle M, < \rangle$ has no endpoints. Consider the following formula:

$$\theta(x) := \forall y_1 \forall y_2 [y_1 < x < y_2 \to \exists t_1 \exists t_2 (y_1 < t_1 < x < t_2 < y_2)].$$

By virtue of the assumption $\theta(M) \neq M$ and $\theta(M) \neq \emptyset$. Whence, by virtue of Fact 1 we again come to a contradiction.

Corollary 1. *Let M be an 1-transitive (weakly) o-minimal linear ordering. Then M is elementarily equivalent to $\langle \mathbb{Q}, < \rangle$ or M is elementarily equivalent to $\langle \omega^* + \omega, < \rangle$.*

Let $\mathcal{T}$ be a family of complete theories of a fixed signature Σ, and let ϕ be an arbitrary Σ-proposition. Then the set

$$\mathcal{T}_\phi := \{T \in \mathcal{T} \mid T \models \phi\}$$

is called a *ϕ-neighborhood* of the family $\mathcal{T}$.

Definition 1. [2] Let $\mathcal{T}$ be a family of complete theories of a fixed signature Σ. We define the *rank* RS for the family of theories as follows:
(1) $\mathrm{RS}(\mathcal{T}) = -1$ if $\mathcal{T} = \emptyset$.

(2) $\mathrm{RS}(\mathcal{T}) = 0$ if $\mathcal{T}$ is a finite nonempty family.

(3) $\mathrm{RS}(\mathcal{T}) \geq 1$ if $\mathcal{T}$ is infinite.

(4) $\mathrm{RS}(\mathcal{T}) \geq \alpha+1$ if there exist pairwise incompatible Σ-propositions $\phi_n, n \in \omega$, such that $\mathrm{RS}(\mathcal{T}_{\phi_n}) \geq \alpha$.

(5) If δ is a limit ordinal then $\mathrm{RS}(\mathcal{T}) \geq \delta$ if $\mathrm{RS}(\mathcal{T}) \geq \beta$ for any $\beta < \delta$.

We assume $\mathrm{RS}(\mathcal{T}) = \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \alpha$ and $\neg[\mathrm{RS}(\mathcal{T}) \geq \alpha + 1]$.

If $\mathrm{RS}(\mathcal{T}) \geq \alpha$ for any α, then we put $\mathrm{RS}(\mathcal{T}) = \infty$.

A family $\mathcal{T}$ is called *e-totally transcendental* or *total transcendental* if $\mathrm{RS}(\mathcal{T})$ is an ordinal.

If a family $\mathcal{T}$ is e-totally transcendental with $\mathrm{RS}(\mathcal{T}) = \alpha \geq 0$, then as the *degree* $\mathrm{ds}(\mathcal{T})$ of the family $\mathcal{T}$ is considered the maximal number of pairwise incompatible sentences ϕ_i for which $\mathrm{RS}(\mathcal{T}_{\varphi_i}) = \alpha$.

Corollary 2. *Let $\mathcal{T}$ be a family of all 1-transitive (weakly) o-minimal linear orders. Then* $\mathrm{RS}(\mathcal{T}) = 0$, $\mathrm{ds}(\mathcal{T}) = 2$.

Proposition 2. *Let T be a linearly ordered theory. If T is 1-transitive, then there is no expansion of the theory T by any constant symbols preserving 1-transitivity.*

Proposition 3. *Let T be an 1-transitive weakly o-minimal theory. Then an expansion of the theory T by an arbitrary unary predicate preserves both the 1-transitivity and weak o-minimality if and only if such a predicate is universal, i.e., it distinguishes the universe of a model of the theory T.*

Proof of Proposition 3. ($\Rightarrow$) Due to weak o-minimality, any unary predicate must determine only finitely many convex sets. If any of the distinguished non-empty convex sets does not coincide with the universe of a model of the theory T, then we get a contradiction with 1-transitivity.

($\Leftarrow$) If a unary predicate is universal, i.e., in particular, determines a convex set then by virtue of Theorem 63 [5] such an expansion preserves the weak o-minimality. And since such a convex set coincides with the universe of a model of the theory T, such an expansion is 1-transitive.

Corollary 3. *Let T be an 1-transitive o-minimal theory. Then there is no non-trivial expansion of the theory T by any unary predicate symbols preserving 1-transitivity.*

Definition 2. [3, 4] Let T be a countable complete theory, $p_1(x_1), \ldots, p_n(x_n) \in S_1(\emptyset)$. We will say that a type $q(x_1, \ldots, x_n) \in S_n(\emptyset)$ is a *$(p_1, \ldots, p_n)$-type* if

$$q(x_1, \ldots, x_n) \supseteq \bigcup_{i=1}^{n} p_i(x_i).$$

The set of all $(p_1, \ldots, p_n)$-types of theory T we denote by $S_{p_1,\ldots,p_n}(T)$.

A theory T is called *almost ω-categorical* or *almost omega-categorical*, if for any $p_1(x_1)$, $\ldots$, $p_n(x_n) \in S_1(\emptyset)$ there exists only a finite number of types $q(x_1, \ldots, x_n) \in S_{p_1,\ldots,p_n}(T)$.

Proposition 4. *Let M be an 1-transitive weakly o-minimal structure. Suppose that $Th(M)$ is almost omega-categorical. Then M is densely ordered.*

Proof of Proposition 4. According to Corollary 1, the reduct of a structure M on a linear order $\{<\}$ is elementarily equivalent either to $\langle \mathbb{Q}, <\rangle$ or $\langle \omega^* + \omega, <\rangle$. Suppose that $M \equiv \langle \omega^* + \omega, <\rangle$. Then consider the following formulas:

$$IS(x) - \text{“}x \text{ has an immediate successor”},$$

$$IP(x) - \text{“}x \text{ has an immediate predecessor”},$$

$$S_1(x, y) - \text{“}y \text{ is an immediate successor of element } x\text{”},$$

$$S_n(x, y) - \text{“}y \text{ is } n\text{th immediate successor of element } x\text{”},\ n \geq 1.$$

By virtue of the assumption made, the following set of formulas is locally consistent:

$$\{IS(x) \wedge IP(x)\} \cup \{\exists y S_n(x, y) \mid n \geq 1\}.$$

Hence, there exists $p \in S_1(\emptyset)$ extending this set of formulas, and an elementary extension M' of the structure M in which type p is realized. Then for each $n < \omega$ the set

$$p(x) \cup p(y) \cup \{S_n(x, y)\}$$

is consistent, whence the number of (p_1, p_2)-types is infinite, where $p_i(x_i) := p(x_i), i \in \{1, 2\}$. We obtain a contradiction with the almost ω-categoricity of $Th(M)$.

Note that an 1-transitive weakly o-minimal densely ordered structure does not necessarily have an almost omega-categorical theory.

Example 3. Let $M = \langle \mathbb{Q}, <, R^2\rangle$ be a linearly ordered structure, where $\mathbb{Q}$ is the set of rational numbers. The relation $R(x, y)$ is defined as follows:

$$R(a, b) \Leftrightarrow a \leq b < a + \sqrt{2}$$

for any $a, b \in \mathbb{Q}$.

It can be established that M is an 1-transitive weakly o-minimal structure which is densely ordered, and $Th(M)$ is not almost omega-categorical.

2 Expansion of theories by equivalence relations

The definition of the rank of convexity of a formula with one free variable was introduced in [6] and extended to an arbitrary set in [7]:

Definition 3. [6, 7] Let T be a weakly o-minimal theory, $M \models T$, $A \subseteq M$. The *convexity rank of the set* A ($RC(A)$) is defined as follows:

1) $RC(A) = -1$ if $A = \emptyset$.

2) $RC(A) = 0$ if A is finite and non-empty.

3) $RC(A) \geq 1$ if A is infinite.

4) $RC(A) \geq \alpha + 1$ if there exist a parametrically definable equivalence relation $E(x, y)$ and $b_i \in A, i \in \omega$, satisfying the following conditions:

- For any $i, j \in \omega$, whenever $i \neq j$ we have $M \models \neg E(b_i, b_j)$
- For each $i \in \omega$ $\quad RC(E(M, b_i)) \geq \alpha$ and $E(M, b_i)$ is a convex subset of A

5) $RC(A) \geq \delta$ if $RC(A) \geq \alpha$ for all $\alpha < \delta$ (δ is limit).

For an ordinal α, let $RC(A) = \alpha$ if $RC(A) \geq \alpha$ and $RC(A) \not\geq \alpha + 1$.

If $RC(A) = \alpha$ for some α, then we say that $RC(A)$ is defined. Otherwise (i.e., if $RC(A)$ $\geq \alpha$ for all ordinals α), we put $RC(A) = \infty$.

The *convexity rank of a formula* $\phi(x, \bar{a})$, where $\bar{a} \in M$, is defined as the convexity rank of the set $\phi(M, \bar{a})$, i.e.,

$$RC(\phi(x, \bar{a})) := RC(\phi(M, \bar{a})).$$

The *rank of the convexity of an 1-type* p is defined as the rank of convexity of the set $p(M)$, i.e., $RC(p) := RC(p(M))$.

Proposition 5. *Let T be an 1-transitive o-minimal theory. Then there is no non-trivial expansion of the theory T by any equivalence relation partitioning the universe of a model of T into infinite convex classes, preserving 1-transitivity and o-minimality.*

Proof of Proposition 5. By virtue of o-minimality, any equivalence relation E with infinite convex classes partitions the underlying set of a model of T into only finitely many E-classes, from which every E-class is $\emptyset$-definable. Since by virtue of o-minimality every non-trivial convex infinite set that is definable must have endpoints, then E must be a universal relation, i.e., for any $a, b \in M$, $E(a, b)$ must hold.

Proposition 6. *Let T be an 1-transitive weakly o-minimal linear ordering. Then expanding the theory of T by an equivalence relation $E(x, y)$ which partitions the universe of a structure into infinite convex classes, preserves the 1-transitivity and weak o-minimality if and only if the induced order on M/E is an 1-transitive linear ordering.*

Proposition 7. *Let $\mathcal{T}$ be the family of all 1-transitive weakly o-minimal densely ordered theories. Then* $\mathrm{RS}(\mathcal{T}) \geq 1$.

Proof of Proposition 7. Note that the 1-transitive weakly o-minimal theories differ by the rank of convexity: if $RC(x = x) = 1$, then there is no definable (possibly with parameters) equivalence relation partitioning the underlying set of a model of T into infinitely many infinite convex classes; if $RC(x = x) = 2$, then such an equivalence relation exists, it partitions $M \models T$ into infinitely many infinite convex classes, and these classes are densely ordered and without the leftmost and rightmost classes, and there is no other such an equivalence relation; if $RC(x = x) = 3$, then there exist equivalence relations $E_1(x, y)$ and $E_2(x, y)$, partitioning the universe of a model of T into infinitely many infinite convex classes, with each E_1-class partitioned into infinitely many E_2-subclasses, which are also densely ordered in the E_1-class and without leftmost and rightmost subclasses; etc. Thus, for every natural $n \geq 1$ there exists an 1-transitive weakly o-minimal theory of convexity rank n, whence we obtain that $\mathrm{RS}(\mathcal{T}) \geq 1$.

3 Expansion of theories by unary functions

Let us recall some notions originally introduced in [1]. Let $Y \subset M^{n+1}$ be $\emptyset$-definable, let $\pi : M^{n+1} \to M^n$ be a projection that drops the last coordinate, and let $Z := \pi(Y)$. For each $\bar{a} \in Z$, let $Y_{\bar{a}} := \{y : (\bar{a}, y) \in Y\}$. Suppose that for for every $\bar{a} \in Z$, the set $Y_{\bar{a}}$ is bounded from above but has no supremum in M. Let $\sim$ be an $\emptyset$-definable equivalence relation on M^n, defined as follows:

$$\bar{a} \sim \bar{b} \text{ for all } \bar{a}, \bar{b} \in M^n \setminus .Z, \text{ and } \bar{a} \sim \bar{b} \Leftrightarrow \sup Y_{\bar{a}} = \sup Y_{\bar{b}}, \text{ if } \bar{a}, \bar{b} \in Z.$$

Let $\overline{Z} := Z/\sim$, and for each tuple $\bar{a} \in Z$, we denote by $[\bar{a}] \sim$ the class of tuple $\bar{a}$. There is a natural $\emptyset$–definable linear ordering on $M \cup \overline{Z}$, defined as follows.

Let $\bar{a} \in Z$ and $c \in M$. Then $[\bar{a}] < c$ if and only if $w < c$ for all $w \in Y_{\bar{a}}$. If $\bar{a} \not\sim \bar{b}$, then there exists some $x \in M$ such that $[\bar{a}] < x < [\bar{b}]$ or $[\bar{b}] < x < [\bar{a}]$, and therefore $<$ induces a linear order on $M \cup \overline{Z}$. We call such a set $\overline{Z}$ *a sort* (in this case, an $\emptyset$–definable sort) in $\overline{M}$, where $\overline{M}$ is the Dedekind completion of the structure M, and we view $\overline{Z}$ as naturally embedded in $\overline{M}$. Similarly, we can obtain a sort in $\overline{M}$ by considering infima instead of suprema.

Thus, we will consider definable functions from M in its Dedekind completion $\overline{M}$, more precisely into definable sorts of the structure $\overline{M}$, representing infima or suprema of definable sets.

Let $A, D \subseteq M$, D be infinite, $Z \subseteq \overline{M}$ be an A–definable sort and $f : D \to Z$ be an A-definable function. We say that f is *locally increasing* (*locally decreasing, locally constant*) on D if for any $a \in D$ there exists an infinite interval $J \subseteq D$ containing $\{a\}$ such that f is strictly increasing (strictly decreasing, constant) on J; we also say that f is *locally monotonic* on D if it is locally increasing or locally decreasing on D.

Let f be an A-definable function on $D \subseteq M$, E be an A-definable equivalence relation on D. We say that f is *strictly increasing (decreasing) on D/E* if for any $a, b \in D$ with conditions $a < b$ and $\neg E(a, b)$ we have $f(a) < f(b)$ $(f(a) > f(b))$.

Proposition 8. [7,8] *Let T be a weakly o-minimal theory, $M \models T$, $A \subseteq M$, $p \in S_1(A)$ be non-algebraic. Suppose that there exists an A-definable function f the domain of which contains the set $p(M)$, and f is not a constant on $p(M)$. Then f is locally monotonic or locally constant on $p(M)$ and there exists an A-definable equivalence relation $E(x, y)$ partitioning $p(M)$ into infinitely many convex classes such that f is strictly monotonic on $p(M)/E$.*

Definition 4. [9, 10] Let M be a weakly o-minimal structure, $B, D \subseteq M$, $A \subseteq \overline{M}$ be a B-definable sort and $f : D \to A$ be a B-definable function that is locally increasing (decreasing) on D. We will say that a function f *has depth* n on the set D if there are equivalence relations $E_1(x, y), \dots, E_n(x, y)$, partitioning D into infinitely many infinite convex classes, so that for any $2 \le i \le n$ every E_i-class is partitioned into infinitely many infinite convex E_{i-1}-subclasses and the following holds:

- f is strictly increasing (decreasing) on each E_1-class,
- f is locally decreasing (increasing) on D/E_k for any odd $k \le n$ (or the same, f is strictly decreasing (increasing) on every $E_{k+1}(a, M)/E_k$ for any $a \in D$),
- f is locally increasing (decreasing) on D/E_k for for any even $k \le n$,
- f is strictly monotonic on D/E_n.

In this case, the function f is called *locally increasing (decreasing) of depth* n.

Obviously, a strictly increasing (decreasing) function is *locally increasing (decreasing) of depth* 0.

Theorem 1. [10] *Let T be a weakly o-minimal theory. Then any definable function into a definable sort has finite depth.*

In [11], Definition 4 was extended by introducing the notion of *a locally constant function of depth* n, i.e., in Definition 4 the function f is a constant on every E_1-class. Note that in this case the function f can be either locally increasing, or locally decreasing on D/E_1. In the following examples, M is a weakly o-minimal structure, and the function f is locally constant.

Example 4. (Example 2.6.1, [1]) Let $M := \langle M, <, P_1^1, P_2^1, f^1 \rangle$ be a linearly ordered structure so that M is a disjoint union of interpretations of the unary predicates P_1 and P_2, with $P_1(M) < P_2(M)$. We identify the interpretation P_2 with $\mathbb{Q}$ ordered as usual, and P_1 with $\mathbb{Q} \times \mathbb{Q}$, ordered lexicographically. The symbol f is interpreted by a partial unary function with $\operatorname{Dom} f = P_1(M)$ and $\operatorname{Range} f = P_2(M)$ and is defined by $f((n, m)) = n$ for for all $(n, m) \in \mathbb{Q} \times \mathbb{Q}$.

Let $p := \{P_1(x)\}$, $q := \{P_2(x)\}$. Obviously, $p, q \in S_1(\emptyset)$. Let us take an arbitrary $a \in p(M)$. Then there exists a unique $b \in q(M)$ such that $f(a) = b$, i.e., $b \in \operatorname{dcl}(\{a\})$.

Consider the following formula:

$$E(x, y) := P_1(x) \wedge P_2(y) \wedge \wedge \exists x[P_2(z) \wedge f(x) = z \wedge f(y) = z].$$

We can see that $E(x, y)$ is an $\emptyset$-definable equivalence relation partitioning $p(M)$ into infinitely many infinite convex classes. We assert that f is locally constant of depth 1 on $P_1(M)$, i.e., f is constant on every E-class and f is strictly increasing on $P_1(M)/E$.

Example 5. Let $M := \langle M, <, P_1^1, P_2^1, E_1^p, E_2^p, E_1^q, f^1 \rangle$ be a linearly ordered structure so that M is a disjoint union of interpretations of the unary predicates P_1 and P_2 with $P_1(M) < P_2(M)$. We identify the interpretation of P_1 with $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$, ordered lexicographically, and P_2 with $\mathbb{Q} \times \mathbb{Q}$, also ordered lexicographically. Interpretations of the binary predicates $E_1^p(x, y)$ and $E_2^p(x, y)$ are equivalence relations on $P_1(M)$ such that for all $x = (n_1, m_1, l_1)$, $y = (n_2, m_2, l_2) \in \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ we have

$$E_1^p(x, y) \Leftrightarrow n_1 = n_2 \wedge m_1 = m_2$$

and

$$E_2^p(x, y) \Leftrightarrow n_1 = n_2.$$

The interpretation of the binary predicate $E_1^q(x, y)$ is similarly defined: it is an equivalence relation on $P_2(M)$ such that for all $x = (n_1, m_1), y = (n_2, m_2) \in \mathbb{Q} \times \mathbb{Q}$ we have

$$E_1^q(x, y) \Leftrightarrow n_1 = n_2.$$

The symbol f is interpreted by a partial unary function such that $\operatorname{Dom} f = P_1(M)$ and $\operatorname{Range} f = P_2(M)$, and is defined by $f((n, m, l)) = (-n, m)$ for all $(n, m, l) \in \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$.

We assert that the function f is locally constant of depth 2 on $P_1(M)$, i.e., f is constant on every E_1^p-class, f is strictly increasing on every $E_2^p(a, M)/E_1^p$, where $a \in p(M)$, and f is strictly decreasing on $P_1(M)/E_2^p$.

Proposition 9. *Let T be an 1-transitive o-minimal linear ordering, $M \models T$. Then an o-minimal expansion of the theory T by a unary function f preserves the 1-transitivity if and only if f is strictly increasing on M.*

Proof of Proposition 9. By virtue of o-minimality, f is piecewise monotonic on its domain. If f is not strictly monotonic over the whole M, then such an expansion loses 1-transitivity.

Suppose the contrary: f is strictly decreasing on M, i.e., for any $a, b \in M$ such that $a < b$ we have $f(b) < f(a)$. Consider the following formulas:

$$\phi_1(x) := x < f(x), \quad \phi_2(x) := x = f(x), \quad \phi_3(x) := f(x) < x.$$

Obviously, by virtue of 1-transitivity, either $\phi_1(M) = M$, or $\phi_2(M) = M$, or $\phi_3(M) = M$. If $\phi_2(M) = M$, then f is strictly increasing. If $\phi_1(M) = M$, then for any $a \in M$ we have

$a < f(a)$. Since f is strictly decreasing, then $f^2(a) < f(a)$, whence $f(a) \notin \phi_1(M)$ which contradicts our assumption. If $\phi_3(M) = M$ then for any $a \in M$ we have $f(a) < a$. Since f is strictly decreasing, then $f(a) < f^2(a)$, whence $f(a) \notin \phi_3(M)$, which again contradicts our assumption.

Proposition 10. *Let T be an 1-transitive weakly o-minimal linear ordering, $M \models T$. Then a weakly o-minimal expansion of the theory T by a unary function f preserves the 1-transitivity if and only if f is locally monotonic or locally constant on M and there exists an $\emptyset$-definable equivalence relation $E(x, y)$ partitioning M into infinitely many convex classes such that f is strictly increasing on M/E.*

Proof of Proposition 10. By virtue of weak o-minimality, f is piecewise locally monotonic on its domain. If f is not locally monotonic or locally constant on the the whole M, then such an expansion loses 1-transitivity. By virtue of Proposition 8 there exists an $\emptyset$-definable equivalence relation $E(x, y)$ that partitions M into infinitely many convex classes such that f is strictly monotonic on M/E.

Suppose the contrary: f is strictly decreasing on M/E, i.e., for any $a, b \in M$ such that $a < b$ and $\neg E(a, b)$ we have $f(b) < f(a)$. Consider the following formulas:

$$\phi_1(x) := x < f(x), \quad \phi_2(x) := x = f(x), \quad \phi_3(x) := f(x) < x.$$

Obviously, by virtue of 1-transitivity, either $\phi_1(M) = M$, or $\phi_2(M) = M$, or $\phi_3(M) = M$. If $\phi_2(M) = M$, then f is strictly increasing on M.

Suppose that $\phi_1(M) = M$. Take arbitrary $a, b \in M$ such that $a < b$ and $\neg E(a, b)$. Then $f(b) < f(a)$. Since $b \in \phi_1(M)$, then $a < b < f(b) < f(a)$, whence we obtain $a < f(a) \wedge \neg E(a, f(a)$. Then by by virtue of strict decreasing of f on M/E we have $f^2(a) < f(a)$, whence $f(a) \notin \phi_1(M)$, which contradicts our assumption.

Now let $\phi_3(M) = M$. Let's take arbitrary $a, b \in M$ such that $a < b$ and $\neg E(a, b)$. Then $f(b) < f(a)$. Since $b \in \phi_3(M)$, then $f(b) < f(a) < a < b$, whence we obtain $f(b) < b \wedge \neg E(f(b), b)$. Then by by virtue of strict decreasing of f on M/E we have $f(b) < f^2(b)$, whence $f(b) \notin \phi_3(M)$, which again contradicts our assumption.

Corollary 4. *Let T be an 1-transitive weakly o-minimal linear ordering, $M \models T$. Then a weakly o-minimal expansion of the theory of T by a unary function f preserves 1-transitivity if and only if either f is locally increasing of depth n on M for some even $n \in \omega$, or f is locally decreasing of depth n on M for some odd $n \in \omega$; or f is locally constant on M and there exists an $\emptyset$-definable equivalence relation $E(x, y)$ partitioning M into infinitely many convex classes such that f is strictly increasing on M/E.*

Corollary 5. *Let $\mathcal{T}$ be the family of all expansions of 1-transitive of weakly o-minimal orderings by a unary function preserving 1-transitivity and weak o-minimality. Then* $\mathrm{RS}(\mathcal{T}) \geq 1$.

Proof of Corollary 5. In a weakly o-minimal theory, a unary function can have depth n for every $n \in \omega$.

4 Expansion of theories by arbitrary binary relations

If we expand an arbitrary weakly o-minimal theory by a new binary relation $R(x,y)$ and this expansion preserves weak o-minimality, then for any $a \in M$ each of the sets $R(a,M)$ and $R(M,a)$ distinguishes a finite number of convex sets in M. Since any of these convex sets is $\{a\}$-definable, we can assume, that each of the sets $R(a,M)$ and $R(M,a)$ is convex for any $a \in M$.

Theorem 2. *Let T be an 1-transitive weakly o-minimal linear ordering, $M \models T$. Suppose that $R(x,y)$ is a new binary relation such that each of the sets $R(a,M)$ and $R(M,a)$ is convex and infinite for any $a \in M$. Then a weakly o-minimal expansion of the theory T by the binary relation $R(x,y)$ preserves 1-transitivity if and only if $g(x) := \inf R(x,M)$, $f(x) := \sup R(x,M)$, $g'(y) := \inf R(M,y)$ and $f'(y) := \sup R(M,y)$ are locally monotonic or locally constant on M and there exists an $\emptyset$-definable equivalence relation $E(x,y)$ partitioning M into infinitely many convex classes such that that all functions g, f, g' and f' are strictly increasing on M/E.*

Proof of Theorem 2. Let us first show that the statement of the theorem is true for functions g and f. By weak o-minimality, each of the functions g and f is piecewise locally monotonic on its domain. If at least one of these functions is not locally monotonic or locally constant over the whole M, then such an expansion loses 1-transitivity. By virtue of Proposition 8 there exist $\emptyset$-definable equivalence relations $E_1(x,y)$ and $E_2(x,y)$, partitioning M into infinitely many convex classes such that g and f are strictly monotonic on M/E_1 and M/E_2 respectively. Clearly, either $E_1(a,M) \subseteq E_2(a,M)$, or $E_2(a,M) \subseteq E_1(a,M)$ for any $a \in M$. Without loss of generality, let us assume the former. Then g is strictly monotonic on M/E_2. Further, since $R(a,M)$ is convex and infinite for any $a \in M$, then $g(a) < f(a)$ for any $a \in M$.

Consider the following formulas:

$$\phi_1(x) := \exists y[x < y \wedge R(x,y) \wedge \forall t(x \le t \le y \to R(x,t))] \wedge \forall z[z < x \to \neg R(x,z)],$$

$$\phi_2(x) := \exists y[x < y \wedge R(x,y) \wedge \exists y_1(x < y_1 < y \wedge \neg R(x,y_1)],$$

$$\phi_3(x) := \exists y_1 \exists y_2[y_1 < x < y_2 \wedge R(x,y_1) \wedge R(x,y_2)],$$

$$\phi_4(x) := R(x,x) \wedge \exists y_1[y_1 < x \wedge R(x,y_1) \wedge \forall y(R(x,y) \to y \le x)],$$

$$\phi_5(x) := \exists y[y < x \wedge R(x,y) \wedge \forall t[R(x,t) \to t < x] \wedge \neg R(x,x).$$

Suppose that $\phi_1(M) = M$, i.e., $a = g(a)$ for any $a \in M$. Then it is obvious that g is strictly increasing on M. Suppose the contrary: f is strictly decreasing on M/E_2. Take arbitrary $a, b \in M$ such that $a < b \wedge \neg E_2(a,b)$. Then we obtain that $f(b) < f(a)$, whence

$$a < b = g(b) < f(b) < f(b) < f(a).$$

Take an arbitrary $c \in M$ such that $f(a) < c$. Then we have: $\neg E_2(a, c)$. Since f is strictly decreasing on M/E_2, then $f(c) < f(a)$, where $c \notin \phi_1(M)$, which contradicts our assumption.

Let now $\phi_2(M) = M$, i.e. $a < g(a)$ for any $a \in M$. Suppose the contrary: g is strictly decreasing on M/E_2 or f is strictly decreasing on M/E_2. Without loss of generality, let us assume the former. Take arbitrary $a, b \in M$ such that $a < b \wedge \neg E_2(a, b)$. Then we obtain that $g(b) < g(a)$, whence

$$a < b < g(b) < g(a).$$

Take an arbitrary $c \in M$ such that $g(a) < c$. Then we have: $\neg E_2(a, c)$. Since g is strictly decreasing on M/E_2, then $g(c) < g(a)$, whence $c \notin \phi_2(M)$, which again contradicts our assumption. The case where f is strictly decreasing on M/E_2 is considered similarly.

Let now $\phi_3(M) = M$, i.e. $g(a) < a < f(a)$ for any $a \in M$. Suppose the contrary: g is strictly decreasing on M/E_2 or f is strictly decreasing on M/E_2. Without loss of generality, let us assume the former. Take arbitrary $a, b \in M$ such that $a < b \wedge \neg E_2(a, b)$. Then we obtain that $g(b) < g(a)$, whence

$$g(b) < g(a) < a < b.$$

Let us take an arbitrary $c \in M$ such that $c < g(b)$. Then we have: $\neg E_2(c, b)$. Since g is strictly decreasing on M/E_2, then $g(b) < g(c)$, whence $c \notin \phi_3(M)$, which again contradicts our assumption. The case where f is strictly decreasing on M/E_2 is considered similarly.

The cases when $\phi_4(M) = M$ or $\phi_5(M) = M$ are considered similarly.

Note that similar reasoning is true for functions g' and f'. Here we establish that g' and f' are strictly increasing on M/E'_2, where $E'_2(x, y)$ is an $\emptyset$-definable equivalence relation partitioning M into infinitely many convex classes. Obviously, either $E_2(a, M) \subseteq E'_2(a, M)$, or $E'_2(a, M) \subseteq E_2(a, M)$ for any $a \in M$. Thus, the theorem is completely proved.

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Кулпешов Б.Ш. 1-ТРАНЗИТИВТIК РЕТТЕЛГЕН ТЕОРИЯЛАРДЫН ТРИВИАЛДЫ ЕМЕС БАЙЫТУЛЫР.

Бул жумыста бiз 1-транзитивтi реттелген теориялардын 1-транзитивтiлiктi сактайтын тривиалды емес байытуларды зерттеймiз. Сондай-ак, алсiз о-минималды сызыктык реттердi эквиваленттер катынастар, унарлы функциялар жане ерiктi бинарлык катынастар бойынша байытулар зерттелдi. Алынган натижелерге суйене отырып, артурлi алсiз о-минималды теориялардын отбасылары ушiн рангiсi мандер табылды.

Туйiндi создер. Алсiз о-минималдык, 1-транзитивтiк, донестiк рангiсi, теорияны байыту, реттелген теория, теориялар отбасына арналган рангiсi.

Кулпешов Б.Ш. НЕТРИВИАЛЬНЫЕ ОБОГАЩЕНИЯ 1-ТРАНЗИТИВНЫХ УПОРЯДОЧЕННЫХ ТЕОРИЙ

В настоящей работе мы исследуем нетривиальные обогащения 1-транзитивных упорядоченных теорий, сохраняющие 1-транзитивность. В частности, были исследованы обогащения слабо о-минимальных линейных порядков отношениями эквивалентности, унарными функциями и произвольными бинарными отношениями. На основе полученных результатов установлены значения рангов для различных семейств слабо о-минимальных теорий.

Ключевые слова: слабая о-минимальность, 1-транзитивность, ранг выпуклости, обогащение теории, упорядоченная теория, ранг для семейства теорий.