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Solution of the boundary value problem for differential equation with piecewise-constant argument of generalized type

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Abstract. We examine a boundary value problem for a differential equation with a piecewise-constant argument of generalized type. An interval $[0, T]$ is divided into N parts, the values of the solution at the interior points of the subintervals are treated as additional parameters, and the boundary value problem for a differential equation with piecewise-constant argument of generalized type is transformed to an equivalent initial value problems with parameters for differential-algebraic equations on subintervals. Differential part of this problem consists of the Cauchy problems for ordinary differential equations on the subintervals. Algebraic part of this problem consists of the algebraic equations with respect to the parameters composed by boundary and continuity conditions at the interior points of the partition. The coefficients and the right-hand sides of this system are determined by solving the Cauchy problems for ordinary differential equations on the subintervals. We demonstrate that the solvability of the boundary value problems is equivalent to the solvability of the composed systems. We propose methods for solving boundary value problems based on the construction and solution of these systems.

Keywords. Differential equations with piecewise-constant argument of generalized type, two-point boundary value problem, parametrization method, differential-algebraic equations, solvability criteria.

1 Introduction

On the interval $[0, T]$, we consider the boundary value problem for a differential equation with a piecewise-constant argument of generalized type in the following form:

$$\dot{x} = a(t)x(t) + a_0(t)x(\gamma(t)) + f(t), \tag{1}$$

$$bx(0) + cx(T) = d_1, \tag{2}$$

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where $x(t)$ is an unknown function, the functions $a(t)$, $a_0(t)$, and $f(t)$ are continuous on $[0,T]$; $\gamma(t) = \zeta_j$ if $t \in [\theta_j, \theta_{j+1})$, $j = \overline{0, N-1}$; $\theta_j \le \zeta_j \le \theta_{j+1}$ for all $j = 0, 1, ..., N-1$; $0 = \theta_0 < \theta_1 < ... < \theta_{N-1} < \theta_N = T$, b, c and d_s are constants; $\|x\| = \max\limits_{i=\overline{1,n}} |x_i|$.

A function $x(t)$ is a solution to problem (1), (2) if:

(i) $x(t)$ is continuous on $[0,T]$;

(ii) the derivative $\dot{x}(t)$ exists at each point $t \in [0,T]$ with the possible exception of the points θ_j, $j = \overline{0, N-1}$, where the one-sided derivatives exist; (iii) equation (1) is satisfied for $x(t)$ on each interval (θ_j, θ_{j+1}), $j = \overline{0, N-1}$, and it holds for the right derivative of $x(t)$ at the points θ_j, $j = \overline{0, N-1}$;

(iv) boundary condition (2) are satisfied for $x(t)$ at the points $t = 0$, $t = T$.

The concept of differential equations with piecewise-constant argument of generalized type (DEPCAG) has been introduced in the works [1–3].

Over the past few decades, there has been significant research into applying these equations to various problems, exploring examples and their diverse applications.

In addition to exploring different properties of differential equations with piecewise-constant argument, several authors have delved into questions regarding the solvability and construction of solutions for boundary value problems associated with these equations on finite intervals. Special emphasis has been placed on periodic and multipoint boundary value problems for differential equations with piecewise-constant argument, given their broad utility in natural sciences and engineering [4–15].

The objective of this paper is to establish a constructive approach for the exploration and solution of the boundary value problem, encompassing the development of an algorithm capable of solving problems (1), (2).

To this end, we use the Dzhumabaev's parametrization method [16].

The paper is organized as follows.

The time interval $[0,T]$ is partitioned into N subintervals based on the partition Δ_N: $\theta_0 = 0 < \theta_1 < \theta_2 < ... < \theta_N = T$. We introduce additional parameters as a value of desired function at the internal points of the partition. The boundary value problem for differential equation with piecewise-constant argument of generalized type is transformed to an equivalent initial value problems with parameters for differential-algebraic equations on subintervals. Differential part of this problem contains of the Cauchy problems for ordinary differential equations on the subintervals. Algebraic part of this problem contains of the algebraic equations with respect to parameters composed by boundary and continuity conditions at interior points of the partition. The coefficients and right-hand sides of this system are determined by solving the Cauchy problems for ordinary differential equations on the subintervals. We demonstrate that the solvability of boundary value problems is equivalent to the solvability of the composed systems. We propose methods for solving boundary value problems based on the construction and solution of these systems.

2 Scheme of the parametrization method and reduction to an equivalent problem

Denote by Δ_N a partition of the interval $[0,T)$:

$$[0,T)=\bigcup_{r=1}^{N}[\theta_{r-1},\theta_r) \text{ by lines } \ t=\theta_j, \ \ j=\overline{1,N-1}.$$

Let
$C([0,T],\mathbb{R})$ be the space of continuous functions $x:[0,T]\to\mathbb{R}$ with norm $\|x\|_1=\max\limits_{t\in[0,T]}|x(t)|$; $C([0,T],\Delta_N,\mathbb{R}^N)$ be the space of functions systems $x[t]=(x_1(t),x_2(t),\dots,x_N(t))'$, where $x_r:[\theta_{r-1},\theta_r)\to\mathbb{R}$ are continuous and have finite left-hand side limits $\lim\limits_{t\to\theta_r-0}x_r(t)$ for all $r=\overline{1,N}$ with norm $\|x[\cdot]\|_2=\max\limits_{r=\overline{1,N}}\sup\limits_{t\in[\theta_{r-1},\theta_r)}|x_r(t)|$.

Denote by $x_r(t)$ a restriction of function $x(t)$ on r-th interval $[\theta_{r-1},\theta_r)$, i.e.

$$x_r(t)=x(t) \text{ for } t\in[\theta_{r-1},\theta_r), \ r=\overline{1,N}.$$

Then the function system $x[t]=(x_1(t),x_2(t),\dots,x_N(t))'$ belongs to $C([0,T],\Delta_N,\mathbb{R}^N)$, and its elements $x_r(t)$, $r=\overline{1,N}$, satisfy the following system of differential equations with piecewise-constant argument of generalized type

$$\frac{dx_r}{dt}=a(t)x_r(t)+a_0(t)x_r(\zeta_{r-1})+f(t), \qquad t\in[\theta_{r-1},\theta_r), \quad r=\overline{1,N}, \tag{3}$$

$$bx_1(0)+c\lim_{t\to T-0}x_N(t)=d, \tag{4}$$

$$\lim_{t\to\theta_p-0}x_p(t)=x_{p+1}(\theta_p), \qquad p=\overline{1,N-1}, \tag{5}$$

where conditions (5) are the continuity conditions on internal points of the interval $[0,T]$.

In (3) we take into account that $\gamma(t)=\zeta_j$ if $t\in[\theta_j,\theta_{j+1})$, $j=\overline{0,N-1}$.

A function system $x[t]=(x_1(t),x_2(t),\ \dots,x_N(t))$ is called a solution to problem (3)–(5), if it belongs to $C([0,T],\Delta_N,\mathbb{R}^N)$ and the functions $\widetilde{x}_r(t)$, $r=\overline{1,N}$, satisfy equations (3), boundary condition (4) and the continuity conditions (5).

We introduce additional parameters $\lambda_r=x_r(\zeta_{r-1})$ for all $r=\overline{1,N}$. Making the substitution $x_r(t)=u_r(t)-\lambda_r$ on the every r-th interval $[\theta_{r-1},\theta_r)$, we obtain the initial value problem for differential-algebraic equations with parameters

$$\frac{du_r}{dt}=a(t)(u_r(t)+\lambda_r)+a_0(t)\lambda_r+f(t), \quad t\in[\theta_{r-1},\theta_r), \quad r=\overline{1,N}, \tag{6}$$

$$u_r(\zeta_{r-1})=0, \quad r=\overline{1,N}, \tag{7}$$

$$b\lambda_1+bu_1(0)+c\lambda_N+c\lim_{t\to T-0}u_N(t)=d, \tag{8}$$

$$\lambda_p + \lim_{t\to\theta_p-0} u_p(t) = u_{p+1}(\theta_p), \qquad p = \overline{1, N-1}. \tag{9}$$

Problems (6), (7) are the initial value problems for differential equations with parameters on the intervals $[\theta_{r-1}, \theta_r)$, $r = \overline{1, N}$. The boundary conditions (8) and continuity conditions (9) together form a system of algebraic equations with respect to the parameters λ_r, $r = \overline{1, N}$.

For any fixed $\lambda_r \in \mathbb{R}$ and r, the initial value problem (6), (7) has a unique solution $u_r(t)$, and the function system $u[t] = (u_1(t), u_2(t), \ldots, u_N(t))$ belongs to $C([0, T], \Delta_N, \mathbb{R}^N)$.

A pair $\{u[t], \lambda\}$ with elements $u[t] = (u_1(t), u_2(t), ..., u_N(t))$, $\lambda = (\lambda_1, \lambda_2, ..., \lambda_N)$ is called a solution to the initial value problems with parameters (6)–(9), if it satisfies differential equations (6), initial conditions (7), boundary condition (8) and continuity conditions (9).

If a function system $\widetilde{x}[t] = (\widetilde{x}_1(t), \widetilde{x}_2(t), ..., \widetilde{x}_N(t))$ belongs to $C([0, T], \Delta_N, \mathbb{R}^N)$, and the functions $\widetilde{x}_r(t)$, $r = \overline{1, N}$, satisfy equations (3), boundary condition (4) and the continuity conditions (5), then the pair $\{\widetilde{u}[t], \widetilde{\lambda}\}$ with the elements

$$\widetilde{u}[t] = (\widetilde{u}_1(t), \widetilde{u}_2(t), \ldots, \widetilde{u}_N(t)), \ \widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, \ldots, \widetilde{\lambda}_N),$$

where $\widetilde{u}_r(t) = \widetilde{x}_r(t) - \widetilde{\lambda}_r$, $\widetilde{\lambda}_r = \widetilde{x}_r(\zeta_{r-1})$, $r = \overline{1, N}$, is a solution to the initial value problems with parameters (6)–(9). Conversely, if a function system $\{u^*[t], \lambda^*\}$ with elements

$$u^*[t] = (u_1^*(t), u_2^*(t), \ldots, u_N^*(t)), \ \lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_N^*)$$

is a solution to problems (6)–(9), then the function system $x^*[t] = (x_1^*(t), x_2^*(t), \ldots, x_N^*(t))$ with $x_r^*(t) = \lambda_r^* + u_r^*(t)$, $r = \overline{1, N}$, belongs to $C([0, T], \Delta_N, \mathbb{R})$, and the functions $x_r^*(t)$, $r = \overline{1, N}$, satisfy equations (3), boundary condition (4) and the continuity conditions (5).

Let

$$A_r(t) = e^{\int\limits_{\theta_{r-1}}^{t} a(\tau)d\tau}, \qquad t \in [\theta_{r-1}, \theta_r], \qquad r = \overline{1, N}.$$

The function $A_r(t)$ is a solution of the differential equation

$$\frac{du_r}{dt} = a(t)u_r(t), \qquad t \in [\theta_{r-1}, \theta_r], \qquad r = \overline{1, N}.$$

We write down the solutions to the initial value problems with parameters (6), (7) in the form:

$$u_r(t) = A_r(t) \int\limits_{\zeta_{r-1}}^{t} A_r^{-1}(\tau)[a(\tau) + a_0(\tau)]d\tau\lambda_r + A_r(t) \int\limits_{\zeta_{r-1}}^{t} A_r^{-1}(\tau)f(\tau)d\tau, \tag{10}$$

$$\text{where } t \in [\theta_{r-1}, \theta_r), \quad r = \overline{1, N}.$$

The solution (10) allows us to compose system of linear algebraic equations with respect to parameters $\lambda_r \in \mathbb{R}^2$, $r = \overline{1, N}$.

For this we substitute the suitable expressions of solution (10) into the boundary condition (8) and continuity conditions (9). We obtain

$$b\Big[1+A_1(\theta_0)\int_{\zeta_0}^{\theta_0} A_1^{-1}(\tau)[a(\tau)+a_0(\tau)]d\tau\Big]\lambda_1 + c\Big[1+A_N(T)\int_{\zeta_{N-1}}^{T} A_N^{-1}(\tau)[a(\tau)+a_0(\tau)]d\tau\Big]\lambda_N =$$

$$= d - bA_1(\theta_0)\int_{\zeta_0}^{\theta_0} A_1^{-1}(\tau)f(\tau)d\tau - A_N(T)\int_{\zeta_{N-1}}^{T} A_N^{-1}(\tau)f(\tau)d\tau, \qquad (11)$$

$$\Big[1+A_p(\theta_p)\int_{\zeta_{p-1}}^{\theta_p} A_p^{-1}(\tau)[a(\tau)+a_0(\tau)]d\tau\Big]\lambda_p - \Big[1+A_{p+1}(\theta_p)\int_{\zeta_p}^{\theta_p} A_{p+1}^{-1}(\tau)[a(\tau)+a_0(\tau)]d\tau\Big]\lambda_{p+1} =$$

$$= -A_p(\theta_p)\int_{\zeta_{p-1}}^{\theta_p} A_p^{-1}(\tau)f(\tau)d\tau + A_{p+1}(\theta_p)\int_{\zeta_p}^{\theta_p} A_{p+1}^{-1}(\tau)f(\tau)d\tau, \quad p=\overline{1,N-1}. \qquad (12)$$

Denote by $Q_*(\Delta_N)$ the $N\times N$ matrix corresponding to the left-hand side of system (11), (12) and write the system as

$$Q_*(\Delta_N)\lambda = -F_*(\Delta_N), \qquad \lambda\in R^N, \qquad (13)$$

where

$$F_*(\Delta_N) = \Big(-d + bA_1(\theta_0)\int_{\zeta_0}^{\theta_0} A_1^{-1}(\tau)f(\tau)d\tau + A_N(T)\int_{\zeta_{N-1}}^{T} A_N^{-1}(\tau)f(\tau)d\tau,$$

$$A_1(\theta_1)\int_{\zeta_0}^{\theta_1} A_1^{-1}(\tau)f(\tau)d\tau + A_2(\theta_1)\int_{\zeta_1}^{\theta_1} A_2^{-1}(\tau)f(\tau)d\tau, ...,$$

$$A_{N-1}(\theta_{N-1})\int_{\zeta_{N-2}}^{\theta_{N-1}} A_{N-1}^{-1}(\tau)f(\tau)d\tau - A_N(\theta_{N-1})\int_{\zeta_{N-1}}^{\theta_{N-1}} A_N^{-1}(\tau)f(\tau)d\tau\Big) \in \mathbb{R}^N.$$

The statement provided below is accurate.

Lemma 1. *If $x^*(t)$ is a solution to problem (1), (2) and $\lambda_r^* = x^*(\zeta_{r-1})$, $r=\overline{1,N}$, then the vector $\lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_N^*) \in \mathbb{R}^N$ is a solution to system (13). Conversely, if $\tilde\lambda = (\tilde\lambda_1, \tilde\lambda_2, \dots, \tilde\lambda_N) \in \mathbb{R}^N$ is a solution to system (13) and $\tilde u[t] = (\tilde u_1(t), \tilde u_2(t), \dots, \tilde u_N(t)$ is* *the solution to initial value problems (6), (7) for the parameter* $\tilde{\lambda} \in \mathbb{R}^N$, *then the function* $\tilde{x}(t)$ *given by the equalities* $\tilde{x}(t) = \tilde{\lambda}_r + u_r(t)$, $t \in [\theta_{r-1}, \theta_r)$, $r = \overline{1, N}$, *and* $\tilde{x}(T) = \tilde{\lambda}_N + \lim\limits_{t \to T-0} u_N(t)$, *is a solution to problem (1), (2).*

3 The main results

Definition 1. *The boundary value problem (1), (2) is called uniquely solvable if for any pair* $(f(t), d)$, *with* $f(t) \in C([0,T], \mathbb{R})$ *and* $d \in \mathbb{R}$, *it has a unique solution.*

Lemma 1 and well known theorems of linear algebra imply the following two assertions.

Theorem 1. *The boundary value problem (1), (2) is solvable if and only if the vector* $F_*(\Delta_N)$ *is orthogonal to the kernel of the transposed matrix* $(Q_*(\Delta_N))'$, *i.e. iff the equality*

$$(F_*(\Delta_N), \eta) = 0$$

is valid for all $\eta \in Ker(Q_*(\Delta_N))'$, *where* $(\cdot, \cdot)$ *is the inner product in* $\mathbb{R}^N$.

Theorem 2. *The boundary value problem (1), (2) is uniquely solvable if and only if* $N \times N$ *matrix* $Q_*(\Delta_N)$ *is invertible.*

As evident from Theorem 2, the key condition for the unique solvability of problem (1), (2) is the invertibility of the $N \times N$ matrix $Q_*(\Delta_N)$. The matrix $Q_*(\Delta_N)$ has a special structure. This allows obtaining conditions for the invertibility of the matrix $Q_*(\Delta_N)$ in terms of the non-vanishing of a certain expression.

We introduce the notations:

$$q_r(t) = 1 + A_r(t) \int\limits_{\zeta_{r-1}}^{t} A_r^{-1}(\tau)[a(\tau) + a_0(\tau)]d\tau, \qquad t \in [\theta_{r-1}, \theta_r], \qquad r = \overline{1, N},$$

$$d_p = -A_p(\theta_p) \int\limits_{\zeta_{p-1}}^{\theta_p} A_p^{-1}(\tau) f(\tau) d\tau + A_{p+1}(\theta_p) \int\limits_{\zeta_p}^{\theta_p} A_{p+1}^{-1}(\tau) f(\tau) d\tau, \quad p = \overline{1, N-1},$$

$$d_N = d - bA_1(\theta_0) \int\limits_{\zeta_0}^{\theta_0} A_1^{-1}(\tau) f(\tau) d\tau - A_N(T) \int\limits_{\zeta_{N-1}}^{T} A_N^{-1}(\tau) f(\tau) d\tau.$$

Consider system of algebraic equations (13) in the following form:

$$bq_1(\theta_0)\lambda_1 + cq_N(\theta_N)\lambda_N = d_N, \tag{14_1}$$

$$q_1(\theta_1)\lambda_1 - q_2(\theta_1)\lambda_2 = d_1, \tag{14_2}$$

$$q_2(\theta_2)\lambda_2 - q_3(\theta_2)\lambda_3 = d_2, \tag{14_3}$$

$$q_r(\theta_r)\lambda_r - q_{r+1}(\theta_r)\lambda_{r+1} = d_r, \quad r = \overline{1, N-1}, \tag{14_r}$$

$$q_{N-1}(\theta_{N-1})\lambda_{N-1} - q_N(\theta_{N-1})\lambda_N = d_{N-1}. \tag{14_N}$$

Assume that $q_{N-1}(\theta_{N-1}) \neq 0$ and from equation (14_N) we represent λ_{N-1} by λ_N:

$$\lambda_{N-1} = \frac{q_N(\theta_{N-1})}{q_{N-1}(\theta_{N-1})}\lambda_N + \frac{1}{q_{N-1}(\theta_{N-1})}d_{N-1}. \tag{15_1}$$

Next, we consider the previous equation (14_{N-1}) and substitute the found expression for λ_{N-1} into:

$$q_{N-2}(\theta_{N-2})\lambda_{N-2} = q_{N-1}(\theta_{N-2})\lambda_{N-1} + d_{N-2} =$$

$$= q_{N-1}(\theta_{N-2}) \cdot \frac{q_N(\theta_{N-1})}{q_{N-1}(\theta_{N-1})}\lambda_N + q_{N-1}(\theta_{N-2}) \cdot \frac{1}{q_{N-1}(\theta_{N-1})}d_{N-1} + d_{N-2}, \tag{16}$$

Assume that $q_{N-2}(\theta_{N-2}) \neq 0$ and from equation (16) we represent λ_{N-2} by λ_N:

$$\lambda_{N-2} = \frac{q_{N-1}(\theta_{N-2})}{q_{N-2}(\theta_{N-2})} \cdot \frac{q_N(\theta_{N-1})}{q_{N-1}(\theta_{N-1})}\lambda_N +$$

$$+\frac{q_{N-1}(\theta_{N-2})}{q_{N-2}(\theta_{N-2})} \cdot \frac{1}{q_{N-1}(\theta_{N-1})}d_{N-1} + \frac{1}{q_{N-2}(\theta_{N-2})}d_{N-2}. \tag{15_2}$$

Further, we consider the equation (14_{N-2}) and substitute the found expression for λ_{N-2} into:

$$q_{N-3}(\theta_{N-3})\lambda_{N-3} = q_{N-2}(\theta_{N-3})\lambda_{N-2} + d_{N-3} =$$

$$= q_{N-2}(\theta_{N-3}) \cdot \frac{q_{N-1}(\theta_{N-2})}{q_{N-2}(\theta_{N-2})} \cdot \frac{q_N(\theta_{N-1})}{q_{N-1}(\theta_{N-1})}\lambda_N +$$

$$+q_{N-2}(\theta_{N-3}) \cdot \frac{q_{N-1}(\theta_{N-2})}{q_{N-2}(\theta_{N-2})} \cdot \frac{1}{q_{N-1}(\theta_{N-1})}d_{N-1} + q_{N-2}(\theta_{N-3}) \cdot \frac{1}{q_{N-2}(\theta_{N-2})}d_{N-2} + d_{N-3},$$

from here, assuming that $q_{N-3}(\theta_{N-3}) \neq 0$ we represent λ_{N-3} by λ_N:

$$\lambda_{N-3} = \frac{q_{N-2}(\theta_{N-3})}{q_{N-3}(\theta_{N-3})} \cdot \frac{q_{N-1}(\theta_{N-2})}{q_{N-2}(\theta_{N-2})} \cdot \frac{q_N(\theta_{N-1})}{q_{N-1}(\theta_{N-1})}\lambda_N +$$

$$+\frac{q_{N-2}(\theta_{N-3})}{q_{N-3}(\theta_{N-3})} \cdot \frac{q_{N-1}(\theta_{N-2})}{q_{N-2}(\theta_{N-2})} \cdot \frac{1}{q_{N-1}(\theta_{N-1})}d_{N-1} + \frac{q_{N-2}(\theta_{N-3})}{q_{N-3}(\theta_{N-3})} \cdot \frac{1}{q_{N-2}(\theta_{N-2})}d_{N-2} +$$

$$+\frac{1}{q_{N-3}(\theta_{N-3})}d_{N-3}. \tag{15_3}$$

And so on, we obtain the following expression for λ_r:

$$q_r(\theta_r)\lambda_r = q_{r+1}(\theta_r)\lambda_{r+1} + d_r = q_{r+1}(\theta_r)\cdot \prod_{i=N-1}^{r+1} \frac{q_{i+1}(\theta_i)}{q_i(\theta_i)}\lambda_N +$$

$$+q_{r+1}(\theta_r)\Big\{ \prod_{s=N-2}^{r+1} \frac{q_{s+1}(\theta_s)}{q_s(\theta_s)} \cdot \frac{1}{q_{N-1}(\theta_{N-1})} d_{N-1} + \prod_{s=N-3}^{r+1} \frac{q_{s+1}(\theta_s)}{q_s(\theta_s)} \cdot \frac{1}{q_{N-2}(\theta_{N-2})} d_{N-2} +$$

$$+ \dots + \prod_{s=r+1}^{r+1} \frac{q_{s+1}(\theta_s)}{q_s(\theta_s)} \cdot \frac{1}{q_{r+2}(\theta_{r+2})} d_{r+2} + \frac{1}{q_{r+1}(\theta_{r+1})} d_{r+1} \Big\} + d_r, \quad r = \overline{1, N-1}.$$

Assume that $q_r(\theta_r) \neq 0$ and we represent λ_r by λ_N:

$$\lambda_r = \frac{q_{r+1}(\theta_r)}{q_r(\theta_r)} \cdot \prod_{i=N-1}^{r+1} \frac{q_{i+1}(\theta_i)}{q_i(\theta_i)}\lambda_N +$$

$$+\frac{q_{r+1}(\theta_r)}{q_r(\theta_r)}\Big\{ \prod_{s=N-2}^{r+1} \frac{q_{s+1}(\theta_s)}{q_s(\theta_s)} \cdot \frac{1}{q_{N-1}(\theta_{N-1})} d_{N-1} + \prod_{s=N-3}^{r+1} \frac{q_{s+1}(\theta_s)}{q_s(\theta_s)} \cdot \frac{1}{q_{N-2}(\theta_{N-2})} d_{N-2} +$$

$$+ \dots + \prod_{s=r+1}^{r+1} \frac{q_{s+1}(\theta_s)}{q_s(\theta_s)} \cdot \frac{1}{q_{r+2}(\theta_{r+2})} d_{r+2} + \frac{1}{q_{r+1}(\theta_{r+1})} d_{r+1} \Big\} + \frac{1}{q_r(\theta_r)} d_r, \quad r = \overline{1, N-1}.$$

From here, we obtain

$$\lambda_r = \prod_{i=N-1}^{r} \frac{q_{i+1}(\theta_i)}{q_i(\theta_i)}\lambda_N + \prod_{s=N-2}^{r} \frac{q_{s+1}(\theta_s)}{q_s(\theta_s)} \cdot \frac{1}{q_{N-1}(\theta_{N-1})} d_{N-1} +$$

$$+ \prod_{s=N-3}^{r} \frac{q_{s+1}(\theta_s)}{q_s(\theta_s)} \cdot \frac{1}{q_{N-2}(\theta_{N-2})} d_{N-2} + \dots + \prod_{s=r+1}^{r} \frac{q_{s+1}(\theta_s)}{q_s(\theta_s)} \cdot \frac{1}{q_{r+2}(\theta_{r+2})} d_{r+2} +$$

$$+ \prod_{s=r}^{r} \frac{q_{s+1}(\theta_s)}{q_s(\theta_s)} \cdot \frac{1}{q_{r+1}(\theta_{r+1})} d_{r+1} + \frac{1}{q_r(\theta_r)} d_r, \quad r = \overline{1, N-1}. \qquad (15_r)$$

From expression (15_r), we have the representation for λ_1 by λ_N:

$$\lambda_1 = \prod_{i=N-1}^{1} \frac{q_{i+1}(\theta_i)}{q_i(\theta_i)}\lambda_N + \prod_{s=N-2}^{1} \frac{q_{s+1}(\theta_s)}{q_s(\theta_s)} \cdot \frac{1}{q_{N-1}(\theta_{N-1})} d_{N-1} +$$

$$+ \prod_{s=N-3}^{1} \frac{q_{s+1}(\theta_s)}{q_s(\theta_s)} \cdot \frac{1}{q_{N-2}(\theta_{N-2})} d_{N-2} + \dots + \prod_{s=2}^{1} \frac{q_{s+1}(\theta_s)}{q_s(\theta_s)} \cdot \frac{1}{q_3(\theta_3)} d_3 +$$

$$+\prod_{s=1}^{1}\frac{q_{s+1}(\theta_s)}{q_s(\theta_s)}\cdot\frac{1}{q_2(\theta_2)}d_2+\frac{1}{q_1(\theta_1)}d_1. \qquad (15_{N-1})$$

Substituting the found expression for λ_1 into equation (14_1), we obtain

$$\Big\{bq_1(\theta_0)\prod_{i=N-1}^{1}\frac{q_{i+1}(\theta_i)}{q_i(\theta_i)}+cq_N(\theta_N)\Big\}\lambda_N=d_N-bq_1(\theta_0)\Big\{\prod_{s=N-2}^{1}\frac{q_{s+1}(\theta_s)}{q_s(\theta_s)}\cdot\frac{1}{q_{N-1}(\theta_{N-1})}d_{N-1}+$$

$$+\prod_{s=N-3}^{1}\frac{q_{s+1}(\theta_s)}{q_s(\theta_s)}\cdot\frac{1}{q_{N-2}(\theta_{N-2})}d_{N-2}+\cdots+\prod_{s=2}^{1}\frac{q_{s+1}(\theta_s)}{q_s(\theta_s)}\cdot\frac{1}{q_3(\theta_3)}d_3+$$

$$+\prod_{s=1}^{1}\frac{q_{s+1}(\theta_s)}{q_s(\theta_s)}\cdot\frac{1}{q_2(\theta_2)}d_2+\frac{1}{q_1(\theta_1)}d_1\Big\}. \qquad (17)$$

Assume that $bq_1(\theta_0)\prod\limits_{i=N-1}^{1}\frac{q_{i+1}(\theta_i)}{q_i(\theta_i)}+cq_N(\theta_N)\neq 0$.

Then, from (17) we will have an explicit expression for the parameter λ_N:

$$\lambda_N=\frac{1}{bq_1(\theta_0)\prod\limits_{i=N-1}^{1}\frac{q_{i+1}(\theta_i)}{q_i(\theta_i)}+cq_N(\theta_N)}d_N-$$

$$-\frac{bq_1(\theta_0)}{bq_1(\theta_0)\prod\limits_{i=N-1}^{1}\frac{q_{i+1}(\theta_i)}{q_i(\theta_i)}+cq_N(\theta_N)}\Big\{\prod_{s=N-2}^{1}\frac{q_{s+1}(\theta_s)}{q_s(\theta_s)}\cdot\frac{1}{q_{N-1}(\theta_{N-1})}d_{N-1}+$$

$$+\prod_{s=N-3}^{1}\frac{q_{s+1}(\theta_s)}{q_s(\theta_s)}\cdot\frac{1}{q_{N-2}(\theta_{N-2})}d_{N-2}+\cdots+\prod_{s=2}^{1}\frac{q_{s+1}(\theta_s)}{q_s(\theta_s)}\cdot\frac{1}{q_3(\theta_3)}d_3+$$

$$+\prod_{s=1}^{1}\frac{q_{s+1}(\theta_s)}{q_s(\theta_s)}\cdot\frac{1}{q_2(\theta_2)}d_2+\frac{1}{q_1(\theta_1)}d_1\Big\}. \qquad (15_N)$$

Thus, the following statements are valid.

Lemma 2. *The $N\times N$ matrix $Q_*(\Delta_N)$ is invertible if only if the following expressions are non-zero:*

$$bq_1(\theta_0)\prod_{i=N-1}^{1}\frac{q_{i+1}(\theta_i)}{q_i(\theta_i)}+cq_N(\theta_N)\neq 0, \qquad q_r(\theta_r)\neq 0 \textit{ for all } r=\overline{1,N-1},$$

where

$$q_r(t) = 1 + A_r(t) \int\limits_{\zeta_{r-1}}^{t} A_r^{-1}(\tau)[a(\tau) + a_0(\tau)]d\tau, \quad t \in [\theta_{r-1}, \theta_r], \qquad r = \overline{1, N},$$

$$A_r(t) = e^{\int\limits_{\theta_{r-1}}^{t} a(\tau)d\tau}, \quad t \in [\theta_{r-1}, \theta_r], \quad r = \overline{1, N}.$$

Lemma 3. *Let*

$$bq_1(\theta_0) \prod_{i=N-1}^{1} \frac{q_{i+1}(\theta_i)}{q_i(\theta_i)} + cq_N(\theta_N) \neq 0, \quad q_r(\theta_r) \neq 0 \textit{ for all } r = \overline{1, N-1}.$$

Then the system of algebraic equations has a unique solution λ^* *and its components* λ_r^* *are explicitly determined in the following form:*

$$\lambda_r = \prod_{i=N-1}^{r} \frac{q_{i+1}(\theta_i)}{q_i(\theta_i)} \lambda_N + \prod_{s=N-2}^{r} \frac{q_{s+1}(\theta_s)}{q_s(\theta_s)} \cdot \frac{1}{q_{N-1}(\theta_{N-1})} d_{N-1} +$$

$$+ \prod_{s=N-3}^{r} \frac{q_{s+1}(\theta_s)}{q_s(\theta_s)} \cdot \frac{1}{q_{N-2}(\theta_{N-2})} d_{N-2} + \dots + \prod_{s=r+1}^{r} \frac{q_{s+1}(\theta_s)}{q_s(\theta_s)} \cdot \frac{1}{q_{r+2}(\theta_{r+2})} d_{r+2} +$$

$$+ \prod_{s=r}^{r} \frac{q_{s+1}(\theta_s)}{q_s(\theta_s)} \cdot \frac{1}{q_{r+1}(\theta_{r+1})} d_{r+1} + \frac{1}{q_r(\theta_r)} d_r, \quad r = \overline{1, N-1},$$

$$\lambda_N = \frac{1}{bq_1(\theta_0) \prod\limits_{i=N-1}^{1} \frac{q_{i+1}(\theta_i)}{q_i(\theta_i)} + cq_N(\theta_N)} d_N -$$

$$- \frac{bq_1(\theta_0)}{bq_1(\theta_0) \prod\limits_{i=N-1}^{1} \frac{q_{i+1}(\theta_i)}{q_i(\theta_i)} + cq_N(\theta_N)} \left\{ \prod_{s=N-2}^{1} \frac{q_{s+1}(\theta_s)}{q_s(\theta_s)} \cdot \frac{1}{q_{N-1}(\theta_{N-1})} d_{N-1} + \right.$$

$$+ \prod_{s=N-3}^{1} \frac{q_{s+1}(\theta_s)}{q_s(\theta_s)} \cdot \frac{1}{q_{N-2}(\theta_{N-2})} d_{N-2} + \dots + \prod_{s=2}^{1} \frac{q_{s+1}(\theta_s)}{q_s(\theta_s)} \cdot \frac{1}{q_3(\theta_3)} d_3 +$$

$$\left. + \prod_{s=1}^{1} \frac{q_{s+1}(\theta_s)}{q_s(\theta_s)} \cdot \frac{1}{q_2(\theta_2)} d_2 + \frac{1}{q_1(\theta_1)} d_1 \right\}.$$

Conclusion. In the present article, the conditions for the unique solvability of boundary value problem for differential equation with piecewise constant argument of generalized type (1), (2) are established. For solving this problem is applied Dzhumabaev's parametrization method. We are shown that solvability of the problem (1), (2) is equivalent to solvability of a system of algebraic equations with respect to parameters. Conditions for the existence of a unique solution to the system of algebraic equations are determined, and explicit expressions for the parameters are found. An analytical solution to the considered problem is constructed.

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Асанова А.Т. ЖАЛПЫЛАНҒАН ТҮРДЕГІ БӨЛІКТІ-ТҰРАҚТЫ АРГУМЕНТІ БАР ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУ ҮШІН ШЕТТІК ЕСЕПТІҢ ШЕШІМІ

Жалпыланған түрдегі бөлікті-тұрақты аргументі бар дифференциалдық теңдеу үшін шеттік есеп қарастырылады. $[0,T]$ аралығы N бөлікке бөлінеді, шешімнің ішкі аралықтардың ішкі нүктелеріндегі мәндері қосымша параметрлер ретінде қарастырылады. Жалпыланған түрдегі бөлікті-тұрақты аргументі бар дифференциалдық теңдеу үшін шеттік есеп ішкі аралықтардағы дифференциалдық-алгебралық теңдеулер үшін бастапқы мандері және параметрлері бар пара-пар есепке түрлендіріледі. Есептің дифференциалдық бөлігі ішкі аралықтардағы сызықты дифференциалдық теңдеулер үшін Коши есептерінен тұрады. Есептің алгебралық бөлігі шеттік шарттан және бөліктеудің ішкі нүктелеріндегі үзіліссіздік шарттарынан ъқұрылған параметрлерге қатысты алгебралық теңдеулерді қамтиды. Осы жүйенің коэффициенттері мен оң жақтары ішкі аралықтарда сызықты жәй дифференциалдық теңдеулер үшін Коши есептерін шешу арқылы табылады. Шеттік есептердің шешілімділігі құрастырылған алгебралық теңдеулер жүйесінің шешілімділігіне пара-пар екені көрсетілді. Осы жүйелерді құруға және шешуге негізделген шеттік есептерді шешудің әдістері ұсынылады.

Кілттік сөздер. Жалпыланған түрдегі бөлікті-тұрақты аргументі бар дифференциалдық теңдеулер, шеттік есеп, параметрлеу әдісі, дифференциалдық-алгебралық теңдеулер, шешілімділік критерийлері.

Асанова А.Т. РЕШЕНИЕ КРАЕВОЙ ЗАДАЧИ ДЛЯ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ С КУСОЧНО-ПОСТОЯННЫМ АРГУМЕНТОМ ОБОБЩЕННОГО ТИПА

Мы рассматриваем краевую задачу для дифференциального уравнения с кусочно-постоянным аргументом обобщенного типа. Интервал $[0, T]$ разбивается на N частей, значения решения во внутренних точках подинтервалов рассматриваются как дополнительные параметры. Краевая задача для дифференциального уравнения с кусочно-постоянным аргументом обобщенного типа преобразуется в эквивалентную задачу с начальными значениями и параметрами для дифференциально-алгебраических уравнений на подинтервалах. Дифференциальная часть этой задачи состоит из задач Коши для обыкновенных дифференциальных уравнений на подинтервалах. Алгебраическая часть этой задачи содержит алгебраические уравнения относительно параметров, составленные из краевого условия и условий непрерывности на внутренних точках разбиения. Коэффициенты и правые части этой системы определяются решениями задач Коши для обыкновенных дифференциальных уравнений на подинтервалах. Мы демонстрируем, что разрешимость краевой задачи эквивалентна разрешимости составленной системы алгебраических уравнений. Мы предлагаем метод решения краевой задачи на основе построения и решения этих систем.

Ключевые слова. Дифференциальные уравнения с кусочно-постоянным аргументом обобщенного типа, краевая задача, метод параметризации, дифференциально-алгебраические уравнения, критерии разрешимости.