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Institute of Mathematics and Mathematical Modeling

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http://kmj.math.kz/

PUBLICATION TYPE:
Peer-reviewed open access journal
Periodical
Published four issues per year

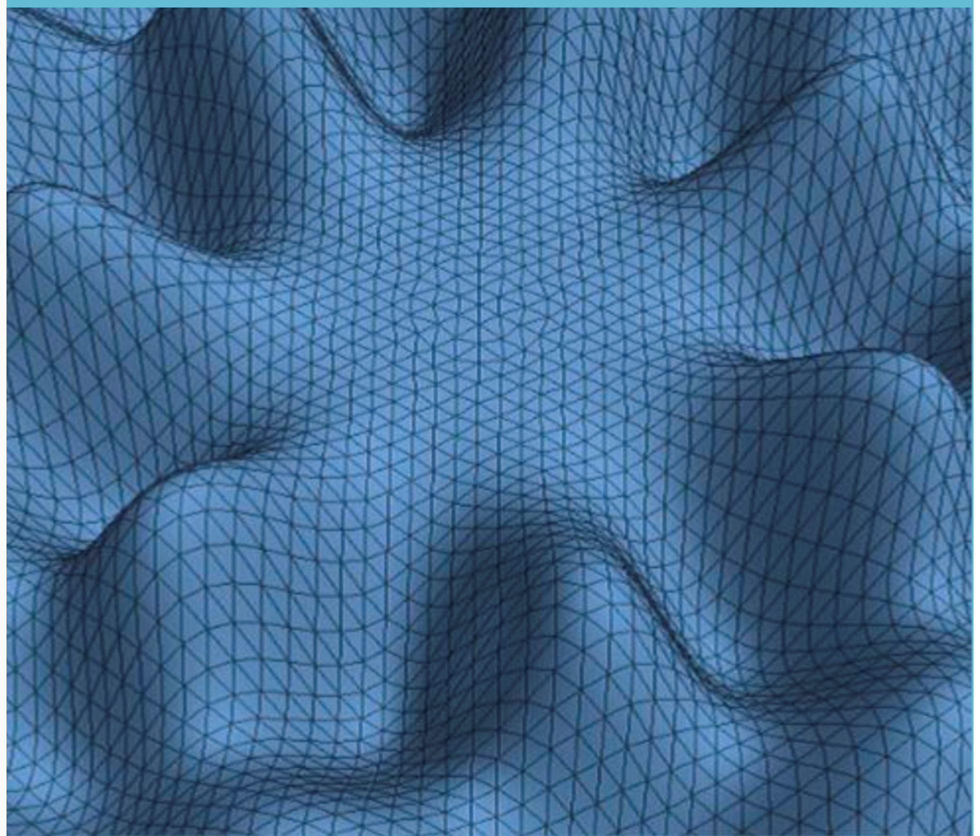
24(4)
2024

The Kazakh Mathematical Journal is registered by the Information Committee under Ministry of Information and Communications of the Republic of Kazakhstan № 17590-Ж certificate dated 13.03. 2019
The journal is based on the Kazakh journal "Mathematical Journal", which is published by the Institute of Mathematics and Mathematical Modeling since 2001 (ISSN 1682-0525).

Institute of Mathematics and Mathematical Modeling

ISSN 2413-6468

KAZAKH MATHEMATICAL JOURNAL



Almaty, Kazakhstan

Institute of Mathematics and Mathematical Modeling

Vol. 24
No. 4
ISSN 2413-6468

https://MathJournal.kz/

Kazakh Mathematical Journal

(founded in 2001 as “Mathematical Journal”)

Official Journal of
Institute of Mathematics and Mathematical Modeling,
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EDITOR IN CHIEF	Makhmud Sadybekov, Institute of Mathematics and Mathematical Modeling
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WEB ADDRESS	http://MathJournal.kz/
PUBLICATION TYPE	Peer-reviewed open-access journal Periodical Published four issues per year ISSN: 2413-6468

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Kazakh Mathematical Journal ISSN 2413–6468

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On algebras of binary isolating formulas for weakly circularly minimal theories of convexity rank 2

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Communicated by: Bektur B. Baizhanov

Received: 21.12.2024 ⋆ Accepted/Published Online: 30.12.2024 ⋆ Final Version: 30.12.2024

Abstract. This paper is devoted to the study of weakly circularly minimal circularly ordered structures. The simplest example of a circular order is a linear order with endpoints, in which the largest element is identified with the smallest. Another example is the order that arises when going around a circle. A circularly ordered structure is called weakly circularly minimal if any of its definable subsets is a finite union of convex sets and points. A theory is called weakly circularly minimal if all its models are weakly circularly minimal. Algebras of binary isolating formulas are described for $\aleph_0$-categorical 1-transitive non-primitive weakly circularly minimal theories of convexity rank 2 with a trivial definable closure having a monotonic-to-right function to the definable completion of a structure and non-having a non-trivial equivalence relation partitioning the universe of a structure into finitely many convex classes.

Keywords. algebra of binary formulas, $\aleph_0$-categorical theory, weak circular minimality, circularly ordered structure, convexity rank.

1 Preliminaries

Algebras of binary formulas are a tool for describing relationships between elements of the sets of realizations of a one-type at the binary level with respect to the superposition of binary definable sets. A *binary isolating formula* is a formula of the form $\varphi(x, y)$ such that for some parameter a the formula $\varphi(a, y)$ isolates a complete type in $S(\{a\})$. The concepts and notations related to these algebras can be found in the papers [1, 2]. In recent years, algebras of binary formulas have been studied intensively and have been continued in the works [3]–[11].

2010 Mathematics Subject Classification: 03C64; 03C35.

Funding: This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. BR20281002).

DOI: https://doi.org/10.70474/kmj24-4-01

Let L be a countable first-order language. Throughout we consider L-structures and assume that L contains a ternary relational symbol K, interpreted as a circular order in these structures (unless otherwise stated).

Let $M = \langle M, \leq \rangle$ be a linearly ordered set. If we connect two endpoints of M (possibly, $-\infty$ and $+\infty$), then we obtain a circular order. More formally, the *circular order* is described by a ternary relation K satisfying the following conditions:

(co1) $\forall x \forall y \forall z (K(x, y, z) \to K(y, z, x))$;

(co2) $\forall x \forall y \forall z (K(x, y, z) \wedge K(y, x, z) \Leftrightarrow x = y \vee y = z \vee z = x)$;

(co3) $\forall x \forall y \forall z (K(x, y, z) \to \forall t [K(x, y, t) \vee K(t, y, z)])$;

(co4) $\forall x \forall y \forall z (K(x, y, z) \vee K(y, x, z))$.

The following observation relates linear and circular orders.

Fact 1. [12] *(i) If $\langle M, \leq \rangle$ is a linear ordering and K is the ternary relation derived from $\leq$ by the rule $K(x, y, z) :\Leftrightarrow (x \leq y \leq z) \vee (z \leq x \leq y) \vee (y \leq z \leq x)$, then K is a circular order relation on M.*

(ii) If $\langle N, K \rangle$ is a circular ordering and $a \in N$, then the relation $\leq_a$ defined on $M := N \setminus \{a\}$ by the rule $y \leq_a z :\Leftrightarrow K(a, y, z)$ is a linear order.

Thus, any linearly ordered structure is circularly ordered, since the relation of circular order is $\emptyset$-definable in an arbitrary linearly ordered structure. However, the opposite is not true. The following example shows that there are circularly ordered structures not being linearly ordered (in the sense that a linear ordering relation is not $\emptyset$-definable in an arbitrary circularly ordered structure).

Example 2. [13, 14] Let $\mathbb{Q}_2^* := \langle \mathbb{Q}_2, K, L \rangle$ be a circularly ordered structure, where $L = \{\sigma_0^2, \sigma_1^2\}$, for which the following conditions hold:

(i) its domain $\mathbb{Q}_2$ is a countable dense subset of the unit circle, no two points making the central angle π;

(ii) for distinct $a, b \in \mathbb{Q}_2$

$$(a, b) \in \sigma_0 \Leftrightarrow 0 < \arg(a/b) < \pi,$$

$$(a, b) \in \sigma_1 \Leftrightarrow \pi < \arg(a/b) < 2\pi,$$

where $\arg(a/b)$ means the value of the central angle between a and b clockwise.

Indeed, one can check that the linear order relation is not $\emptyset$-definable in this structure.

The notion of *weak circular minimality* was studied initially in [15]. Let $A \subseteq M$, where M is a circularly ordered structure. The set A is called *convex* if for any $a, b \in A$ the following property is satisfied: for any $c \in M$ with $K(a, c, b)$, $c \in A$ holds, or for any $c \in M$ with $K(b, c, a)$, $c \in A$ holds. A *weakly circularly minimal structure* is a circularly ordered structure $M = \langle M, K, \ldots \rangle$ such that any definable (with parameters) subset of M is a union of finitely many convex sets in M. The study of weakly circularly minimal structures was continued in the papers [16]–[21].

Let M be an $\aleph_0$-categorical weakly circularly minimal structure, $G := \mathrm{Aut}(M)$. Following the standard group theory terminology, the group G is called *k-transitive* if for any pairwise distinct $a_1, a_2, \ldots, a_k \in M$ and pairwise distinct $b_1, b_2, \ldots, b_k \in M$ there exists $g \in G$ such that $g(a_1) = b_1, g(a_2) = b_2, \ldots, g(a_k) = b_k$. A *congruence* on M is an arbitrary G-invariant equivalence relation on M. The group G is called *primitive* if G is 1-transitive and there are no non-trivial proper congruences on M.

(1) $K_0(x, y, z) := K(x, y, z) \wedge y \neq x \wedge y \neq z \wedge x \neq z$.

(2) $K(u_1, \ldots, u_n)$ denotes a formula saying that all subtuples of the tuple $\langle u_1, \ldots, u_n \rangle$ having the length 3 (in ascending order) satisfy K; similar notations are used for K_0.

(3) Let A, B, C be disjoint convex subsets of a circularly ordered structure M. We write $K(A, B, C)$ if for any $a, b, c \in M$ with $a \in A$, $b \in B$, $c \in C$ we have $K(a, b, c)$. We extend naturally that notation using, for instance, the notation $K_0(A, d, B, C)$ if $d \notin A \cup B \cup C$ and $K_0(A, d, B) \wedge K_0(d, B, C)$ holds.

Further we need the notion of the definable completion of a circularly ordered structure, introduced in [15]. Its linear analog was introduced in [22]. A *cut* $C(x)$ in a circularly ordered structure M is a maximal consistent set of formulas of the form $K(a, x, b)$, where $a, b \in M$. A cut is said to be *algebraic* if there exists $c \in M$ that realizes it. Otherwise, such a cut is said to be *non-algebraic*. Let $C(x)$ be a non-algebraic cut. If there is some $a \in M$ such that either for all $b \in M$ the formula $K(a, x, b) \in C(x)$, or for all $b \in M$ the formula $K(b, x, a) \in C(x)$, then $C(x)$ is said to be *rational*. Otherwise, such a cut is said to be *irrational*. A *definable cut* in M is a cut $C(x)$ with the following property: there exist $a, b \in M$ such that $K(a, x, b) \in C(x)$ and the set $\{c \in M \mid K(a, c, b) \text{ and } K(a, x, c) \in C(x)\}$ is definable. The *definable completion* $\overline{M}$ of a structure M consists of M together with all definable cuts in M that are irrational (essentially $\overline{M}$ consists of endpoints of definable subsets of the structure M).

[15] Let $F(x, y)$ be an L-formula such that $F(M, b)$ is convex infinite co-infinite for each $b \in M$. Let $F^\ell(y)$ be the formula saying y is a left endpoint of $F(M, y)$:

$$\exists z_1 \exists z_2 [K_0(z_1, y, z_2) \wedge \forall t_1 (K(z_1, t_1, y) \wedge t_1 \neq y \to \neg F(t_1, y)) \wedge$$

$$\forall t_2 (K(y, t_2, z_2) \wedge t_2 \neq y \to F(t_2, y))].$$

We say that $F(x, y)$ is *convex-to-right* if

$$M \models \forall y \forall x [F(x, y) \to F^l(y) \wedge \forall z (K(y, z, x) \to F(z, y))].$$

If $F_1(x, y), F_2(x, y)$ are arbitrary convex-to-right formulas we say F_2 is *bigger than* F_1 if there is $a \in M$ with $F_1(M, a) \subset F_2(M, a)$. If M is 1-transitive and this holds for some a, it holds for all a. This gives a total ordering on the (finite) set of all convex-to-right formulas $F(x, y)$ (viewed up to equivalence modulo $Th(M)$).

Consider $F(M, a)$ for arbitrary $a \in M$. In general, $F(M, a)$ has no right endpoint in M. For example, if $dcl(\{a\}) = \{a\}$ holds for some $a \in M$ then for any convex-to-right formula $F(x, y)$ and any $a \in M$ the formula $F(M, a)$ has no right endpoint in M. We write $f(y) := \text{rend}\, F(M, y)$, assuming that $f(y)$ is the right endpoint of the set $F(M, y)$ that lies in general in the definable completion $\overline{M}$ of M. Then f is a function mapping M in $\overline{M}$.

Let $F(x, y)$ be a convex-to-right formula. We say that $F(x, y)$ is *equivalence-generating* if for any $a, b \in M$ such that $M \models F(b, a)$ the following holds:

$$M \models \forall x(K(b, x, a) \land x \neq a \to [F(x, a) \leftrightarrow F(x, b)]).$$

Lemma 3. [20] *Let M be an $\aleph_0$-categorical 1-transitive weakly circularly minimal structure, $F(x, y)$ be a convex-to-right formula that is equivalence-generating. Then $E(x, y) := F(x, y) \lor F(y, x)$ is an equivalence relation partitioning M into infinite convex classes.*

Let $E(x, y)$ be an $\emptyset$–definable equivalence relation partitioning M into infinite convex classes. Suppose that y lies in $\overline{M}$ (non-obligatory in M). Then

$$E^*(x, y) := \exists y_1 \exists y_2 [y_1 \neq y_2 \land \forall t(K(y_1, t, y_2) \to E(t, x)) \land K_0(y_1, y, y_2)].$$

Let M, N be circularly ordered structures. The 2-*reduct* of M is a circularly ordered structure with the same universe of M and consisting of predicates for each $\emptyset$-definable relation on M of arity ≤ 2 as well as of the ternary predicate K for the circular order, but does not have other predicates of arities more than two. We say that the structure M is *isomorphic* to N *up to binarity* or *binarily isomorphic to* N if the 2-reduct of M is isomorphic to the 2-reduct of N.

Let f be a unary function from M to $\overline{M}$. We say that f is *monotonic-to-right (left) on* M if it preserves (reverses) the relation K_0, i.e. for any $a, b, c \in M$ such that $K_0(a, b, c)$, we have $K_0(f(a), f(b), f(c))$ $(K_0(f(c), f(b), f(a)))$.

The following definition can be used in a circular ordered structure as well.

Definition 4. [23], [24] Let T be a weakly o-minimal theory, M be a sufficiently saturated model of T, $A \subseteq M$. *The rank of convexity of the set* A $(RC(A))$ is defined as follows:

1) $RC(A) = -1$ if $A = \emptyset$.

2) $RC(A) = 0$ if A is finite and non-empty.

3) $RC(A) \geq 1$ if A is infinite.

4) $RC(A) \geq \alpha + 1$ if there exist a parametrically definable equivalence relation $E(x, y)$ and an infinite sequence of elements $b_i \in A, i \in \omega$, such that:

- For every $i, j \in \omega$ whenever $i \neq j$ we have $M \models \neg E(b_i, b_j)$;
- For every $i \in \omega$, $RC(E(x, b_i)) \geq \alpha$ and $E(M, b_i)$ is a convex subset of A.

5) $RC(A) \geq \delta$ if $RC(A) \geq \alpha$ for all $\alpha < \delta$, where δ is a limit ordinal.

If $RC(A) = \alpha$ for some α, we say that $RC(A)$ is defined. Otherwise (i.e. if $RC(A)) \geq \alpha$ for all α), we put $RC(A) = \infty$.

The rank of convexity of a formula $\phi(x, \bar{a})$, where $\bar{a} \in M$, is defined as the rank of convexity of the set $\phi(M, \bar{a})$, i.e. $RC(\phi(x, \bar{a})) := RC(\phi(M, \bar{a}))$.

The rank of convexity of a 1-type p is defined as the rank of convexity of the set $p(M)$, i.e. $RC(p) := RC(p(M))$.

In particular, a theory has convexity rank 1 if there is no definable (with parameters) equivalence relations with infinitely many infinite convex classes.

The following theorem characterizes up to binarity $\aleph_0$–categorical 1-transitive non-primitive weakly circularly minimal structures M of convexity rank greater than 1 having both a trivial definable closure and a convex-to-right formula $R(x, y)$ such that $r(y) := R(M, y)$ is monotonic-to-right on M:

Theorem 5. [16] *Let M be an $\aleph_0$–categorical 1-transitive non-primitive weakly circularly minimal structure of convexity rank greater than 1, $dcl(\{a\}) = \{a\}$ for some $a \in M$. Suppose that there exists a convex-to-right formula $R(x, y)$ such that $r(y) := R(M, y)$ is monotonic-to-right on M. Then M is isomorphic up to binarity to*

$$M'_{s,m,k} := \langle M, K^3, E_1^2, E_2^2, \ldots, E_s^2, E_{s+1}^2, R^2 \rangle,$$

where M is a circularly ordered structure, M is densely ordered, $s \geq 1$; E_{s+1} is an equivalence relation partitioning M into m infinite convex classes without endpoints; E_i for every $1 \leq i \leq s$ is an equivalence relation partitioning every E_{i+1}-class into infinitely many infinite convex E_i-subclasses without endpoints so that the induced order on E_i-subclasses is dense without endpoints; $R(M, a)$ has no right endpoint in M and $r^k(a) = a$ for all $a \in M$ and some $k \geq 2$, where $r^k(y) := r(r^{k-1}(y))$; for every $1 \leq i \leq s+1$ and any $a \in M$

$$M'_{s,m,k} \models \neg E_i^*(a, r(a)) \wedge \forall y(E_i(y, a) \to \exists u[E_i^*(u, r(a)) \wedge E_i^*(u, r(y))]),$$

$m = 1$ or k divides m.

In [7] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories with a primitive automorphism group. In [8] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank 1 with a 1-transitive non-primitive automorphism group and a non-trivial definable closure. In [9]–[10] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank greater than 1 with a 1-transitive non-primitive automorphism group and a non-trivial definable closure. In [11] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank

1 with a 1-transitive non-primitive automorphism group and a trivial definable closure. Here we describe algebras of binary isolating formulas for $\aleph_0$-categorical 1-transitive non-primitive weakly circularly minimal theories of convexity rank 2 with a trivial definable closure having a monotonic-to-right function to the definable completion of a structure and non-having a non-trivial equivalence relation partitioning the universe of a structure into finitely many convex classes.

2 Results

Definition 6. [2] Let $p \in S_1(\emptyset)$ be non-algebraic. The algebra $\mathcal{P}_{\nu(p)}$ is said to be *deterministic* if $u_1 \cdot u_2$ is a singleton for any labels $u_1, u_2 \in \rho_{\nu(p)}$.

Generalizing the last definition, we say that the algebra $\mathcal{P}_{\nu(p)}$ is *m-deterministic* if the product $u_1 \cdot u_2$ consists of at most m elements for any labels $u_1, u_2 \in \rho_{\nu(p)}$. We also say that an m-deterministic algebra $\mathcal{P}_{\nu(p)}$ is *strictly m-deterministic* if it is not $(m-1)$-deterministic. Obviously, strict 1-determinacy of an algebra is equivalent its determinacy.

Example 7. Consider the structure $M'_{1,1,2} := \langle M, K^3, E_1^2, R^2 \rangle$ from Theorem 5 with the condition that the function $r(y) := R(M, y)$ is monotonic-to-right on M.

We assert that $Th(M'_{1,1,2})$ has seven binary isolating formulas:

$$\theta_0(x, y) := x = y,$$

$$\theta_1(x, y) := K_0(x, y, r(x)) \wedge E_1(x, y),$$

$$\theta_2(x, y) := K_0(x, y, r(x)) \wedge \neg E_1(x, y) \wedge \neg E_1^*(y, r(x)),$$

$$\theta_3(x, y) := K_0(x, y, r(x)) \wedge \neg E_1(x, y) \wedge E_1^*(y, r(x)),$$

$$\theta_4(x, y) := K_0(r(x), y, x) \wedge \neg E_1(x, y) \wedge E_1^*(y, r(x)),$$

$$\theta_5(x, y) := K_0(r(x), y, x) \wedge \neg E_1(x, y) \wedge \neg E_1^*(y, r(x)),$$

$$\theta_6(x, y) := K_0(r(x), y, x) \wedge E_1(x, y),$$

and the following holds for any $a \in M$:

$$K_0(\theta_0(a, M), \theta_1(a, M), \theta_2(a, M), \theta_3(a, M), \theta_4(a, M), \theta_5(a, M), \theta_6(a, M)).$$

Define labels for these formulas as follows:

$$\text{label } k \text{ for } \theta_k(x, y), \text{ where } 0 \leq k \leq 6.$$

It is easy to check that for the algebra $\mathfrak{P}_{M'_{1,1,2}}$ the Cayley table has the following form:

$\cdot$	0	1	2	3	4	5	6
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$	$\{6\}$
1	$\{1\}$	$\{1\}$	$\{2\}$	$\{3,4\}$	$\{4\}$	$\{5\}$	$\{0,1,6\}$
2	$\{2\}$	$\{2\}$	$\{2,3,4,5\}$	$\{5\}$	$\{5\}$	$\{0,1,2,5,6\}$	$\{2\}$
3	$\{3\}$	$\{3,4\}$	$\{5\}$	$\{6\}$	$\{0,1,6\}$	$\{2\}$	$\{3\}$
4	$\{4\}$	$\{4\}$	$\{5\}$	$\{0,1,6\}$	$\{1\}$	$\{2\}$	$\{3,4\}$
5	$\{5\}$	$\{5\}$	$\{0,1,2,5,6\}$	$\{2\}$	$\{2\}$	$\{2,3,4,5\}$	$\{5\}$
6	$\{6\}$	$\{0,1,6\}$	$\{2\}$	$\{3\}$	$\{3,4\}$	$\{5\}$	$\{6\}$

By the Cayley table the algebra $\mathfrak{P}_{M'_{1,1,2}}$ is commutative and strictly 5-deterministic.

Theorem 8. *The algebra $\mathfrak{P}_{M'_{1,1,k}}$ of binary isolating formulas with monotonic-to-right function r has $3k+1$ labels, is commutative and strictly 5-deterministic for every $k \geq 2$.*

Proof of Theorem 8. We assert that the algebra $\mathfrak{P}_{M'_{1,1,k}}$ has $3k+1$ binary isolating formulas:

$$\theta_0(x,y) := x = y,$$

$$\theta_1(x,y) := K_0(x,y,r(x)) \wedge E_1(x,y),$$

$$\theta_2(x,y) := K_0(x,y,r(x)) \wedge \neg E_1(x,y) \wedge \neg E_1^*(y,r(x)),$$

$$\theta_3(x,y) := K_0(x,y,r(x)) \wedge E_1^*(y,r(x)),$$

$$\theta_{3l-2}(x,y) := K_0(r^{l-1}(x),y,r^l(x)) \wedge E_1^*(y,r^{l-1}(x)), \text{ where } 2 \leq l \leq k-1,$$

$$\theta_{3l-1}(x,y) := K_0(r^{l-1}(x),y,r^l(x)) \wedge \neg E_1^*(y,r^{l-1}(x)) \wedge \neg E_1^*(y,r^l(x)), \text{ where } 2 \leq l \leq k-1,$$

$$\theta_{3l}(x,y) := K_0(r^{l-1}(x),y,r^l(x)) \wedge E_1^*(y,r^l(x)), \text{ where } 2 \leq l \leq k-1,$$

$$\theta_{3k-2}(x,y) := K_0(r^{k-1}(x),y,x) \wedge E_1^*(y,r^{k-1}(x)),$$

$$\theta_{3k-1}(x,y) := K_0(r^{k-1}(x),y,x) \wedge \neg E_1^*(y,r^{k-1}(x)) \wedge \neg E_1(y,x),$$

$$\theta_{3k}(x,y) := K_0(r^{k-1}(x),y,x) \wedge E_1(y,x).$$

Thus, we have $1+3+3(k-2)+3 = 3k+1$ binary isolating formulas. Moreover, we have defined the formulas so that for any $a \in M$ the following holds:

$$K_0(\theta_0(a,M),\theta_1(a,M),\theta_2(a,M),\ldots,\theta_{3k-1}(a,M),\theta_{3k}(a,M)).$$

Prove now that the algebra $\mathfrak{P}_{M'_{1,1,k}}$ is commutative and strictly 5-deterministic for every $k \geq 2$.

Firstly, obviously that $0 \cdot l = l \cdot 0 = \{l\}$ for any $0 \leq l \leq 3k$. Suppose further that $l_1 \neq 0$ and $l_2 \neq 0$.

Consider the following formula

$$\exists t[\theta_{l_1}(x,t) \wedge \theta_{l_2}(t,y)].$$

Case 1: $l_1 = 3m_1 - 2$ for some $1 \leq m_1 \leq k-1$.

We have: $K_0(r^{m_1-1}(x), t, r^{m_1}(x))$ and $E_1^*(t, r^{m_1-1}(x))$.

Let $l_2 = 3m_2 - 2$ for some $1 \leq m_2 \leq k-1$, i.e. $K_0(r^{m_2-1}(t), y, r^{m_2}(t))$ and $E_1^*(y, r^{m_2-1}(t))$. Then we obtain the following:

$$K_0(r^{m_1+m_2-2}(x), y, r^{m_1+m_2-1}(x)) \text{ and } E_1^*(y, r^{m_1+m_2-2}(x)).$$

Suppose firstly that $(3m_1-2)+(3m_2-2) < 3k+1$. We assert that in this case $m_1+m_2-1 \leq k$. Then we have $l_1 \cdot l_2 = \{3(m_1+m_2-1)-2\}$.

Obviously, $(3m_1-2)+(3m_2-2) \neq 3k+1$.

Suppose now that $(3m_1-2)+(3m_2-2) > 3k+1$. Let $s = (m_1+m_2-1)[\text{mod k}]$. Obviously, $0 \leq s \leq k-1$. If $s = 0$, we have $K_0(r^{k-1}(x), y, x)$ and $E_1^*(y, r^{k-1}(x))$, i.e. $l_1 \cdot l_2 = \{3k-2\}$. If $1 \leq s \leq k-1$ then we have $K_0(r^{s-1}(x), y, r^s(x))$ and $E_1^*(y, r^{s-1}(x))$, i.e. $l_1 \cdot l_2 = \{3s-2\}$.

Let now $l_2 = 3m_2 - 1$ for some $1 \leq m_2 \leq k-1$. Then we have the following: $K_0(r^{m_2-1}(t), y, r^{m_2}(t))$, $\neg E_1^*(y, r^{m_2-1}(t))$ and $\neg E_1^*(y, r^{m_2}(t))$. Whence we obtain:

$$K_0(r^{m_1+m_2-2}(x), y, r^{m_1+m_2-1}(x)), \neg E_1^*(y, r^{m_1+m_2-2}(x)) \text{ and } \neg E_1^*(y, r^{m_1+m_2-1}(x)).$$

Suppose firstly $(3m_1-2)+(3m_2-1) < 3k+1$. We assert that $m_1+m_2-1 \leq k$. In this case we have $l_1 \cdot l_2 = \{3(m_1+m_2-1)-1\}$.

Obviously, also $(3m_1-2)+(3m_2-1) \neq 3k+1$.

Suppose now $(3m_1-2)+(3m_2-1) > 3k+1$. Let $s = (m_1+m_2-1)[\text{mod k}]$. Obviously, $0 \leq s \leq k-1$. If $s = 0$, we have $K_0(r^{k-1}(x), y, x)$, $\neg E_1^*(y, r^{k-1}(x))$ and $\neg E_1(y, x)$, i.e. $l_1 \cdot l_2 = \{3k-1\}$. If $1 \leq s \leq k-1$ then we have $K_0(r^{s-1}(x), y, r^s(x))$, $\neg E_1^*(y, r^{s-1}(x))$ and $\neg E_1^*(y, r^s(x))$, i.e. $l_1 \cdot l_2 = \{3s-1\}$.

Consider the product $l_2 \cdot l_1$. We have: $K_0(r^{m_2-1}(x), t, r^{m_2}(x))$, $\neg E_1^*(t, r^{m_2-1}(x))$, $\neg E_1^*(t, r^{m_2}(x))$, $K_0(r^{m_1-1}(t), y, r^{m_1}(t))$ and $E_1^*(y, r^{m_1-1}(t))$. Whence we obtain:

$$K_0(r^{m_1+m_2-2}(x), y, r^{m_1+m_2-1}(x)), \neg E_1^*(y, r^{m_1+m_2-2}(x)) \text{ and } \neg E_1^*(y, r^{m_1+m_2-1}(x)).$$

If $(3m_1-2)+(3m_2-1) < 3k+1$ then $l_2 \cdot l_1 = \{3(m_1+m_2-1)-1\}$. If $(3m_1-2)+(3m_2-1) > 3k+1$ then in case $s = 0$ we obtain $l_2 \cdot l_1 = \{3k-1\}$, and in case $1 \leq s \leq k-1$ we obtain $l_2 \cdot l_1 = \{3s-1\}$.

Let now $l_2 = 3m_2$ for some $1 \leq m_2 \leq k-1$. Then we have: $K_0(r^{m_2-1}(t), y, r^{m_2}(t))$ and $E_1^*(y, r^{m_2}(t))$. Whence we obtain:

$$E_1^*(y, r^{m_1+m_2-1}(x)), \text{ and either } K_0(r^{m_1+m_2-2}(x), y, r^{m_1+m_2-1}(x)) \text{ or}$$

$$K_0(r^{m_1+m_2-1}(x), y, r^{m_1+m_2}(x)).$$

Suppose firstly that $(3m_1 - 2) + 3m_2 < 3k + 1$. Then we assert that $m_1 + m_2 - 1 < k$. Whence we obtain: $l_1 \cdot l_2 = \{3(m_1 + m_2 - 1), 3(m_1 + m_2 - 1) + 1\}$.

Suppose now that $(3m_1-2)+3m_2 = 3k+1$. This case is possible since $(3m_1-2)+3m_2 = 3(m_1 + m_2 - 1) + 1$. We also have: $(3m_1 - 2) + 3m_2 = 3k + 1$ iff $m_1 + m_2 - 1 = k$. Thus, we obtain: $E_1(y, x)$ and either $K_0(r^{k-1}(x), y, x)$ or $K_0(x, y, r(x))$, i.e. $l_1 \cdot l_2 = \{3k, 0, 1\}$.

Let now $(3m_1 - 2) + 3m_2 > 3k + 1$. Consider $s = (m_1 + m_2 - 1)$[mod k]. We prove that $0 < s < k - 1$. Indeed, $(3m_1 - 2) + 3m_2 = 3(m_1 + m_2 - 1) + 1 > 3k + 1$ iff $m_1 + m_2 - 1 > k$. Since $m_1 \leq k - 1$ and $m_2 \leq k - 1$, $m_1 + m_2 - 1 \leq (k - 1) + (k - 1) = 1 = 2k - 3$. Thus, $k < m_1 + m_2 - 1 \leq 2k - 3$, whence $0 < s < k - 1$. We have:

$$E_1^*(y, r^s(x)) \text{ and either } K_0(r^{s-1}(x), y, r^s(x)) \text{ or } K_0(r^s(x), y, r^{s+1}(x)).$$

Whence we obtain: $l_1 \cdot l_2 = \{3s, 3s + 1\}$.

Consider the product $l_2 \cdot l_1$. We have: $K_0(r^{m_2-1}(x), t, r^{m_2}(x))$, $E_1^*(t, r^{m_2}(x))$, $K_0(r^{m_1-1}(t), y, r^{m_1}(t))$, and $E_1^*(y, r^{m_1-1}(t))$. Whence we obtain the following:

$$E_1^*(y, r^{m_1+m_2-1}(x)), \text{ and either } K_0(r^{m_1+m_2-2}(x), y, r^{m_1+m_2-1}(x)) \text{ or}$$

$$K_0(r^{m_1+m_2-1}(x), y, r^{m_1+m_2}(x)).$$

If $3m_2 + (3m_1 - 1) < 3k + 1$ then $l_2 \cdot l_1 = \{3(m_1 + m_2 - 1), 3(m_1 + m_2 - 1) + 1\}$. If $3m_2 + (3m_1 - 1) = 3k + 1$ then $l_2 \cdot l_1 = \{3k, 0, 1\}$. If $3m_2 + (3m_1 - 1) > 3k + 1$ then $l_2 \cdot l_1 = \{3s, 3s + 1\}$.

Let now $l_2 = 3k - 2$, i.e. $K_0(r^{k-1}(t), y, t)$ and $E_1^*(y, r^{k-1}(t))$. Whence we obtain:

$$K_0(r^{m_1+k-2}(x), y, r^{m_1-1}(x)) \text{ and } E_1^*(y, r^{m_1+k-2}(x)),$$

i.e. $K_0(r^{m_1-2}(x), y, r^{m_1-1}(x))$ and $E_1^*(y, r^{m_1-2}(x))$. Consequently, $l_1 \cdot l_2 = \{3(m_1 - 1) - 2\}$.

Consider the product $l_2 \cdot l_1$. We have: $K_0(r^{k-1}(x), t, x)$, $E_1^*(t, r^{k-1}(x))$, $K_0(r^{m_1-1}(t), y, r^{m_1}(t))$, and $E_1^*(y, r^{m_1-1}(t))$. Whence we obtain:

$$K_0(r^{m_1-2}(x), y, r^{m_1-1}(x)) \text{ and } E_1^*(y, r^{m_1-2}(x)),$$

i.e. $l_2 \cdot l_1 = \{3(m_1 - 1) - 2\}$.

Let now $l_2 = 3k - 1$, i.e. $K_0(r^{k-1}(t), y, t)$, $\neg E_1^*(y, r^{k-1}(t))$ and $\neg E_1^*(y, t)$. Whence we obtain:

$$K_0(r^{m_1-2}(x), y, r^{m_1-1}(x)), \neg E_1^*(y, r^{m_1-2}(x)) \text{ and } \neg E_1^*(y, r^{m_1-1}(x)).$$

Thus, $l_1 \cdot l_2 = \{3(m_1 - 1) - 1\}$.

Consider the product $l_2 \cdot l_1$. Then we have: $K_0(r^{k-1}(x), t, x)$, $\neg E_1^*(t, r^{k-1}(x))$, $\neg E_1^*(x, t)$, $K_0(r^{m_1-1}(t), y, r^{m_1}(t))$, and $E_1^*(y, r^{m_1-1}(t))$. Whence we obtain:

$$K_0(r^{m_1-2}(x), y, r^{m_1-1}(x)), \neg E_1^*(y, r^{m_1-2}(x)) \text{ and } \neg E_1^*(y, r^{m_1-1}(x)),$$

i.e. $l_2 \cdot l_1 = \{3(m_1-1)-1\}$.

Let now $l_2 = 3k$, i.e. $K_0(r^{k-1}(t), y, t)$, and $E_1^*(y,t)$. Whence we obtain:

$$E_1^*(y, r^{m_1-1}(x)) \text{ and either } K_0(r^{m_1-2}(x), y, r^{m_1-1}(x)) \text{ or } K_0(r^{m_1-1}(x), y, r^{m_1}(x)),$$

i.e. $l_1 \cdot l_2 = \{3(m_1-1), 3m_1-2\}$.

Consider the product $l_2 \cdot l_1$. Then we have: $K_0(r^{k-1}(x), t, x)$, $E_1^*(x,t)$, $K_0(r^{m_1-1}(t), y, r^{m_1}(t))$, and $E_1^*(y, r^{m_1-1}(t))$. Whence we obtain:

$$E_1^*(y, r^{m_1-1}(x)) \text{ and either } K_0(r^{m_1-2}(x), y, r^{m_1-1}(x)) \text{ or } K_0(r^{m_1-1}(x), y, r^{m_1}(x)),$$

i.e. $l_2 \cdot l_1 = \{3(m_1-1), 3m_1-2\}$.

Case 2. $l_1 = 3m_1 - 1$ for some $1 \le m_1 \le k-1$.

We have the following: $K_0(r^{m_1-1}(x), t, r^{m_1}(x))$, $\neg E_1^*(t, r^{m_1-1}(x))$ and $\neg E_1^*(t, r^{m_1}(x))$.

Let $l_2 = 3m_2 - 1$ for some $1 \le m_2 \le k-1$, i.e. $K_0(r^{m_2-1}(t), y, r^{m_2}(t))$, $\neg E_1^*(y, r^{m_2-1}(t))$ and $\neg E_1^*(y, r^{m_2}(t))$. Whence we obtain:

$$\text{either } K_0(r^{m_1+m_2-2}(x), y, r^{m_1+m_2-1}(x)) \text{ or } K_0(r^{m_1+m_2-1}(x), y, r^{m_1+m_2}(x)).$$

Suppose firstly that $(3m_1-1)+(3m_2-1) < 3k+1$. It can be checked that $(3m_1-1)+(3m_2-1) < 3k+1$ iff $m_1+m_2-1 < k$. Then

$$l_1 \cdot l_2 = \{3(m_1+m_2-1)-1, 3(m_1+m_2-1), 3(m_1+m_2-1)+1, 3(m_1+m_2-1)+2\}.$$

Let now $(3m_1-1)+(3m_2-1) = 3k+1$. This case is possible, and $(3m_1-1)+(3m_2-1) = 3k+1$ iff $m_1+m_2-1 = k$. Then we have: either $K_0(r^{k-1}(x), y, x)$ or $K_0(x, y, r(x))$. Consequently, $l_1 \cdot l_2 = \{3k-1, 3k, 0, 1, 2\}$.

Let now $(3m_1-1)+(3m_2-1) > 3k+1$. Clearly, $(3m_1-1)+(3m_2-1) > 3k+1$ iff $m_1+m_2-1 > k$. Let $s = (m_1+m_2-1)[\text{mod k}]$. Since $k < m_1+m_2-1 \le k-1+k-1-1 = 2k-3$, we have $0 < s \le k-3$. Thus, we obtain: either $K_0(r^{s-1}(x), y, r^s(x))$ or $K_0(r^s(x), y, r^{s+1}(x)))$, whence $l_1 \cdot l_2 = \{3s-1, 3s, 3s+1, 3s+2\}$.

Let $l_2 = 3m_2$ for some $1 \le m_2 \le k-1$, i.e. $K_0(r^{m_2-1}(t), y, r^{m_2}(t))$ and $E_1^*(y, r^{m_2}(t))$. Whence we obtain:

$$K_0(r^{m_1+m_2-1}(x), y, r^{m_1+m_2}(x)), \neg E_1^*(y, r^{m_1+m_2-1}(x)) \text{ and } \neg E_1^*(y, r^{m_1+m_2}(x)).$$

Suppose firstly that $(3m_1-1)+3m_2 < 3k+1$. It can be checked that $(3m_1-1)+3m_2 < 3k+1$ iff $m_1+m_2-1 < k$. In this case $l_1 \cdot l_2 = \{3(m_1+m_2)-1\}$.

The case $(3m_1-1)+3m_2 = 3k+1$ is impossible. Suppose that $(3m_1-1)+3m_2 > 3k+1$. It can be checked that $(3m_1-1)+3m_2 > 3k+1$ iff $m_1+m_2-1 \ge k$. Let $s = (m_1+m_2-1)[\text{mod k}]$. Since $k \le m_1+m_2-1 \le k-1+k-1-1 = 2k-3$, we have $0 \le s \le k-3$. Thus, we obtain:

$$K_0(r^s(x), y, r^{s+1}(x)), \neg E_1^*(y, r^s(x)) \text{ and } \neg E_1^*(y, r^{s+1}(x)),$$

whence $l_1 \cdot l_2 = \{3(s+1)-1\}$.

Consider the product $l_2 \cdot l_1$. We have: $K_0(r^{m_2-1}(x), t, r^{m_2}(x))$, $E_1^*(t, r^{m_2}(x))$, $K_0(r^{m_1-1}(t), y, r^{m_1}(t))$, $\neg E_1^*(y, r^{m_1-1}(t))$ and $\neg E_1^*(y, r^{m_1}(t))$. Whence we obtain:

$$K_0(r^{m_1+m_2-1}(x), y, r^{m_1+m_2}(x)), \neg E_1^*(y, r^{m_1+m_2-1}(x)) \text{ and } \neg E_1^*(y, r^{m_1+m_2}(x)).$$

If $(3m_1-1)+3m_2 < 3k+1$ then $l_2 \cdot l_1 = \{3(m_1+m_2)-1\}$. If $(3m_1-1)+3m_2 > 3k+1$ then $l_2 \cdot l_1 = \{3(s+1)-1\}$, where $s = (m_1+m_2-1)[\text{mod k}]$.

Let now $l_2 = 3k-2$, i.e. $K_0(r^{k-1}(t), y, t)$ and $E_1^*(y, r^{k-1}(t))$. Whence we obtain:

$$K_0(r^{m_1-2}(x), y, r^{m_1-1}(x)), \neg E_1^*(y, r^{m_1-2}(x)) \text{ and } \neg E_1^*(y, r^{m_1-1}(x)).$$

Consequently, $l_1 \cdot l_2 = \{3(m_1-1)-1\}$.

Consider the product $l_2 \cdot l_1$. We have: $K_0(r^{m_1-2}(x), y, r^{m_1-1}(x))$, $\neg E_1^*(y, r^{m_1-2}(x))$ and $\neg E_1^*(y, r^{m_1-1}(x))$, whence $l_2 \cdot l_1 = \{3(m_1-1)-1\}$.

Let now $l_2 = 3k-1$, i.e. $K_0(r^{k-1}(t), y, t)$, $\neg E_1^*(y, r^{k-1}(t))$ and $\neg E_1^*(y, t)$. Whence we obtain:

$$\text{either } K_0(r^{m_1-2}(x), y, r^{m_1-1}(x)) \text{ or } K_0(r^{m_1-1}(x), y, r^{m_1}(x)).$$

Clearly, $(3m_1-1)+3(k-1) > 3k+1$. Let $s = 3(m_1-2)$, whence $4 \le s \le k-5$. Then $l_1 \cdot l_2 = \{3(m_1-1)-1, 3(m_1-1), 3m_2-2, 3m_2-1\}$.

Consider the product $l_2 \cdot l_1$. Then we have: $K_0(r^{k-1}(x), t, x)$, $\neg E_1^*(t, r^{k-1}(x))$, $\neg E_1^*(x, t)$, $K_0(r^{m_1-1}(t), y, r^{m_1}(t))$, $\neg E_1^*(y, r^{m_1-1}(t))$, and $\neg E_1^*(y, r^{m_1}(t))$. Whence we obtain:

$$\text{either } K_0(r^{m_1-2}(x), y, r^{m_1-1}(x)) \text{ or } K_0(r^{m_1-1}(x), y, r^{m_1}(x)),$$

i.e. $l_2 \cdot l_1 = \{3(m_1-1)-1, 3(m_1-1), 3m_2-2, 3m_2-1\}$.

Let now $l_2 = 3k$, i.e. $K_0(r^{k-1}(t), y, t)$, and $E_1^*(y, t)$. Whence we obtain:

$$K_0(r^{m_1-2}(x), y, r^{m_1-1}(x)), \neg E_1^*(y, r^{m_1-1}(x)) \text{ and } \neg E_1^*(y, r^{m_1}(x)),$$

i.e. $l_1 \cdot l_2 = \{3m_1-1\}$.

Consider the product $l_2 \cdot l_1$. Then we have: $K_0(r^{k-1}(x), t, x)$, $E_1^*(x, t)$, $K_0(r^{m_1-1}(t), y, r^{m_1}(t))$, $\neg E_1^*(y, r^{m_1-1}(x))$ and $\neg E_1^*(y, r^{m_1}(x))$. Whence we obtain:

$$K_0(r^{m_1-1}(x), y, r^{m_1}(x)), \neg E_1^*(y, r^{m_1-1}(x)) \text{ and } \neg E_1^*(y, r^{m_1}(x)),$$

i.e. $l_2 \cdot l_1 = \{3m_1-1\}$.

Case 3. $l_1 = 3m_1$ for some $1 \le m_1 \le k-1$.

We have: $K_0(r^{m_1-1}(x), t, r^{m_1}(x))$ and $E_1^*(t, r^{m_1}(x))$.

Let $l_2 = 3m_2$ for some $1 \le m_2 \le k-1$, i.e. $K_0(r^{m_2-1}(t), y, r^{m_2}(t))$ and $E_1^*(y, r^{m_2}(t))$. Whence we obtain the following:

$$K_0(r^{m_1+m_2-1}(x), y, r^{m_1+m_2}(x)) \text{ and } E_1^*(y, r^{m_1+m_2}(x)).$$

Suppose firstly that $3m_1+3m_2<3k+1$. It can be checked that $3m_1+3m_2<3k+1$ iff $m_1+m_2\leq k$. If $m_1+m_2=k$ then $l_1\cdot l_2=\{3k\}$. If $m_1+m_2<k$ then $l_1\cdot l_2=\{3(m_1+m_2)\}$.

The case $3m_1+3m_2=3k+1$ is impossible. Suppose that $3m_1+3m_2>3k+1$. It can be checked that $3m_1+3m_2>3k+1$ iff $m_1+m_2>k$ iff $m_1+m_2-1\geq k$. Let $s=(m_1+m_2-1)[\text{mod k}]$. Since $k\leq m_1+m_2-1\leq k-1+k-1-1=2k-3$, we have $0\leq s\leq k-3$. Thus, we obtain:

$$K_0(r^s(x),y,r^{s+1}(x)) \text{ and } E_1^*(y,r^{s+1}(x)),$$

whence $l_1\cdot l_2=\{3s\}$.

Let now $l_2=3k-2$, i.e. $K_0(r^{k-1}(t),y,t)$ and $E_1^*(y,r^{k-1}(t))$. Whence we obtain:

$$E_1^*(y,r^{m_1-1}(x)) \text{ and either } K_0(r^{m_1-2}(x),y,r^{m_1-1}(x)) \text{ or } K_0(r^{m_1-1}(x),y,r^{m_1}(x)).$$

Consequently, $l_1\cdot l_2=\{3(m_1-1),3m_1-2\}$.

Consider the product $l_2\cdot l_1$. We have: $K_0(r^{k-1}(x),t,x)$, $E_1^*(t,r^{k-1}(x))$, $K_0(r^{m_1-1}(t),y,r^{m_1}(t))$, and $E_1^*(y,r^{m_1}(t))$. Whence we obtain:

$$E_1^*(y,r^{m_1-1}(x)) \text{ and either } K_0(r^{m_1-2}(x),y,r^{m_1-1}(x)) \text{ or } K_0(r^{m_1-1}(x),y,r^{m_1}(x)),$$

i.e. $l_2\cdot l_1=\{3(m_1-1),3m_1-2\}$.

Let now $l_2=3k-1$, i.e. $K_0(r^{k-1}(t),y,t)$, $\neg E_1^*(y,r^{k-1}(t))$ and $\neg E_1(y,t)$. Whence we obtain:

$$K_0(r^{m_1-1}(x),y,r^{m_1}(x)), \neg E_1^*(y,r^{m_1-1}(t)), \text{ and } \neg E_1^*(y,r^{m_1}(t)),$$

i.e. $l_1\cdot l_2=\{3m_1-1\}$.

Consider the product $l_2\cdot l_1$. We have the following: $K_0(r^{k-1}(x),t,x)$, $\neg E_1^*(t,r^{k-1}(x))$, $\neg E_1^*(x,t)$, $K_0(r^{m_1-1}(t),y,r^{m_1}(t))$ and $E_1^*(y,r^{m_1}(t))$. Whence we obtain:

$$K_0(r^{m_1-1}(x),y,r^{m_1}(x)), \neg E_1^*(y,r^{m_1-1}(t)), \text{ and } \neg E_1^*(y,r^{m_1}(t)),$$

i.e. $l_2\cdot l_1=\{3m_1-1\}$.

Let now $l_2=3k$, i.e. $K_0(r^{k-1}(t),y,t)$, and $E_1(y,t)$. Whence we obtain:

$$K_0(r^{m_1-1}(x),y,r^{m_1}(x)) \text{ and } E_1(y,r^{m_1}(x)),$$

i.e. $l_1\cdot l_2=\{3m_1\}$.

Consider the product $l_2\cdot l_1$. Then we have: $K_0(r^{k-1}(x),t,x)$, $E_1(x,t)$, $K_0(r^{m_1-1}(t),y,r^{m_1}(t))$ and $E_1^*(y,r^{m_1}(x))$. Whence we obtain: $K_0(r^{m_1-1}(x),y,r^{m_1}(x))$ and $E_1^*(y,r^{m_1}(x))$, i.e. $l_2\cdot l_1=\{3m_1\}$.

Case 4. $l_1=3k-2$.

We have: $K_0(r^{k-1}(x),t,x)$ and $E_1^*(t,r^{k-1}(x))$.

Let $l_2 = 3k-2$, i.e. $K_0(r^{k-1}(t), y, t)$ and $E_1^*(y, r^{k-1}(t))$. Whence we obtain the following: $K_0(r^{k-2}(x), y, r^{k-1}(x))$ and $E_1^*(y, r^{k-1}(x))$, i.e. $l_1 \cdot l_2 = \{3(k-1)-2\}$.

Let now $l_2 = 3k-1$, i.e. $K_0(r^{k-1}(t), y, t)$, $\neg E_1^*(y, r^{k-1}(t))$ and $\neg E_1(y, t)$. Whence we obtain:

$$K_0(r^{k-2}(x), y, r^{k-1}(x)), \neg E_1^*(y, r^{k-2}(x)), \text{ and } \neg E_1(y, r^{k-1}(x)),$$

i.e. $l_1 \cdot l_2 = \{3(k-1)-1\}$.

Consider the product $l_2 \cdot l_1$. We have the following: $K_0(r^{k-1}(x), t, x)$, $\neg E_1^*(t, r^{k-1}(x))$, $\neg E_1^*(x, t)$, $K_0(r^{m_1-1}(t), y,\ r^{m_1}(t))$ and $E_1^*(y, r^{m_1}(t))$. Whence we obtain:

$$\neg E_1^*(y, r^{k-2}(x)), K_0(r^{k-2}(x), y, r^{k-1}(x)) \text{ and } \neg E_1(y, r^{k-1}(x)),$$

i.e. $l_2 \cdot l_1 = \{3(k-1)-1\}$.

Let now $l_2 = 3k$, i.e. $K_0(r^{k-1}(t), y, t)$, and $E_1(y, t)$. Whence we obtain:

$$E_1^*(y, r^{k-1}(x)) \text{ and either } K_0(r^{k-2}(x), y, r^{k-1}(x)) \text{ or } K_0(r^{k-1}(x), y, x),$$

i.e. $l_1 \cdot l_2 = \{3k-2, 3(k-1)\}$.

Consider the product $l_2 \cdot l_1$. Then we have: $K_0(r^{k-1}(x), t, x)$, $E_1(x, t)$, $K_0(r^{k-1}(t), y, t)$ and $E_1^*(y, r^{k-1}(t))$. Whence we obtain:

$$E_1^*(y, r^{k-1}(x)) \text{ and either } K_0(r^{k-2}(x), y, r^{k-1}(x)) \text{ or } K_0(r^{k-1}(x), y, x),$$

i.e. $l_2 \cdot l_1 = \{3k-2, 3(k-1)\}$.

Case 5. $l_1 = 3k-1$.

We have: $K_0(r^{k-1}(x), t, x)$, $\neg E_1^*(t, r^{k-1}(x))$ and $\neg E_1(t, x)$.

Let now $l_2 = 3k-1$, i.e. $K_0(r^{k-1}(t), y, t)$, $\neg E_1^*(y, r^{k-1}(t))$ and $\neg E_1(y, t)$. Whence we obtain:

$$K_0(r^{k-2}(x), y, x), \neg E_1^*(y, r^{k-2}(x)), \text{ and } \neg E_1(y, x),$$

i.e. $l_1 \cdot l_2 = \{3k-4, 3k-3, 3k-2, 3k-1\}$.

Let now $l_2 = 3k$, i.e. $K_0(r^{k-1}(t), y, t)$, and $E_1(y, t)$. Whence we obtain:

$$K_0(r^{k-1}(x), t, x), \neg E_1^*(t, r^{k-1}(x)) \text{ and } \neg E_1(t, x),$$

i.e. $l_1 \cdot l_2 = \{3k-1\}$.

Consider the product $l_2 \cdot l_1$. Then we have: $K_0(r^{k-1}(x), t, x)$, $E_1(x, t)$, $K_0(r^{k-1}(t), y, t)$, $\neg E_1^*(y, r^{k-1}(t))$ and $\neg E_1(y, t)$. Whence we obtain:

$$K_0(r^{k-1}(x), t, x), \neg E_1^*(t, r^{k-1}(x)) \text{ and } \neg E_1(t, x),$$

i.e. $l_2 \cdot l_1 = \{3k-1\}$.

Case 6. $l_1 = 3k$.

We have: $K_0(r^{k-1}(x), t, x)$ and $E_1(t, x)$.

Let $l_2 = 3k$, i.e. $K_0(r^{k-1}(t), y, t)$ and $E_1(y, t)$. Whence we obtain: $K_0(r^{k-1}(x), t, x)$ and $E_1(y, x)$, i.e. $l_1 \cdot l_2 = \{3k\}$.

Thus, we established that the algebra $\mathfrak{P}_{M'_{1,1,k}}$ is commutative and strictly 5-deterministic for every $k \geq 2$.

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Кулпешов Б.Ш., Судоплатов С.В. ДӨҢЕСТІК РАНГІСІ 2 ӘЛСІЗ ЦИКЛДІК МИНИМАЛДЫ ТЕОРИЯЛАР ҮШІН БИНАРЛЫҚ ОҚШАУЛАУ ФОРМУЛАЛАРЫ АЛГЕБРАЛАРЫНДА

Бұл жұмыс циклдік реттелген әлсіз циклді минималды құрылымдарды зерттеуге арналған. Циклдік тәртіптің ең қарапайым мысалы — соңғы нүктелері бар сызықтық тәртіп, онда ең үлкен элемент ең кішімен сәйкестендіріледі. Тағы бір мысал, шеңбер бойымен жүру кезінде пайда болатын тәртіп. Циклдік реттелген құрылым, егер оның формулалық ішкі жиындарының кез келгені дөңес жиындар мен нүктелердің ақырлы бірлестігі болса, оны әлсіз циклдік минималды деп атайды. Теория әлсіз циклдік минималды деп аталады, егер оның барлық модельдері әлсіз циклдік минималды болса. Біз құрылымның анықталатын аяқталуына оң-монотонды функцияға ие және құрылымның негізгі жиынын дөңес класстарына шектеулі санына бөлетін тривиалды емес эквиваленттік қатынас бар тривиалды анықталатын түйықталуына ие дөңестік рангісі 2 санаулы категориялық 1-отпелі примитивтік емес әлсіз циклдік минималды теориялары ушін бинарлық оқшаулау формулаларынын алгебрасын сипаттаймыз.

Түйін сөздер: бинарлық формулалар алгебрасы, $\aleph_0$-категориялық теория, әлсіз циклдік минималдылық, циклдік реттелген құрылым, дөңестік рангісі.

Кулпешов Б.Ш., Судоплатов С.В. ОБ АЛГЕБРАХ БИНАРНЫХ ИЗОЛИРУЮЩИХ ФОРМУЛ ДЛЯ СЛАБО ЦИКЛИЧЕСКИ МИНИМАЛЬНЫХ ТЕОРИЙ РАНГА ВЫПУКЛОСТИ 2

Данная работа посвящена исследованию слабо циклически минимальных циклически упорядоченных структур. Простейший пример циклического порядка — это линейный порядок с концевыми точками, в котором наибольший элемент отождествили с наименьшим. Другой пример — это порядок, возникающий при обходе окружности. Циклически упорядоченная структура называется слабо циклически минимальной, если любое ее формульное подмножество является конечным объединением выпуклых множеств и точек. Теория называется слабо циклически минимальной, если все ее модели являются слабо циклически минимальными. Описываются алгебры бинарных изолирующих формул для счетно категоричных 1-транзитивных непримитивных слабо циклически минимальных теорий ранга выпуклости 2 с тривиальным определимым замыканием, имеющих монотонную вправо функцию в определимое пополнение структуры и не имеющих нетривиального отношения эквивалентности, разбивающего основное множество структуры на конечное число выпуклых классов.

Ключевые слова. алгебра бинарных формул, $\aleph_0$-категоричная теория, слабая циклическая минимальность, циклически упорядоченная структура, ранг выпуклости.

Kazakh Mathematical Journal ISSN 1682–0525

24:4 (2024) 22–36

On the solvability of the Dirichlet problem for the viscous Burgers equation

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Communicated by: Marat I. Tleubergenov

Received: 06.12.2024 ⋆ Accepted/Published Online: 03.01.2025 ⋆ Final Version: 03.01.2025

Abstract. In this work, we study a Dirichlet problem for the viscous Burgers equation in a domain with moving boundaries that degenerates at the initial moment. The primary method of investigation is the Galerkin method, for which we construct an orthonormal basis suitable for domains with moving boundaries. Uniform a priori estimates are obtained, and based on these, theorems on the unique solvability of the problem are proven using methods of functional analysis. The viscous Burgers equation serves as a simplified model for studying fundamental aspects of nonlinear systems. It bridges the gap between purely theoretical nonlinear equations (like the inviscid Burgers equation) and more complex systems like the Navier-Stokes equations, making it a valuable tool in mathematical and physical research.

Keywords. Burgers equation, a priori estimates, Galerkin method.

1 Introduction

Let $\Omega = \{x, t \,|\, \varphi_1(t) < x < \varphi_2(t),\ 0 < t < T < \infty\}$ be a domain that degenerates into a point. The functions $\varphi_1(t)$ and $\varphi_2(t)$ are defined on $[0, T]$ and are strictly monotonically decreasing and increasing functions, respectively, which belong to $C^1(0, T)$ with $\varphi_1(0) = \varphi_2(0)$ and $\Omega_t = (\varphi_1(t), \varphi_2(t))$ for $t \in (0, T)$.

The study of solvability issues for initial-boundary value problems in domains with moving boundaries, namely, in domains whose boundaries change over time, has been the focus of numerous works; we note only a few of them [1, 2, 3, 4, 5]. In these works, we observed that the lack of a suitable basis applicable to such domains necessitates transforming these domains into ones with stationary boundaries. This transformation leads to the need to study several auxiliary problems, significantly complicating the research process. Previously,

2010 Mathematics Subject Classification: 35K55; 35R37, 65N30, 35B45.

Funding: This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. BR20281002).

DOI: https://doi.org/10.70474/f392sv46

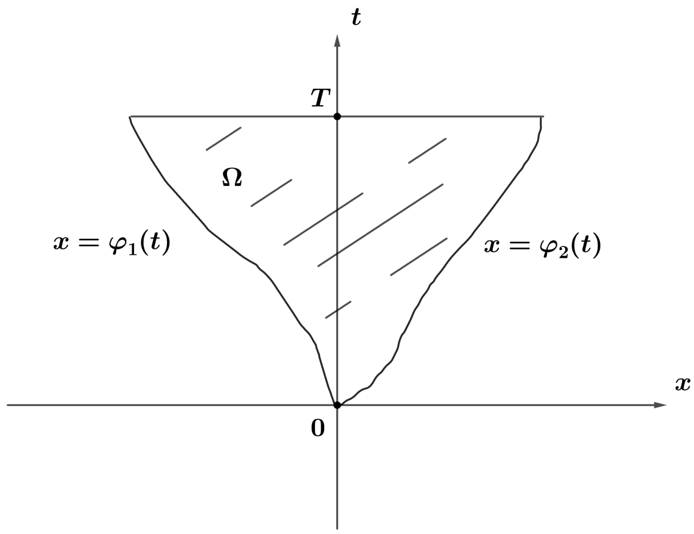

Figure 1: The degenerating domain Ω.

in work [6], we constructed an orthonormal basis and demonstrated its application to solving initial-boundary value problems in degenerate domains.

In this paper, in the domain Ω we are studying the solvability issues of the following boundary value problem for viscous Burgers equation:

$$\partial_t u(x,t) + u(x,t)\partial_x u(x,t) - \nu\partial_x^2 u(x,t) + \partial_x u(x,t) = f(x,t), \ \ (x,t) \in \Omega, \tag{1}$$

with homogeneous boundary conditions

$$u(\varphi_1(t), t) = u(\varphi_2(t), t) = 0, \ \ t \in (0, T). \tag{2}$$

We look for some conditions for functions $\varphi_1(t)$ and $\varphi_2(t)$ such that the problem (1)–(2) admits a unique solution. So, to establish the unique solvability of the problem (1)–(2) we suppose that

$$|\varphi'(t)| \leq \gamma \text{ for all } t \in [0, T], \ \varphi(t) = \varphi_2(t) - \varphi_1(t), \, \gamma = \text{const} > 0. \tag{3}$$

Here is our main result on the problem(1)–(2):

Theorem 1. *Let $f(x,t) \in L^2(\Omega)$ and conditions* (3) *be satisfied. Then boundary value problem* (1)–(2) *has a unique solution*

$$u \in H_0^{2,1}(Q) \equiv \left\{ L^2(0, T; H_0^2(\Omega_t)) \cap H^1(0, T; L^2(\Omega_t)) \right\}.$$

In work [7], the homogeneous version of problem (1)–(2) was previously studied in a non-degenerate domain, where theoretical and numerical results were obtained by the authors. In works [8, 9], the authors investigated the existence of solutions to boundary value problems for the Burgers equation in both degenerate and non-degenerate domains.

The paper is divided as follows: Section 2 investigates one auxiliary initial boundary value problem for the Burgers equation in the non-degenerate domain, where $\varphi_1(1/n) \neq \varphi_2(1/n)$. In Section 3, we obtain the necessary a priori estimates. In Section 4, we solve one spectral problem and construct the necessary orthonormal basis, then, based on the obtained basis, we introduce an approximate solution. In this section, we also prove the solvability of the Cauchy problem for the coefficients of the approximate solution. The unique solvability of the auxiliary problem is given in Section 5. Section 6 is devoted to the proof of the main result. A brief conclusion completes the work.

2 Statement of auxiliary problem

We introduce the family of domains $\Omega^n = \{x, t \,|\, \varphi_1(t) < x < \varphi_2(t),\ 1/n < t < T\}$, $n \in \mathbb{N}^*$, $n \in \mathbb{N}^* \equiv \{n \in \mathbb{N} \,| n \geq n_1,\ 1/n_1 < T\}$. These domains Ω^n are "curvilinear" trapezoids for which $\varphi_1(1/n) \neq \varphi_2(1/n)$ holds and now the domains do not degenerate at the point $t = 1/n$. We aslo note that between the initial domain Ω and domains Ω^n there are strict embeddings $\Omega^n \subset \Omega^{n+1} \subset ... \subset \Omega$ and, obviously, that $\lim\limits_{n\to\infty} \Omega^n = \Omega$.

In the domains Ω^n, we will consider the following initial boundary value problems for the Burgers equation with respect to the functions $u_n(x,t)$:

$$\partial_t u_n(x,t) + u_n(x,t)\partial_x u_n(x,t) - \nu\partial_x^2 u_n(x,t) + \partial_x u_n(x,t) = f_n(x,t), \tag{4}$$

with homogeneous boundary

$$u_n(\varphi_1(t),t) = u_n(\varphi_2(t),t) = 0, \ \ t \in (1/n, T), \tag{5}$$

and initial conditions

$$u_n(x, 1/n) = 0, \ \ \ x \in \Omega_{1/n} = (\varphi_1(1/n), \varphi_2(1/n)). \tag{6}$$

Obviously, if $f(x,t) \in L^2(\Omega)$, then $f_n(x,t) \in L^2(\Omega^n)$, where $f_n(x,t)$ is the restriction of function $f(x,t) \in L^2(\Omega)$ to domains Ω^n.

For the problem (4)–(6) we have the following

Theorem 2. *For every fixed $n \in \mathbb{N}^*$ the initial-boundary value problem* (4)–(6) *is uniquely solvable in the space $u_n(x,t) \in H_0^{2,1}(\Omega^n)$.*

3 A priori estimates

Lemma 3. *There is a positive, independent of* n*, constants* K_1*,* K_2 *and* K_3*, such that for all* $t \in [1/n, T]$ *we have estimates*

$$\|u_n(x,t)\|^2_{L^2(\Omega_t)} + \int\limits_{1/n}^{t} \|\partial_x u_n(x,\tau)\|^2_{L^2(\Omega_\tau)} d\tau \le K_1 \|f_n(x,t)\|^2_{L^2(\Omega^n)}, \tag{7}$$

$$\|\partial_x u_n(x,t)\|^2_{L^2(\Omega_t)} + \int\limits_{1/n}^{t} \|\partial_x^2 u_n(x,\tau)\|^2_{L^2(\Omega_\tau)} d\tau \le K_2 \|f_n(x,t)\|^2_{L^2(\Omega^n)}, \tag{8}$$

$$\|\partial_t u_n(x,t)\|^2_{L^2(\Omega^n)} \le K_3 \|f_n(x,t)\|^2_{L^2(\Omega^n)}. \tag{9}$$

Proof. We start with the proof of the first a priori estimate. Multiplying the equation (4) by the function $u_n(x,t)$ scalarly in $L^2(\Omega_t)$ and using the ε-Cauchy inequlity we get

$$\frac{d}{dt}\|u_n(x,t)\|^2_{L^2(\Omega_t)} + 2\nu \|\partial_x u_n(x,t)\|^2_{L^2(\Omega_t)} \le \|f_n(x,t)\|^2_{L^2(\Omega_t)} + \|u_n(x,t)\|^2_{L^2(\Omega_t)}. \tag{10}$$

By applying the Gronwall inequality to (10), we obtain the estimate (7).

Let us proceed to the proof of the second a priori estimate. Multiplying the equation (4) by $-\partial_x^2 u_n(x,t)$ scalarly in $L^2(\Omega_t)$ we get

$$\frac{d}{dt}\|\partial_x u_n(x,t)\|^2_{L^2(\Omega_t)} + 2\nu \|\partial_x^2 u_n(x,t)\|^2_{L^2(\Omega_t)} \le 2\left| \int\limits_{\varphi_1(t)}^{\varphi_2(t)} u_n(x,t)\partial_x u_n(x,t)\partial_x^2 u_n(x,t) dx \right|$$

$$+2\left| \int\limits_{\varphi_1(t)}^{\varphi_2(t)} f_n(x,t)\partial_x^2 u_n(x,t) dx \right| + 2\left| \int\limits_{\varphi_1(t)}^{\varphi_2(t)} \partial_x u(x,t)\partial_x^2 u(x,t) dx \right|$$

$$+\gamma \left(|\partial_x u_n(\varphi_2(t),t)|^2 + |\partial_x u_n(\varphi_1(t),t)|^2 \right). \tag{11}$$

To estimate the nonlinear term in the right-hand side of (11) we use the following inequality ([10], Theorems 5.8–5.9, p.140–141)

$$\|\partial_x u_n(x,t)\|_{L^4(\Omega_t)} \le K \|\partial_x u_n(x,t)\|^{1/2}_{H^1(\Omega_t)} \|\partial_x u_n(x,t)\|^{1/2}_{L^2(\Omega_t)}, \ \forall\, \partial_x u_n(x,t) \in H^1(\Omega_t),$$

Young's inequality $(r^{-1} + s^{-1} = 1)$:

$$|UV| = \left| \left(\Theta^{1/r} U\right) \left(\Theta^{1/s} \frac{V}{\Theta}\right) \right| \le \frac{\Theta}{r} |U|^r + \frac{\Theta}{s\Theta^s} |V|^s,$$

with $\Theta = \nu/6,\ r = 4/3,\ s = 4,$

$$U = \|\partial_x u_n(x,t)\|^{3/2}_{H^1(\Omega_t)},\ \ V = K\,\|u_n(x,t)\|_{L^4(\Omega_t)}\,\|\partial_x u_n(x,t)\|^{1/2}_{L^2(\Omega_t)}.$$

After which we will get

$$\Big|\int\limits_{\varphi_1(t)}^{\varphi_2(t)} u_n(x,t)\partial_x u_n(x,t)\partial_x^2 u_n(x,t)dx\Big| \le \frac{\nu}{8}\|\partial_x^2 u_n(x,t)\|^2_{L^2(\Omega_t)}$$

$$+\left[\frac{\nu}{8}+\frac{54}{\nu^3}K^4\|u_n(x,t)\|^4_{L^4(\Omega_t)}\right]\|\partial_x u_n(x,t)\|^2_{L^2(\Omega_t)}. \tag{12}$$

For the remaining terms in (11) we have

$$\Big|\int\limits_{\varphi_1(t)}^{\varphi_2(t)} f_n(x,t)\partial_x^2 u_n(x,t)dx\Big| \le \frac{\nu}{8}\|\partial_x^2 u_n(x,t)\|^2_{L^2(\Omega_t)}+\frac{2}{\nu}\|f_n(x,t)\|^2_{L^2(\Omega_t)}, \tag{13}$$

$$\Big|\int\limits_{\varphi_1(t)}^{\varphi_2(t)} \partial_x u(x,t)\partial_x^2 u(x,t)dx\Big| \le \frac{2}{\nu}\|\partial_x u_n(x,t)\|^2_{L^2(\Omega_t)}+\frac{\nu}{8}\|\partial_x^2 u_n(x,t)\|^2_{L^2(\Omega_t)}, \tag{14}$$

$$\gamma|\partial_x u_n(\varphi_i(t),t)|^2 \le \gamma\|\partial_x u_n(x,t)\|^2_{L^\infty(\Omega_t)} \le K^2\gamma\|\partial_x u_n(x,t)\|_{H^1(\Omega_t)}\|\partial_x u_n(x,t)\|_{L^2(\Omega_t)}$$

$$= K^2\gamma\|\partial_x u_n(x,t)\|_{L^2(\Omega_t)}\left[\|\partial_x u_n(x,t)\|_{L^2(\Omega_t)}+\|\partial_x^2 u_n(x,t)\|_{L^2(\Omega_t)}\right]$$

$$\le \frac{\nu}{8}\|\partial_x^2 u_n(x,t)\|^2_{L^2(\Omega_t)}+\left[K^2\gamma+\frac{K^4\gamma^2}{2\nu}\right]\|\partial_x u_n(x,t)\|^2_{L^2(\Omega_t)},\ i=1,2. \tag{15}$$

Based on inequalities (11)–(15) we obtain:

$$\frac{d}{d\,t}\|\partial_x u_n(x,t)\|^2_{L^2(\Omega_t)}+\nu\|\partial_x^2 u_n(x,t)\|^2_{L^2(\Omega_t)} \le C_1\|f_n(x,t)\|^2_{L^2(\Omega_t)}+C_2\|\partial_x u_n(x,t)\|^2_{L^2(\Omega_t)}, \tag{16}$$

where $C_1 = \frac{4}{\nu},\ C_2 = \frac{\nu}{4}+\frac{4}{\nu}+\frac{108K^4}{\nu^3}C_5^4+\frac{\gamma K^2\nu+2\gamma^2K^4}{\nu}$, since

$$\|u_n(x,t)\|_{L^4(\Omega_t)} \le \max_{1/n\le t\le T}\sqrt[4]{\varphi(t)}\|u_n(x,t)\|_{L^\infty(\Omega_t)} \le C_4\|u_n(x,t)\|_{H^1(\Omega_t)} \le C_5,$$

where

$$C_4 = \max_{1/n\le t\le T}\sqrt[4]{\varphi(t)}C_3.$$

From inequality (16) similarly as in the proof of the first a priori estimate we obtain the required estimate (8).

Now, let us proceed to the proof of the final a priori estimate. From equation (4) we have

$$\|\partial_t u_n(x,t)\|_{L^2(\Omega^n)} \leq \nu\|\partial_x^2 u_n(x,t)\|_{L^2(\Omega^n)} + \|f_n(x,t)\|_{L^2(\Omega^n)}$$
$$+\|\partial_x u_n(x,t)\|_{L^2(\Omega^n)} + \|u_n(x,t)\partial_x u_n(x,t)\|_{L^2(\Omega^n)}. \tag{17}$$

According to (8) we need to estimate the last term in (17) only. Using the embedding $H^1(\Omega_t) \hookrightarrow L^\infty(\Omega_t)$ and estimates (7) and (8) we have

$$\|u_n(x,t)\partial_x u_n(x,t)\|^2_{L^2(\Omega^n)} \leq C_6 \int\limits_{1/n}^{T} \|u_n(x,t)\|^2_{H^1(\Omega_t)}\|\partial_x u_n(x,t)\|^2_{L^2(\Omega_t)}dt$$
$$\leq C_6\|u_n(x,t)\|^2_{L^\infty(1/n,T;H^1(\Omega_t))}\|\partial_x u_n(x,t)\|^2_{L^2(Q^n)} \leq C_7\|f_n(x,t)\|^2_{L^2(\Omega^n)}, \tag{18}$$

where $C_7 = K_1K_2C_6T$, and K_1, K_2 are the constants from (7) and (8).

Based on inequalities (17)–(18) we establish the estimate (9). This completes the proof of Lemma 3. □

4 Spectral problem and approximate solution

4.1 Spectral problem

To apply the Faedo-Galerkin approach, it is necessary to resolve the corresponding spectral problem

$$-\partial_x^2 Y_k(x,t) = \lambda_k^2(t)Y_k(x,t),\ (x,t) \in \Omega^n,\ k \in \mathbb{N}_0, \tag{19}$$
$$Y_k(\varphi_1(t),t) = Y_k(\varphi_2(t),t) = 0. \tag{20}$$

The solution to this problem is sought in the form

$$Y_k(x,t) = A_k(t)\cos(\lambda_k(t)x) + B_k(t)\sin(\lambda_k(t)x). \tag{21}$$

Using the conditions (20) from (21) we get:

$$\begin{cases} A_k(t)\cos(\lambda_k(t)\varphi_1(t)) + B_k(t)\sin(\lambda_k(t)\varphi_1(t)) = 0, \\ A_k(t)\cos(\lambda_k(t)\varphi_2(t)) + B_k(t)\sin(\lambda_k(t)\varphi_2(t)) = 0. \end{cases} \tag{22}$$

For the system (22) to admit a nontrivial solution, the following condition must hold:

$$\begin{vmatrix} \cos(\lambda_k(t)\varphi_1(t)) & \sin(\lambda_k(t)\varphi_1(t)) \\ \sin(\lambda_k(t)\varphi_2(t)) & \cos(\lambda_k(t)\varphi_2(t)) \end{vmatrix} = 0,$$

From where we obtain

$$\sin\left(\lambda_k(t)\varphi(t)\right) = 0,\ \ k \in \mathbb{N}_0,$$

hence

$$\lambda_k(t) = \frac{\pi k}{\varphi(t)},\ \ k \in \mathbb{N}_0. \tag{23}$$

From (22) we also obtain

$$A_k(t) = -B_k(t)\frac{\sin\lambda_k(t)\varphi_1(t)}{\cos\lambda_k(t)\varphi_1(t)}. \tag{24}$$

Substituting (24) into (21) and choosing

$$B_k(t) = \frac{\sqrt{2}\cos\lambda_k(t)\varphi_1(t)}{\sqrt{\varphi(t)}},$$

we have

$$Y_k(x,t) = \frac{\sqrt{2}}{\sqrt{\varphi(t)}}\sin\left(\lambda_k(t)(x-\varphi_1(t))\right),\ \ \lambda_k^2(t) = \left(\frac{\pi k}{\varphi(t)}\right)^2,\ \ k \in \mathbb{N}_0. \tag{25}$$

4.2 Approximate solution

The following approximate solution

$$u_n^N(x,t) = \sum_{j=1}^{N} c_j(t)Y_j(x,t),\ \ u_n^N(x,1/n) = \sum_{j=1}^{N} c_j(1/n)Y_j(x,1/n) = 0, \tag{26}$$

is introduced and utilized to solve the problem (4)–(6):

$$\int\limits_{\varphi_1(t)}^{\varphi_2(t)}\left[\partial_t u_n^N(x,t)dx + u_n^N(x,t)\partial_x u_n^N(x,t) - \nu\partial_x^2 u_n^N(x,t) + \partial_x u_n^N(x,t)\right]Y_k(x,t)dx$$

$$= \int\limits_{\varphi_1(t)}^{\varphi_2(t)} f_n(x,t)Y_k(x,t)dx, \tag{27}$$

$$u_n^N(x,1/n) = 0,\ \ x \in \Omega_{1/n}, \tag{28}$$

for all $k = 1,\ldots,N$ and $t \in [1/n,T]$.

Lemma 4. *The problem* (27)–(28) *has a unique solution* $C(t) = \{c_j(t)\}_{j=1}^N$.

Proof. Given that the system of functions $\{Y_k(x,t)\}_{k\in\mathbb{N}_0}$ forms an orthonormal basis in $L^2(\Omega_t)$ for $\mathbb{N}_0 = \{0,1,2,\dots\}$, it follows that for any finite N:

$$\int\limits_{\varphi_1(t)}^{\varphi_2(t)} \partial_t u_n^N(x,t) Y_k(x,t)dx = \sum_{j=1}^{N} c_j'(t) \int\limits_{\varphi_1(t)}^{\varphi_2(t)} Y_j(x,t)Y_k(x,t)dx$$

$$+\sum_{j=1}^{N} c_j(t) \int\limits_{\varphi_1(t)}^{\varphi_2(t)} Y_j'(x,t)Y_k(x,t)dx = c_j'(t) + S_1(t)c_j(t),$$

where for all $k = 1,\dots,N$

$$S_1(t)c_j(t) = \left(I_1(t) + I_2(t) + I_3(t)\right) c_j(t),$$

$$I_1(t)c_j(t) = -\frac{\varphi'(t)}{\varphi^2(t)} \sum_{j=1}^{N} c_j(t) \int\limits_{\varphi_1(t)}^{\varphi_2(t)} \sin\left(\lambda_j(t)(x-\varphi_1(t))\right) \sin\left(\lambda_k(t)(x-\varphi_1(t))\right) dx,$$

$$I_2(t)c_j(t) = \left(\frac{2\pi}{\varphi^2(t)} \left(\frac{\varphi'(t)\varphi_1(t)}{\varphi(t)} - \varphi_1'(t)\right)\right) \sum_{j=1}^{N} j c_j(t)$$

$$\cdot \int\limits_{\varphi_1(t)}^{\varphi_2(t)} \cos\left(\lambda_j(t)(x-\varphi_1(t))\right) \sin\left(\lambda_k(t)(x-\varphi_1(t))\right) dx,$$

$$I_3(t)c_j(t) = -\frac{2\pi\varphi'(t)}{\varphi^3(t)} \sum_{j=1}^{N} j c_j(t) \int\limits_{\varphi_1(t)}^{\varphi_2(t)} x\cos\left(\lambda_j(t)(x-\varphi_1(t))\right) \sin\left(\lambda_k(t)(x-\varphi_1(t))\right) dx.$$

From (19) we have $\partial_x^2 u_n^N(x,t) = -\sum\limits_{j=1}^{N} \lambda_j^2(t)c_j(t)Y_j(x,t)$. Then, for all $t \in [1/n,\, T]$,

$$-\int\limits_{\varphi_1(t)}^{\varphi_2(t)} \partial_x^2 u_n^N(x,t)Y_k(x,t)dx = \sum_{j=1}^{N} \lambda_j^2(t)c_j(t) \int\limits_{\varphi_1(t)}^{\varphi_2(t)} Y_j(x,t)Y_k(x,t)dx = \lambda_j^2(t)c_j(t).$$

For the nonlinear term we have

$$\int\limits_{\varphi_1(t)}^{\varphi_2(t)} u_n^N(x,t)\partial_x u_n^N(x,t)Y_k(x,t)dx$$

$$= \int\limits_{\varphi_1(t)}^{\varphi_2(t)} \sum_{l=1}^{N} c_l(t) Y_l(x,t) \sum_{m=1}^{N} c_m(t) \partial_x Y_m(x,t) Y_k(x,t) dx = S_2(t) c_{lm}(t).$$

For the last term we have

$$\int\limits_{\varphi_1(t)}^{\varphi_2(t)} \partial_x u_n^N(x,t) Y_k(x,t) dx = \sum_{j=1}^{N} c_j(t) \int\limits_{\varphi_1(t)}^{\varphi_2(t)} \partial_x Y_j(x,t) Y_k(x,t) dx = S_3(t) c_j(t)$$

For $j \in \mathbb{N}$, the problem (27)–(28) can be reformulated as the following Cauchy problem for a finite system of nonlinear ordinary differential equations:

$$c_j'(t) = \left(-S_1(t) - \nu \lambda_j^2(t) - S_3(t)\right) c_j(t) - S_2(t) c_{lm}(t) + g_j(t), \;\; c_j(1/n) = 0, \tag{29}$$

where

$$g_j(t) = \int\limits_{\varphi_1(t)}^{\varphi_2(t)} f_n(x,t) Y_j(x,t) dx, \;\; j \in \mathbb{N}.$$

Since $f(x,t) \in L^2(Q)$, it follows that $g_k(t)$ is a square-integrable function, and function $S_1(t)$ is well defined. Consequently, the Cauchy problem (29) has a unique solution on some interval $[1/n, T']$, where $T' \leq T$. Moreover, due to the a priori estimates provided in Lemma 3 in Section 3, the solution $C(t)$ can be extended up to the finite time T.

Thus, for any fixed finite N, the functions $C(t) = \{c_j(t)\}_{j=1}^{N}$ are determined as the solution to the Cauchy problem (29). Along with these, the unique approximate solution $u_n^N(x,t)$ to problem (27)–(28) is obtained. This concludes the proof of Lemma 4. □

5 Solvatility of auxiliary problem

5.1 Proof of Theorem 2. Existence

By virtue of Lemma 3 we can extract weakly convergent subsequences from bounded sequences $\{u_n^N(x,t),\, \partial_t u_n^N(x,t)\, N = 1, 2, ...\}$:

$$u_n^N(x,t) \to u_n(x,t) \;\; \text{weakly in } L^2(1/n, T; H_0^2(\Omega_t)) \cap H^1(1/n, T; L^2(\Omega_t)), \tag{30}$$

$$u_n^N(x,t) \to u_n(x,t) \;\; \text{strongly in } L^2(1/n, T; L^2(\Omega_t)) \;\; \text{and a.e. in } \Omega^n. \tag{31}$$

We introduce the new function $w_j(x,t) = \psi(t) Y_j(x,t)$, where $Y_j(x,t) \in H_0^2(\Omega_t)$ and $\psi(t) \in C^1([1/n, T])$. Next, we multiply the identity (27) by $\psi(t) \in C^1([1/n, T])$ and after that we integrate the resulting expression with respect to t over the interval $[1/n, T]$ for

$j = 1, \dots, N$ and use the fact that the set of all linear combinations of $\{w_j(x,t)\}$ is dense in $L^2(1/n, T; H_0^2(\Omega_t))$. Thus, we obtain:

$$\int_{1/n}^{T} \int_{\varphi_1(t)}^{\varphi_2(t)} \big[\partial_t u_n^N(x,t) + u_n^N(x,t)\partial_x u_n^N(x,t) - \nu \partial_x^2 u_n^N(x,t) + \partial_x u_n^N(x,t) - f(x,t)\big] w(x,t) dx dt = 0,$$

$$\forall\, w(x,t) \in L^2(1/n, T; H_0^2(\Omega_t)). \tag{32}$$

In the identity (32) we take the limit as $N \to \infty$. For the linear terms in equation (4), the passage to the limit is performed using the relations (30) and (31). Regarding the nonlinear term, as $N \to \infty$ we arrive at the following result:

$$\int_{1/n}^{T} \int_{\varphi_1(t)}^{\varphi_2(t)} [u_n^N(x,t) - u_n(x,t)] \partial_x u_n^N(x,t) w(x,t)\, dx\, dt$$

$$+ \int_{1/n}^{T} \int_{\varphi_1(t)}^{\varphi_2(t)} u_n(x,t) \partial_x u_n^N(x,t) w(x,t)\, dx\, dt \to \int_{1/n}^{T} \int_{\varphi_1(t)}^{\varphi_2(t)} u_n(x,t) \partial_x u_n(x,t) w(x,t)\, dx\, dt, \tag{33}$$

since according to (30)–(31) there exists a limiting relationship

$$\int_{1/n}^{T} \int_{\varphi_1(t)}^{\varphi_2(t)} [u_n^N(x,t) - u_n(x,t)] \partial_x u_n^N(x,t) w(x,t)\, dx\, dt \to 0.$$

Thus, by passing to the limit as $N \to \infty$ in the identity (32), and taking into account the limiting relation (33) along with the initial condition (28), we obtain:

$$\int_{1/n}^{T} \int_{\varphi_1(t)}^{\varphi_2(t)} \big[\partial_t u_n(x,t) + u_n(x,t)\partial_x u_n(x,t) - \nu \partial_x^2 u_n(x,t) + \partial_x u_n(x,t) - f(x,t)\big]\, w(x,t) dx dt = 0,$$

$$\forall\, w(x,t) \in L^2(1/n, T; H_0^2(\Omega_t)), \tag{34}$$

$$\int_{\varphi_1(1/n)}^{\varphi_2(1/n)} u_n(x, 1/n) \theta(x)\, dx = 0, \ \ \forall\, \theta(x) \in L^2(\Omega_{1/n}). \tag{35}$$

Thus, from (34)–(35), it follows that the limiting function $u_n(x,t)$ satisfies equation (4) along with the boundary and initial conditions (5)–(6).

5.2 Proof of Theorem 2. Uniqueness

We suppose that the initial boundary value problem (4)–(6) has two distinct solutions, denoted by $u_n^{(1)}(x,t)$ and $u_n^{(2)}(x,t)$. Then, their difference, given by $u_n(x,t) = u_n^{(1)}(x,t) - u_n^{(2)}(x,t)$, fulfills the following problem:

$$\partial_t u_n(x,t) + u_n(x,t)\partial_x u_n^{(1)}(x,t) + u_n^{(2)}(x,t)\partial_x u_n(x,t) - \nu\partial_x^2 u_n(x,t) + \partial_x u_n(x,t) = 0, \quad (36)$$

$$u_n(\varphi_1(t),t) = u_n(\varphi_2(t),t) = 0, \;\; t \in (1/n, T), \quad (37)$$

$$u_n(x,1/n) = 0, \;\; x \in \Omega_{1/n}. \quad (38)$$

By Lemma 3, it follows that

$$u_n^{(k)}(x,t) \in L^\infty(1/n,T;H^1(\Omega_t)) \cap L^2(1/n,T;H_0^2(\Omega_t)), \; k = 1,2. \quad (39)$$

Consider equality

$$\frac{1}{2}\frac{d}{dt}\|u_n(x,t)\|^2_{L^2(\Omega_t)} + \nu\,\|\partial_x u_n(x,t)\|^2_{L^2(\Omega_t)} =$$

$$-\int\limits_{\varphi_1(t)}^{\varphi_2(t)} \left[u_n(x,t)\partial_x u_n^{(1)}(x,t)u_n(x,t) + u_n^{(2)}(x,t)\partial_x u_n(x,t)u_n(x,t)\right] dx, \quad (40)$$

derived by taking the scalar product of equation (36) with the function $u_n(x,t)$ in the space $L^2(\Omega_t)$.

From (39), we derive an estimate for the right-hand side of (40):

$$\int\limits_{\varphi_1(t)}^{\varphi_2(t)} \left[|u_n(x,t)|^2\partial_x u_n^{(1)}(x,t) + u_n^{(2)}(x,t)\partial_x u_n(x,t)u_n(x,t)\right] dx$$

$$= \int\limits_{\varphi_1(t)}^{\varphi_2(t)} \left[-2u_n^1(x,t)u_n(x,t)\partial_x u_n(x,t) + u_n^{(2)}(x,t)\partial_x u_n(x,t)u_n(x,t)\right] dx$$

$$\leq \frac{1}{2\nu}\left[2\|u_n^{(1)}(x,t)\|_{L^\infty(1/n,T;(\Omega_t))} + \|u_n^{(2)}(x,t)\|_{L^\infty(1/n,T;(\Omega_t))}\right]^2 \|u_n(x,t)\|^2_{L^2(\Omega_t)}$$

$$+\frac{\nu}{2}\|\partial_x u_n(x,t)\|^2_{L^2(\Omega_t)} = C_8\|u_n(x,t)\|^2_{L^2(\Omega_t)} + \frac{\nu}{2}\|\partial_x u_n(x,t)\|^2_{L^2(\Omega_t)}, \quad (41)$$

where

$$C_8 = \frac{1}{2\nu}\left[2\|u_n^{(1)}(x,t)\|_{L^\infty(1/n,T;(\Omega_t))} + \|u_n^{(2)}(x,t)\|_{L^\infty(1/n,T;(\Omega_t))}\right]^2.$$

Using relation (40), we deduce:

$$\frac{d}{dt}\|u_n(x,t)\|^2_{L^2(\Omega_t)} + \nu\,\|\partial_x u_n(x,t)\|^2_{L^2(\Omega_t)} \le C_9\|u(x,t)\|^2_{L^2(\Omega_t)},\ \ \forall\, t \in (1/n, T], \tag{42}$$

where $C_9 = 2C_8$. From (42), by applying Gronwall's inequality, we obtain:

$$\|u_n(x,t)\|^2_{L^2(\Omega_t)} \equiv 0,\ \forall\, t \in (1/n, T].$$

This implies that $u_n^{(1)}(x,t) \equiv u_n^{(2)}(x,t)$ in $L^2(\Omega^n)$, meaning the solution to the initial boundary value problem (4)–(6) is unique. Hence, the uniqueness has been established, and Theorem 2 is proven.

6 Proof of the main result

6.1 Proof of Theorem 1. Existence

In the boundary value problems (4)–(6), we extend each element of the sequence $\{u_n(x,t) : (x,t) \in \Omega^n,\ n \in \mathbb{N}^*\}$ by zero to the entire domain Ω. As a result, we obtain a bounded sequence of functions $\left\{\widetilde{u_n(x,t)},\ n \in \mathbb{N}^*\right\}$, from which a convergent subsequence can be extracted (retaining n as the index for this subsequence), i.e.

$$\widetilde{u_n(x,t)} \to u(x,t) \ \text{ weakly in } \ H_0^{2,1}(\Omega), \tag{43}$$

$$\widetilde{u_n(x,t)} \to u(x,t) \ \text{ strongly in } \ L^2(\Omega). \tag{44}$$

Then, based on (43)–(44), we can pass to the limit as $n \to \infty$ in the following integral identity for all $\psi(x,t) \in L^2(\Omega)$

$$\int\limits_Q \Big[\partial_t \widetilde{u_n(x,t)} + \widetilde{u_n(x,t)}\partial_x \widetilde{u_n(x,t)} - \nu\partial_x^2 \widetilde{u_n(x,t)} + \partial_x \widetilde{u_n(x,t)} - f_n(x,t)\Big]\,\psi(x,t)\,dx\,dt \to$$

$$\to \int\limits_Q \left[\partial_t u(x,t) + u(x,t)\partial_x u(x,t) - \nu\partial_x^2 u(x,t) + \partial_x u(x,t) - f(x,t)\right]\psi(x,t)\,dx\,dt = 0, \tag{45}$$

and $\widetilde{u_n(x,1/n)} \to 0,$ as $n \to \infty$. Thus, it has been shown that the boundary value problem (1)–(2) possesses a solution $u(x,t) \in H_0^{2,1}(\Omega)$, as defined by the integral identity (45). This proves the existence of a solution, thereby confirming Theorem 1.

6.2 Proof of Theorem 1. Uniqueness

Suppose that the boundary value problem (1)–(2) has two distinct solutions, denoted $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$. Then, their difference $u(x,t) = u^{(1)}(x,t) - u^{(2)}(x,t)$ will fulfill the following problem:

$$\partial_t u(x,t) + u(x,t)\partial_x u^{(1)}(x,t) + u^{(2)}(x,t)\partial_x u(x,t) - \nu\partial_x^2 u(x,t) + \partial_x u(x,t) = 0, \tag{46}$$

$$u(\varphi_1(t),t) = u(\varphi_2(t),t) = 0,\ t \in (0,T). \tag{47}$$

By similar reasoning as in Lemma 3, the following inequality can be established:

$$\|u^{(k)}(x,t)\|_{L^\infty(0,T;H^1(\Omega_t))} \le M = K_2\|f(x,t)\|_{L^2(\Omega)},\ k = 1,2. \tag{48}$$

Consider the equality

$$\frac{1}{2}\frac{d}{dt}\|u(x,t)\|^2_{L^2(\Omega_t)} + \nu\,\|\partial_x u(x,t)\|^2_{L^2(\Omega_t)} =$$

$$-\int\limits_{\varphi_1(t)}^{\varphi_2(t)} \Big[|u(x,t)|^2\partial_x u^{(1)}(x,t) + u^{(2)}(x,t)\partial_x u(x,t)u(x,t)\Big]\,dx, \tag{49}$$

which is obtained by multiplying the equation (46) by function $u(x,t)$ scalarly in $L^2(\Omega_t)$.

From (48), we obtain an estimate for the right side of (49)

$$\int\limits_{\varphi_1(t)}^{\varphi_2(t)} \Big[|u(x,t)|^2\partial_x u^{(1)}(x,t) + u^{(2)}(x,t)\partial_x u(x,t)u(x,t)\Big]\,dx$$

$$\le C_{10}\|u(x,t)\|^2_{L^2(\Omega_t)} + \frac{\nu}{2}\|\partial_x u(x,t)\|^2_{L^2(\Omega_t)}, \tag{50}$$

where

$$\frac{1}{2\nu}\Big[2\|u^{(1)}(x,t)\|_{L^\infty(0,T;\,(\Omega_t))} + \|u^{(2)}(x,t)\|_{L^\infty(0,T;(\Omega_t))}\Big]^2 \le \frac{9M^2}{2\nu} = C_{10},$$

and M is the constant from (48).

Based on relations (49)–(50) we obtain

$$\frac{d}{dt}\|u(x,t)\|^2_{L^2(\Omega_t)} + \nu\,\|\partial_x u(x,t)\|^2_{L^2(\Omega_t)} \le C_{11}\|u(x,t)\|^2_{L^2(\Omega_t)},\ \ \forall\, t \in (0,T], \tag{51}$$

where $C_{11} = 2C_{10}$. From (51), applying the Gronwall inequality, we obtain that

$$\|u(x,t)\|^2_{L^2(\Omega_t)} \equiv 0,\ \ \forall\, t \in (0,T].$$

This implies that $u^{(1)}(x,t) \equiv u^{(2)}(x,t)$ in $L^2(\Omega)$, i.e. solution to the boundary value problem (1)–(2) is unique. Thus, we have proved the main result of the work, namely, Theorem 1.

7 Conclusion

In this work, we studied a Dirichlet problem for the Burgers equation in a domain with moving boundaries that degenerates at the initial moment. An orthonormal basis suitable for domains with moving boundaries was constructed. Uniform a priori estimates were obtained, based on which theorems on the unique solvability of the considered problem were proven using methods of functional analysis.

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Сарыбай Т.А., Ерғалиев М.Ғ., Жақсыбай Ы.К. ТҰТҚЫР БЮРГЕРС ТЕҢДЕУІ ҮШІН ҚОЙЫЛҒАН ДИРИХЛЕ ЕСЕБІНІҢ ШЕШІМДІЛІГІ ТУРАЛЫ

Жұмыста біз уақыттың бастапқы мезетінде жойылмалы және шекаралары қозғалмалы облыста Бюргерс теңдеуі үшін қойылған Дирихле есебін зерттейміз. Зерттеудің негізгі әдісі — Гарекин әдісі болғандықтан, біз шекаралары қозғалмалы облыстар үшін қолдануға болатын ортонормаланған базис құрылады. Бірқалыпты априорлы бағалаулар алынып, олардың негізінде қарастырылып отырған есептің бірмәнді шешімділігі туралы теоремалар функционалдық талдау әдістері көмегімен дәлелденді. Тұтқыр Бургерс теңдеуі сызықты емес жүйелердің іргелі аспектілерін зерттеу үшін жеңілдетілген үлгі ретінде қызмет етеді. Ол таза теориялық сызықты емес теңдеулер (мысалы, бұрыс Бургерс теңдеуі) мен Навье-Стокс теңдеулері сияқты күрделі жүйелер арасындағы алшақтықты жояды, бұл оны математикалық және физикалық зерттеулерде құнды құрал етеді.

Түйін сөздер: Бюргерс теңдеуі, априорлы бағалаулар, Галеркин әдісі.

Сарыбай Т.А., Ергалиев М.Г., Жаксыбай Ы.К. О РАЗРЕШИМОСТИ ЗАДАЧИ ДИРИХЛЕ ДЛЯ ВЯЗКОГО УРАВНЕНИЯ БЮРГЕРСА

В работе нами исследуется одна задача Дирихле для уравнения Бюргерса в области с подвижными границами, которая вырождается в начальный момент времени. Основным методом исследования является метод Галеркина, для применения которого нами в работе строится ортонормированный базис, применимый для областей с подвижными границами. Получены равномерные априорные оценки на основе которых методами функционального анализа доказаны теоремы однозначной разрешимости рассматриваемой задачи. Вязкое уравнение Бюргерса служит упрощенной моделью для изучения фундаментальных аспектов нелинейных систем. Оно заполняет пробел между чисто теоретическими нелинейными уравнениями (такими как невязкое уравнение Бюргерса) и более сложными системами, такими как уравнения Навье-Стокса, что делает его ценным инструментом в математических и физических исследованиях.

Ключевые слова: уравнение Бюргерса, априорные оценки, метод Галеркина.

Kazakh Mathematical Journal ISSN 1682–0525

24:4 (2024) 37–51

3-nil alternative, pre-Lie, and assosymmetric operads

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Communicated by: Viktor V. Verbovskiy

Received: 11.12.2024 ⋆ Accepted/Published Online: 09.01.2025 ⋆ Final Version: 09.01.2025

Abstract. Alternative algebras are vital for studying and modeling systems that deviate from strict associativity but maintain enough structure to be useful. Indeed, alternative algebras generalize associative algebras by relaxing the strict associativity condition. Alternative algebras naturally include the octonions, which are a key example of a non-associative division algebra. The octonions are part of the Cayley-Dickson construction and play a critical role in geometry, topology, and theoretical physics, especially in string theory and exceptional Lie groups. The origin of alternative algebras lies in the historical exploration of division algebras and their applications extend to various mathematical and physical disciplines, especially in understanding non-associative algebraic structures. In this paper, we consider free alternative algebra with the additional identity $x^3 = 0$. For motivation, we refer to the dual operad of the alternative operad. Also, we obtain pre-Lie algebra with the identity $x^3 = 0$ from binary perm algebra. Finally, we consider assosymmetric algebra with identity $x^3 = 0$.

Keywords. alternative algebra, pre-Lie algebra, assosymmetric algebra, polynomial identities.

1 Introduction

An algebra is called alternative if it satisfies the following identities:

$$(ab)c - a(bc) = -(ac)b + a(cb), \tag{1}$$

$$(ab)c - a(bc) = -(ba)c + b(ac). \tag{2}$$

A natural source of alternative algebras is Artin's theorem, which states that its every two-generated subalgebra of alternative algebra is associative [11]. Let us demonstrate some works related to the subvarieties of the variety of alternative algebras. In [9], the authors

2010 Mathematics Subject Classification: 17A30, 17A50, 17D05.

Funding: This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP26193761).

DOI: https://doi.org/10.70474/41eaqa09

constructed a basis of the free alternative algebra with identity $[a,b][c,d]=0$ and proved that every metabelian Malcev algebra can be embedded into appropriate alternative algebra under commutator. In [5], the authors considered a variety of alternative algebras with the identity

$$(ab)c+(cb)a=(ac)b+(ca)b, \tag{3}$$

which coincides with a variety of binary perm algebras. There is given a basis of the free alternative algebra with identity (3) and described a complete list of identities of algebras that appear under commutator and anti-commutator.

The variety of alternative algebras is a natural generalization of the variety of associative algebras. On the other side, the dual operad of the alternative operad is an associative operad with additional identity $x^3=0$. So, we obtain

$$\mathcal{A}lt^! = \mathcal{A}s + \{x^3=0\}.$$

Also, we obtain the following trivial result which immediately follows from the definitions given above:

Theorem 1. *Let $\mathcal{A}lt_3$ be a variety of alternative algebras defined by identity $x^3=0$. Then every two-generated algebra from $\mathcal{A}lt_3$ lies in $\mathcal{A}lt^!$, i.e., $\mathcal{A}lt+\{x^3=0\}=\mathcal{A}lt^!{}_2$.*

All described motivations can be illustrated as inclusions of the varieties as follows:

$$\begin{array}{ccc} \mathcal{A}s & \supset & \mathcal{A}s+\{x^3=0\}=\mathcal{A}lt^! \\ \cap & & \cap \qquad\qquad\quad \cap \\ \mathcal{A}lt & \supset & \mathcal{A}lt+\{x^3=0\}=\mathcal{A}lt^!{}_2 \end{array}$$

Also, we consider Koszul dual operad $\mathcal{P}_2{}^!$, where $\mathcal{P}_2$ is a variety of binary perm algebra, i.e., it is an alternative operad with additional identity (3). It turns out that $\mathcal{P}_2{}^!$ is a variety of pre-Lie algebras with two additional independent identities, where one of them is $x^3=0$. In addition, it is observed the fact that an algebra from $\mathcal{P}_2{}^!$ is a Lie algebra with an additional independent identity of degree 5. The situation looks like for the Novikov algebras under commutator [3]. For Novikov algebras, there occurs a standard identity of degree 5

$$\sum_{\sigma\in S_4}(-1)^\sigma[x_{\sigma(1)},[x_{\sigma(2)},[x_{\sigma(3)},[x_{\sigma(4)},x_5]]]]=0.$$

Finally, we consider assosymmetric algebra with identities generated by 1-dimensional invariant basis vectors which are described in [7]. These identities are

$$\sum_{\sigma\in S_3}(-1)^\sigma x_{\sigma(1)}(x_{\sigma(2)}x_{\sigma(3)}) \ \text{ and } \ \sum_{\sigma\in S_3}x_{\sigma(1)}(x_{\sigma(2)}x_{\sigma(3)}).$$

Indeed, considered alternative and pre-Lie algebras with identity $x^3 = 0$ is equivalent to the alternative and pre-Lie algebras with identity generated by 1-dimensional invariant basis vectors of identities space of degree 3, see [7]. For more details on assosymmetric and pre-Lie algebras, see [1, 6, 8, 10].

We consider all algebras over a field K of characteristic 0.

2 Some properties of algebras with identity $x^3 = 0$

Definition 2. An alternative algebra with additional identity $x^3 = 0$ is called a 3-nil alternative algebra. We denote by Alt_3 and $Alt_3\langle X\rangle$ the variety of 3-nil alternative algebras and free algebra if the variety Alt_3, respectively.

In characteristic 0, the identity $x^3 = 0$ comes to

$$(xy)z + (yx)z + (xz)y + (zx)y + (yz)x + (zy)x = 0. \quad (4)$$

By using (1) and (2), the identity (4) can be rewritten as

$$x(yz) + x(zy) + y(xz) + y(zx) + z(xy) + z(yx) = 0.$$

Both identities that are obtained from $x^3 = 0$ give

$$\{\{a, b\}, c\} + \{\{b, c\}, a\} + \{\{c, a\}, b\} = 0. \quad (5)$$

Proposition 3. *The polarization of 3-nil alternative algebra gives*

$$[x, \{y, z\}] = \{[x, y], z\} + \{[x, z], y\}$$

and

$$\{\{x, y\}, z\} = 1/3([x, [y, z]] - [[x, z], y]).$$

Proof. It can be stated by straightforward calculations. □

Indeed, the identity (4) and its consequence in alternative algebra can be rewritten as

$$\sum_{\sigma\in S_3} (x_{\sigma(1)}x_{\sigma(2)})x_{\sigma(3)} = 0$$

and

$$\sum_{\sigma\in S_3} x_{\sigma(1)}(x_{\sigma(2)}x_{\sigma(3)}) = 0.$$

These identities correspond to the 1-dimensional invariant basis vectors.

Theorem 4. *An operad $\mathcal{A}lt_3$ is self-dual.*

Proof. Firstly, let us fix a multilinear basis of algebra $\mathcal{A}lt_3$ of degree 3. That is

$$c(ba) = -(ac)b + (cb)a + a(cb),$$

$$c(ab) = (ca)b + (ac)b - a(cb),$$
$$b(ca) = (ac)b + (bc)a - a(cb),$$
$$(ab)c = -(ac)b + a(cb) + a(bc),$$
$$b(ac) = -(ca)b - (ac)b - (cb)a - (bc)a - a(bc)$$

and

$$(ba)c = -(ca)b - (cb)a - (bc)a - a(cb) - a(bc).$$

The Lie-admissibility condition for $S \otimes U$ gives the defining identities of the operad $\mathcal{A}lt_3^!$, where S is a 3-nil alternative algebra. Then

$$[[a\otimes u, b\otimes v], c\otimes w] = (ab)c\otimes (uv)w - (ba)c\otimes (vu)w - c(ab)\otimes w(uv) + c(ba)\otimes w(vu) = (-(ac)b + a(cb) + a(bc))\otimes (uv)w - (-(ca)b - (cb)a - (bc)a - a(cb) - a(bc))\otimes (vu)w - ((ca)b + (ac)b - a(cb))\otimes w(uv) + (-(ac)b + (cb)a + a(cb))\otimes w(vu).$$

Also, we obtain

$$[[b\otimes v, c\otimes w], a\otimes u] = (bc)a\otimes (vw)u - (cb)a\otimes (wv)u - a(bc)\otimes u(vw) + a(cb)\otimes u(wv).$$

and

$$[[c\otimes w, a\otimes u], b\otimes v] = (ca)b\otimes (wu)v - (ac)b\otimes (uw)v - b(ca)\otimes v(wu) + b(ac)\otimes v(uw) = (ca)b\otimes (wu)v - (ac)b\otimes (uw)v - ((ac)b + (bc)a - a(cb))\otimes v(wu) + (-(ca)b - (ac)b - (cb)a - (bc)a - a(bc))\otimes v(uw).$$

Calculating the sum and collecting the same basis monomials, we obtain

$$[[a\otimes u, b\otimes v], c\otimes w] + [[b\otimes v, c\otimes w], a\otimes u] + [[c\otimes w, a\otimes u], b\otimes v] = (ac)b\otimes (-(uv)w - w(uv) - w(vu) - (uw)v - v(wu) - v(uw)) + a(cb)\otimes ((uv)w + (vu)w + w(uv) + w(vu) + u(wv) + v(wu)) + a(bc)\otimes ((uv)w + (vu)w - u(vw) - v(uw)) + (ca)b\otimes ((vu)w - w(uv) + (wu)v - v(uw)) + (cb)a\otimes ((vu)w + w(vu) - (wv)u - v(uw)) + (bc)a\otimes ((vu)w + (vw)u - v(wu) - v(uw)) = 0.$$

From the right sides of the tensors, we obtain the identities (1), (2) and (4) which means that the operad $\mathcal{A}lt_3$ is self-dual. □

We denote by $\mathcal{P}_2$ and $\mathcal{P}_2\langle X\rangle$ the variety of binary perm algebras and free binary perm algebra. Let us calculate the dual operad of binary perm algebra ${\mathcal{P}_2}^!$. As above, we first fix the multilinear basis of binary perm algebra of degree 3. That is

$$(bc)a = (ba)c - (ac)b + (ab)c,$$

$$(cb)a = (ca)b + (ac)b - (ab)c,$$
$$a(bc) = c(ab) - (ca)b + (ab)c,$$
$$a(cb) = -c(ab) + (ca)b + (ac)b,$$
$$b(ac) = -c(ab) + (ca)b + (ba)c,$$
$$c(ba) = -c(ab) + 2(ca)b + (ac)b - (ab)c$$

and

$$b(ca) = c(ab) - (ca)b + (ba)c - (ac)b + (ab)c.$$

Performing similar calculations as above, we obtain

$$[[a\otimes u, b\otimes v], c\otimes w] + [[b\otimes v, c\otimes w], a\otimes u] + [[c\otimes w, a\otimes u], b\otimes v] =$$
$$(ab)c\otimes(uv)w - (ba)c\otimes(vu)w - c(ab)\otimes w(uv) + c(ba)\otimes w(vu)$$
$$+ (bc)a\otimes(vw)u - (cb)a\otimes(wv)u - a(bc)\otimes u(vw) + a(cb)\otimes u(wv)$$
$$+ (ca)b\otimes(wu)v - (ac)b\otimes(uw)v - b(ca)\otimes v(wu) + b(ac)\otimes v(uw) =$$
$$(ab)c\otimes(uv)w - (ba)c\otimes(vu)w - c(ab)\otimes w(uv) + (-c(ab) + 2(ca)b + (ac)b - (ab)c)\otimes w(vu)$$

$$+ ((ba)c - (ac)b + (ab)c)\otimes(vw)u - ((ca)b + (ac)b - (ab)c)\otimes(wv)u$$
$$-(c(ab)-(ca)b+(ab)c)\otimes u(vw)+(-c(ab)+(ca)b+(ac)b)\otimes u(wv)+(ca)b\otimes(wu)v-(ac)b\otimes(uw)v$$
$$- (c(ab) - (ca)b + (ba)c - (ac)b + (ab)c)\otimes v(wu) + (-c(ab) + (ca)b + (ba)c)\otimes v(uw) =$$
$$(ab)c\otimes((uv)w - w(vu) + (vw)u + (wv)u - u(vw) - v(wu))$$
$$+ (ba)c\otimes(-(vu)w + (vw)u - v(wu) + v(uw))$$
$$+ c(ab)\otimes(-w(uv) - w(vu) - u(vw) - u(wv) - v(wu) - v(uw))$$
$$+ (ca)b\otimes(2w(vu) - (wv)u + u(vw) + u(wv) + (wu)v + v(wu) + v(uw))$$
$$(ac)b\otimes(w(vu) - (vw)u - (wv)u + u(wv) - (uw)v + v(wu) + v(uw)) = 0.$$

From all calculations, we obtain the following result:

Theorem 5. *The following identities define an operad which corresponds to* ${\mathcal{P}_2}^!$*:*

$$(v, w, u) = (v, u, w),$$

$$(u,w,v)+(w,v,u)+(v,w,u)=0$$

and

$$w(uv)+w(vu)+u(vw)+u(wv)+v(wu)+v(uw)=0,$$

where (v,w,u) *stands for associator.*

Theorem 6. *The operad* ${\mathcal{P}_2}^!$ *is not Koszul.*

Proof. Calculating the dimension of the operad ${\mathcal{P}_2}^!$ by means of the package [2], we get the following result:

n	1	2	3	4	5
$\dim({\mathcal{P}_2}^!(n))$	1	2	7	26	67

According to the obtained table and [5], the first few terms of the Hilbert series of the operads ${\mathcal{P}_2}^!$ and $\mathcal{P}_2$ are

$$H(t)=-t+t^2-5t^3/6+6t^4/24-5t^5/120+O(t^6)$$

and

$$H^!(t)=-t+t^2-7t^3/6+26t^4/24-67t^5/120+O(t^6)$$

Thus,

$$H(H^!(t))=t+31t^5/60+O(t^6)\neq t.$$

By [4], the operad ${\mathcal{P}_2}^!$ is not Koszul. □

Proposition 7. *The polarization of* ${\mathcal{P}_2}^!$ *algebra gives*

$$[[a,b],c]+[[b,c],a]+[[c,a],b]=0,$$

$$\{\{a,b\},c\}+\{\{b,c\},a\}+\{\{c,a\},b\}=0,$$

$$[\{a,b\},c]+[\{b,c\},a]+[\{c,a\},b]=0$$

and

$$\{[b,c],a\}=\{\{a,b\},c\}+\{[c,a],b\}+[\{a,b\},c]+1/3[[b,c],a]-1/3[[c,a],b].$$

Proof. It can be stated by straightforward calculations. □

The next natural operad that we have to consider is an assosymmetric operad with identity $x^3=0$.

Definition 8. An algebra is called a 3-nil assosymmetric if it satisfies the following identities:

$$(x,y,z)=(x,z,y),$$

$$(x,y,z)=(y,x,z)$$

and

$$x(yz)+x(zy)+y(xz)+y(zx)+z(xy)+z(yx)=0.$$

In other words, this is an assosymmetric algebra with identity generated by a 1-dimensional invariant basis vector
$$\sum_{\sigma\in S_3} x_{\sigma(1)}(x_{\sigma(2)}x_{\sigma(3)}).$$

Remark 9. *A 3-nil assosymmetric algebra does not satisfy the identity (4) under anti-commutator.*

Let us calculate $S\otimes U$, where S is a 3-nil assosymmetric algebra.

$$\begin{gathered}
[[a\otimes u, b\otimes v], c\otimes w] + [[b\otimes v, c\otimes w], a\otimes u] + [[c\otimes w, a\otimes u], b\otimes v] = \\
(ab)c\otimes(uv)w - (ba)c\otimes(vu)w - c(ab)\otimes w(uv) + c(ba)\otimes w(vu) \\
+ (-c(ba) - 5c(ab) + 4(ca)b - (ba)c - (ac)b - (ab)c)\otimes(vw)u \\
-(c(ba)-c(ab)+(ca)b)\otimes(wv)u-(c(ab)-(ca)b+(ab)c)\otimes u(vw)+(c(ab)-(ca)b+(ac)b)\otimes u(wv) \\
+(ca)b\otimes(wu)v - (ac)b\otimes(uw)v - (-c(ba) - 4c(ab) + 3(ca)b - (ba)c - (ac)b - (ab)c)\otimes v(wu) \\
+ (c(ab) - (ca)b + (ba)c)\otimes v(uw) = \\
(ab)c\otimes((uv)w - (vw)u - u(vw) + v(wu)) + (ba)c\otimes(-(vu)w - (vw)u + v(wu) + v(uw)) \\
+ c(ab)\otimes(-w(uv) - 5(vw)u + (wv)u - u(vw) + u(wv) - 4v(wu) + v(uw)) \\
+ c(ba)\otimes(w(vu) - (vw)u - (wv)u + v(wu)) \\
+ (ca)b\otimes(4(vw)u - (wv)u + u(vw) - u(wv) + (wu)v - 3v(wu) - v(uw)) \\
+ (ac)b\otimes(-(vw)u + u(wv) - (uw)v + v(wu)).
\end{gathered}$$

The above calculations give the following result:

Theorem 10. *The dual operad of 3-nil assosymmetric operad is an alternative operad with the additional identity:*

$$(uv)w - (vu)w - (uw)v + (wu)v + (vw)u - (wv)u = 0.$$

So, this is an alternative algebra with the additional identity generated by a 1-dimensional invariant basis vector
$$\sum_{\sigma\in S_3}(-1)^{\sigma}(x_{\sigma(1)}x_{\sigma(2)})x_{\sigma(3)}.$$

Theorem 11. *The operad governed by the variety of 3-nil assosymmetric algebras is not Koszul.*

Proof. Calculating the dimension of these operads by means of the package [2], the first few terms of the Hilbert series of these operads are

$$H(t) = -t + t^2 - t^3 + 13t^4/24 - 15t^5/120 + O(t^6)$$

and

$$H^!(t) = -t + t^2 - t^3 + 13t^4/24 - 9t^5/120 + O(t^6).$$

Thus,

$$H(H^!(t)) = t + 19t^5/20 + O(t^6) \neq t.$$

By [4], such operad is not Koszul. □

Let us define an assosymmetric algebra with the additional identity which is generated by the 1-dimensional invariant basis vector

$$\sum_{\sigma \in S_3} (-1)^\sigma (x_{\sigma(1)} x_{\sigma(2)}) x_{\sigma(3)}.$$

Such algebra also satisfies another identity

$$\sum_{\sigma \in S_3} (-1)^\sigma x_{\sigma(1)} (x_{\sigma(2)} x_{\sigma(3)}).$$

Theorem 12. *An assosymmetric operad with identity* $\sum_{\sigma \in S_3} (-1)^\sigma (x_{\sigma(1)} x_{\sigma(2)}) x_{\sigma(3)}$ *is self-dual.*

Proof. As before

$$\begin{gathered}
[[a\otimes u, b\otimes v], c\otimes w] + [[b\otimes v, c\otimes w], a\otimes u] + [[c\otimes w, a\otimes u], b\otimes v] = \\
(ab)c\otimes(uv)w - (ba)c\otimes(vu)w - c(ab)\otimes w(uv) + c(ba)\otimes w(vu) \\
+ (c(ba) - c(ab) + (ba)c + (ac)b - (ab)c)\otimes(vw)u \\
-(c(ba)-c(ab)+(ca)b)\otimes(wv)u-(c(ab)-(ca)b+(ab)c)\otimes u(vw)+(c(ab)-(ca)b+(ac)b)\otimes u(wv) \\
+ (ca)b\otimes(wu)v - (ac)b\otimes(uw)v \\
- (c(ba) - (ca)b + (ba)c + (ac)b - (ab)c)\otimes v(wu) + (c(ab) - (ca)b + (ba)c)\otimes v(uw) = \\
(ab)c\otimes((uv)w - (vw)u - u(vw) + v(wu)) + (ba)c\otimes(-(vu)w + (vw)u - v(wu) + v(uw)) \\
+ c(ab)\otimes(-w(uv) - (vw)u + (wv)u - u(vw) + u(wv) + v(uw)) \\
+ c(ba)\otimes(w(vu) + (vw)u - (wv)u - v(wu)) + (ac)b\otimes((vw)u + u(wv) - (uw)v - v(wu)) \\
+ (ca)b)\otimes(-(wv)u + u(vw) - u(wv) + (wu)v + v(wu) - v(uw)).
\end{gathered}$$

The right parts of the tensors are equal to 0 if and only if the given operad is self-dual. □

Theorem 13. *The operad governed by the variety of assosymmetric algebras with identity*

$$\sum_{\sigma \in S_3} (-1)^\sigma (x_{\sigma(1)} x_{\sigma(2)}) x_{\sigma(3)}$$

is not Koszul.

Proof. Calculating the dimension of this operad by means of the package [2], the first few terms of the Hilbert series of this operad is

$$H(t) = H^!(t) = -t + t^2 - t^3 + 14t^4/24 - 12t^5/120 + O(t^6)$$

Thus,

$$H(H^!(t)) = t + 7t^5/10 + O(t^6) \neq t.$$

As before, such operad is not Koszul. □

The last remaining algebra is assosymmetric algebra which admits the identity

$$\{\{x, y\}, z\} + \{\{y, z\}, x\} + \{\{z, x\}, y\} = 0$$

under anti-commutator. For such an operad, let us calculate its Koszul dual operad:

$$\begin{gathered}
[[a \otimes u, b \otimes v], c \otimes w] + [[b \otimes v, c \otimes w], a \otimes u] + [[c \otimes w, a \otimes u], b \otimes v] = \\
(ab)c \otimes (uv)w - (ba)c \otimes (vu)w - c(ab) \otimes w(uv) + c(ba) \otimes w(vu) \\
+ (-c(ba) - 2c(ab) + (ca)b - (ba)c - (ac)b - (ab)c) \otimes (vw)u - (c(ba) - c(ab) + (ca)b) \otimes (wv)u \\
- (c(ab) - (ca)b + (ab)c) \otimes u(vw) + (c(ab) - (ca)b + (ac)b) \otimes u(wv) \\
+ (ca)b \otimes (wu)v - (ac)b \otimes (uw)v \\
- (-c(ba) - c(ab) - (ba)c - (ac)b - (ab)c) \otimes v(wu) + (c(ab) - (ca)b + (ba)c) \otimes v(uw) = \\
(ab)c \otimes ((uv)w - (vw)u - u(vw) + v(wu)) + (ba)c \otimes (-(vu)w - (vw)u + v(wu) + v(uw)) \\
c(ab) \otimes (-w(uv) - 2(vw)u + (wv)u - u(vw) + u(wv) + v(wu) + v(uw)) \\
+ c(ba) \otimes (w(vu) - (vw)u - (wv)u + v(wu)) + (ac)b \otimes (-(vw)u + u(wv) - (uw)v + v(wu)) \\
+ (ca)b \otimes ((vw)u - (wv)u + u(vw) - u(wv) + (wu)v - v(uw)).
\end{gathered}$$

We obtain an alternative operad with the identity

$$(vw)u - (wv)u + (wu)v - u(wv) - v(uw) + u(vw) = 0.$$

Let us check Koszulness condition for the last considered operad:

Theorem 14. *An assosymmetric operad with identity* $\{\{x, y\}, z\} + \{\{y, z\}, x\} + \{\{z, x\}, y\} = 0$ *is not Koszul.*

Proof. Calculating the dimension of these operads by means of the package [2], the first few terms of the Hilbert series of these operads are

$$H(t) = -t + t^2 - t^3 + 12t^4/24 - 15t^5/120 + O(t^6)$$

and

$$H^!(t) = -t + t^2 - t^3 + 12t^4/24 - 9t^5/120 + O(t^6)$$

Thus,

$$H(H^!(t)) = t + 6t^5/5 + O(t^6) \neq t.$$

So, such operad is not Koszul. □

3 Some identities under commutator

For $\mathcal{A}\langle X\rangle$, we define commutator algebra $\mathcal{A}^{(-)}\langle X\rangle$ which is obtained from $\mathcal{A}\langle X\rangle$ under the operation

$$[x, y] = xy - yx.$$

Analogically, we define anti-commutator algebra $\mathcal{A}^{(+)}\langle X\rangle$ under the operation $\{x, y\} = xy + yx$.

Theorem 15. *An algebra ${\mathcal{P}_2}^{!\,(-)}\langle X\rangle$ satisfies the following identities:*

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0,$$

$$\frac{1}{3}([[[[a, b], d], c], e] + [[[[a, b], d], e], c] + [[[[a, d], b], c], e] + [[[[a, d], b], e], c]) \\ + [[[[a, c], b], e], d] + [[[[a, c], d], e], b] + [[[[a, e], b], c], d] + [[[[a, e], d], c], b] = 0,$$

and

$$\frac{1}{2}([[[[a, b], c], d], e] + [[[[a, b], d], c], e] + [[[[a, c], b], d], e] \\ + [[[[a, c], d], b], e] + [[[[a, d], b], c], e] + [[[[a, d], c], b], e]) = \\ [[[[a, b], c], e], d] + [[[[a, b], d], e], c] + [[[[a, c], b], e], d] + \\ [[[[a, c], d], e], b] + [[[[a, d], b], e], c] + [[[[a, d], c], e], b].$$

Proof. The Jacobi identity follows from the fact that every pre-Lie algebra under commutator is a Lie. The identities of degree 5 can be obtained using means of the package [2]. Since both identities are written as a sum of basis monomials of the free Lie algebra, they are independent of anti-commutative and Jacobi identities. □

Theorem 16. *An algebra $Alt_3^{(-)}\langle X\rangle$ satisfies the following identities:*

$$[[a, c], [b, d]] = [[[a, b], c], d] + [[[b, c], d], a] + [[[c, d], a], b] + [[[d, a], b], c]$$

and

$$[[[[a, b], d], e], c] + [[[[a, b], e], d], c] + [[[[a, c], d], e], b] + [[[[a, c], e], d], b] \\ - [[[[a, d], b], c], e] - [[[[a, d], c], b], e] - [[[[a, e], b], c], d] - [[[[a, e], c], b], d] = 0.$$

Proof. The first identity corresponds to the Malcev identity. The identity of degree 5 can be obtained using means of the package [2]. □

4 Some identities under anti-commutator

Theorem 17. *An algebra $Alt_3^{(+)}\langle X\rangle$ satisfies the following identities:*

$$\{a,\{b,c\}\}+\{\{a,c\},b\}+\{\{a,b\},c\}=0, \tag{6}$$

$$\{\{a,d\},\{b,c\}\}=-\{\{a,c\},\{b,d\}\}+\{\{a,\{c,d\}\},b\}+\{a,\{b,\{c,d\}\}\}, \tag{7}$$

$$\{\{\{a,d\},c\},b\}+\{\{\{a,c\},d\},b\}+\{\{\{a,d\},b\},c\}+\{\{\{a,c\},b\},d\} \\ +\{\{\{a,b\},d\},c\}+\{\{\{a,b\},c\},d\}=0, \tag{8}$$

$$-\{\{\{\{a,e\},c\},d\},b\}+\{\{\{\{a,c\},d\},e\},b\}+\{\{\{\{a,d\},e\},b\},c\}+\{\{\{\{a,c\},e\},b\},d\} \\ +\{\{\{\{a,e\},b\},c\},d\}-\{\{\{\{a,d\},b\},c\},e\}+2\{\{\{\{a,c\},b\},e\},d\}-\{\{\{\{a,b\},d\},e\},c\} \\ -2\{\{\{\{a,b\},d\},c\},e\}-\{\{\{\{a,b\},c\},d\},e\}=0, \tag{9}$$

$$-\{\{\{\{a,d\},c\},e\},b\}-\{\{\{\{a,c\},d\},e\},b\}-\{\{\{\{a,d\},e\},b\},c\}-\{\{\{\{a,c\},e\},b\},d\} \\ -\{\{\{\{a,e\},b\},d\},c\}-\{\{\{\{a,d\},b\},e\},c\}-\{\{\{\{a,e\},b\},c\},d\}+\{\{\{\{a,d\},b\},c\},e\} \\ -\{\{\{\{a,c\},b\},e\},d\}+\{\{\{\{a,c\},b\},d\},e\}+\{\{\{\{a,b\},d\},e\},c\}+2\{\{\{\{a,b\},d\},c\},e\} \\ +\{\{\{\{a,b\},c\},e\},d\}+2\{\{\{\{a,b\},c\},d\},e\}=0. \tag{10}$$

Proof. The identity (6) is taken from (5). Other identities can be obtained using means of the package [2].

Proposition 18. *The identities (7), (8), (9) and (10) are consequence of commutative identity and (6).*

We use the identity (6) in two different ways on monomial $\{\{\{a,b\},c\},d\}$ as follows:

$$\{\{\{a,b\},c\},d\}=^{(6)}-\{\{\{a,c\},b\},d\}-\{\{a,\{b,c\}\},d\}=^{(6)}\{\{\{a,c\},d\},b\}+\{\{a,c\},\{b,d\}\} \\ +\{a,\{\{b,c\},d\}+\{\{a,d\},\{b,c\}\}=^{(6)}-\{\{\{a,d\},c\},b\}-\{\{a,\{c,d\}\},b\}+\{\{a,c\},\{b,d\}\} \\ +\{\{a,d\},\{b,c\}\}-\{a,\{\{b,d\},c\}\}-\{a,\{b,\{c,d\}\}\},$$

$$\{\{\{a,b\},c\},d\}=^{(6)}-\{\{\{a,b\},d\},c\}-\{\{a,b\},\{c,d\}\}=^{(6)}\{\{\{a,b\},d\},c\}+\{\{a,\{c,d\}\},b\} \\ +\{a,\{b,\{c,d\}\}\}=^{(6)}-\{\{\{a,d\},c\},b\}-\{\{a,d\},\{b,c\}\}-\{\{a,c\},\{b,d\}\}-\{a,\{\{b,d\},c\}\} \\ +\{\{a,\{c,d\}\},b\}+\{a,\{b,\{c,d\}\}\}.$$

By equating and collecting similar monomials, we get

$$\{\{a,d\},\{b,c\}\}+\{\{a,c\},\{b,d\}\}-\{\{a,\{c,d\}\},b\}-\{a,\{b,\{c,d\}\}\}=0,$$

which correspond to (7).

To find the next identity, we use the previously obtained identities:

$$\{a,\{b,\{c,d\}\}\} =^{(6)} -\{a,\{\{b,d\},c\}\} - \{a,\{\{b,c\},d\}\} =^{(6)} \{\{a,c\},\{b,d\}\} + \{\{a,\{b,d\}\},c\}$$
$$+ \{\{a,d\},\{b,c\}\} + \{\{a,\{b,c\}\},d\} =^{(6)} -\{\{\{a,c\},d\},b\} - \{\{\{a,c\},b\},d\} - \{\{\{a,d\},b\},c\}$$
$$- \{\{\{a,b\},d\},c\} - \{\{\{a,d\},c\},b\} - \{\{\{a,d\},b\},c\} - \{\{\{a,c\},b\},d\} - \{\{\{a,b\},c\},d\},$$

$$\{a,\{b,\{c,d\}\}\} =^{(7)} \{\{a,\{c,d\}\},b\} - \{\{a,b\},\{c,d\}\} = \{\{\{a,d\},c\},b\}$$
$$+ \{\{\{a,c\},d\},b\} + \{\{\{a,b\},d\},c\} + \{\{\{a,b\},c\},d\}.$$

By equalizing them, we obtain the identity (8). In the same way, we obtain the remained identities. More explicitly, there identity (9) can be prove as follows:

$$\{\{\{a,d\},c\},\{b,e\}\} =^{(6)} -\{\{\{\{a,d\},c\},e\},b\} - \{\{\{\{a,d\},c\},b\},e\} =^{(8)} -\{\{\{\{a,d\},c\},e\},b\}$$
$$+ \{\{\{\{a,c\},d\},b\},e\} + \{\{\{\{a,d\},b\},c\},e\} + \{\{\{\{a,c\},b\},d\},e\} + \{\{\{\{a,b\},d\},c\},e\}$$
$$+ \{\{\{\{a,b\},c\},d\},e\},$$
$$\{\{\{a,d\},c\},\{b,e\}\} =^{(8)} -\{\{\{a,c\},d\},\{b,e\}\} - \{\{\{a,d\},\{b,e\}\},c\} - \{\{\{a,c\},\{b,e\}\},d\}$$
$$- \{\{\{a,\{b,e\}\},d\},c\} - \{\{\{a,\{b,e\}\},c\},d\} =^{(6)} \{\{\{\{a,c\},d\},e\},b\} + \{\{\{\{a,c\},d\},b\},e\}$$
$$+ \{\{\{\{a,d\},e\},b\},c\} + \{\{\{\{a,d\},e\},b\},c\} + \{\{\{\{a,e\},b\},c\},d\} + \{\{\{\{a,b\},e\},c\},d\}$$
$$+ \{\{\{\{a,e\},b\},d\},c\} + \{\{\{\{a,b\},e\},d\},c\} + \{\{\{\{a,c\},e\},b\},d\} + \{\{\{\{a,c\},b\},e\},d\}$$
$$=^{(8)} \{\{\{\{a,c\},d\},e\},b\} + \{\{\{\{a,c\},d\},b\},e\} + \{\{\{\{a,d\},e\},b\},c\} + \{\{\{\{a,d\},e\},b\},c\}$$
$$+ \{\{\{\{a,e\},b\},c\},d\} + \{\{\{\{a,b\},e\},c\},d\} + \{\{\{\{a,e\},b\},d\},c\} - \{\{\{\{a,b\},d\},e\},c\}$$
$$- \{\{\{\{a,b\},e\},c\},d\} - \{\{\{\{a,b\},d\},c\},e\} - \{\{\{\{a,b\},c\},e\},d\} - \{\{\{\{a,b\},c\},d\},e\}$$
$$+ \{\{\{\{a,c\},e\},b\},d\} + \{\{\{\{a,c\},b\},e\},d\}.$$

As before, we equalize them. Reducing the same monomials, we obtain (9).

For the last identity, we use previous identities to the monomial $\{\{\{a,e\},c\},\{b,d\}\}$ in two different ways as follows:

$$\{\{\{a,e\},c\},\{b,d\}\} =^{(6)} -\{\{\{\{a,e\},c\},d\},b\} - \{\{\{\{a,e\},c\},b\},d\} =^{(8)} -\{\{\{\{a,e\},c\},d\},b\}$$
$$+ \{\{\{\{a,c\},e\},b\},d\} + \{\{\{\{a,e\},b\},d\},c\} + \{\{\{\{a,c\},b\},e\},d\} + \{\{\{\{a,b\},e\},c\},d\}$$
$$+ \{\{\{\{a,b\},c\},e\},d\},$$

$$\{\{\{a,e\},c\},\{b,d\}\} =^{(8)} -\{\{\{a,c\},e\},\{b,d\}\} - \{\{\{a,e\},\{b,c\}\},d\} - \{\{\{a,c\},\{b,d\}\},e\}$$
$$- \{\{\{a,\{b,d\}\},e\},c\} - \{\{\{a,\{b,d\}\},c\},e\} =^{(8)} \{\{\{\{a,c\},e\},b\},d\} + \{\{\{\{a,c\},e\},d\},b\}$$
$$+ \{\{\{\{a,e\},d\},b\},c\} + \{\{\{\{a,e\},b\},d\},c\} + \{\{\{\{a,c\},d\},b\},e\} + \{\{\{\{a,c\},b\},d\},e\}$$
$$+ \{\{\{\{a,d\},b\},e\},c\} + \{\{\{\{a,b\},d\},e\},c\} + \{\{\{\{a,d\},b\},c\},e\} + \{\{\{\{a,b\},d\},c\},e\}$$
$$=^{(6),(8)} -\{\{\{\{a,c\},d\},e\},b\} - \{\{\{\{a,c\},e\},b\},d\} - \{\{\{\{a,c\},d\},b\},e\} - \{\{\{\{a,c\},b\},e\},d\}$$
$$- \{\{\{\{a,c\},b\},d\},e\} + \{\{\{\{a,c\},e\},b\},d\} - \{\{\{\{a,d\},e\},b\},c\} - \{\{\{\{a,e\},b\},d\},c\}$$
$$- \{\{\{\{a,d\},b\},e\},c\} - \{\{\{\{a,b\},d\},e\},c\} + \{\{\{\{a,e\},b\},d\},c\} + \{\{\{\{a,c\},d\},b\},e\}$$
$$+ \{\{\{\{a,c\},b\},d\},e\} + \{\{\{\{a,d\},b\},e\},c\} + \{\{\{\{a,b\},d\},e\},c\} + \{\{\{\{a,b\},e\},c\},d\}$$
$$+ \{\{\{\{a,b\},d\},c\},e\} + \{\{\{\{a,b\},c\},e\},d\} + \{\{\{\{a,b\},c\},d\},e\}.$$

Finally, we obtain the needed result. □

Remark 19. *An algebra* $\mathcal{P}_2^{!\,(+)}\langle X\rangle$ *satisfies the following identity:*

$$\{a,\{b,c\}\} + \{\{a,c\},b\} + \{\{a,b\},c\} = 0,$$

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Дүйсенбай Е. Қ., Сартаев Б. К., Текебай А. А. 3-НӨЛДІК АЛЬТЕРНАТИВТІ, ПРЕ-ЛИ ЖӘНЕ АССИММЕТРИЯЛЫҚ ОПЕРАДАЛАР

Альтернативті алгебралар қатаң ассоциативтіктен ауытқыған, бірақ пайдалы болу үшін жеткілікті құрылымды сақтайтын жүйелерді зерттеу және модельдеу үшін өте маңызды. Шынында да, альтернативті алгебралар ассоциативті алгебраларды қатаң ассоциативтік шартты босаңсыту арқылы жалпылайды. Альтернативті алгебралар, әрине, ассоциативті емес бөліну алгебрасының негізгі мысалы болып табылатын октониондарды қамтиды. Октониондар Кейли-Диксон құрылымының бөлігі болып табылады және геометрияда, топологияда және теориялық физикада, әсіресе желі теориясында және ерекше Ли топтарында маңызды рөл атқарады. Альтернативті алгебраның шығу тегі бөліну алгебрасын тарихи зерттеуде жатыр және олардың қолданылуы әртүрлі математикалық және физикалық пәндерге, әсіресе ассоциативті емес алгебралық құрылымдарды түсінуге таралады. Бұл жұмыста біз $x^3 = 0$ қосымша сәйкестігімен еркін альтернативті алгебраны қарастырамыз. Мотивация үшін альтернативті операданың қос операсына жүгінеміз. Сондай-ақ, біз екілік перм алгебрасынан $x^3 = 0$ сәйкестігі бар пре-Ли алгебрасын аламыз. Соңында, $x^3 = 0$ сәйкестігі бар ассосимметриялық алгебраны қарастырамыз.

Түйін сөздер: альтернативті алгебра, пре-Ли алгебра, ассосимметриялық алгебра, көпмүшелік сәйкестіктер.

Дуйсенбай Е. К., Сартаев Б. К., Текебай А. А. 3-НУЛЕВАЯ АЛЬТЕРНАТИВНАЯ, ПРЕ-ЛИ И АССОСИММЕТРИЧЕСКАЯ ОПЕРАДЫ

Альтернативные алгебры имеют решающее значение для изучения и моделирования систем, которые отклоняются от строгой ассоциативности, но сохраняют достаточную структуру, чтобы быть полезными в алгебре. Действительно, альтернативные алгебры обобщают ассоциативные алгебры, ослабляя условие строгой ассоциативности. Альтернативные алгебры естественным образом включают октонионы, которые являются ключевым примером неассоциативной алгебры с делением. Октонионы являются частью конструкции Кэли-Диксона и играют важную роль в геометрии, топологии и теоретической физике, особенно в теории струн и исключительных группах Ли. Происхождение альтернативных алгебр лежит в историческом исследовании алгебр с делением, и их приложения распространяются на различные математические и физические дисциплины, особенно в понимании неассоциативных алгебраических структур. В этой статье мы рассматриваем свободную альтернативную алгебру с дополнительным тождеством $x^3 = 0$. Для мотивации мы ссылаемся на двойственную операду альтернативной операды. Также мы получаем пре-Ли алгебру с тождеством $x^3 = 0$ из бинарной перм алгебры. Наконец, мы рассматриваем асосимметрическую алгебру с тождеством $x^3 = 0$.

Ключевые слова: альтернативная алгебра, пре-Ли алгебра, ассосимметрическая алгебра, полиномиальные тождества.

Kazakh Mathematical Journal ISSN 1682–0525

24:4 (2024) 52–62

Transformation of degenerate indirect control systems in the vicinity of a program manifold

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Communicated by: Anar T. Assanova

Received: 13.12.2024 ⋆ Accepted/Published Online: 10.01.2025 ⋆ Final Version: 08.01.2025

Abstract. We consider one of the classes of implicit differential systems, systems of ordinary differential equations that are not resolved with respect to the highest derivative. Such equations are often found in everyday life in mechanics, physics, economics, biology, etc. The problems of constructing automatic control systems according to a given smooth program manifold are also come down to such equations.This is the case when the dimension of the systems of equations under construction is greater than the dimension of the program manifold. Then systems of algebraic equations with a rectangular matrix arise. We consider a system with a square matrix, the discriminant of which is zero. The general problem of constructing systems of differential equations for a given manifold is considered. A necessary and sufficient condition is drawn up that the manifold is integral to the system of equations. The Yerugin function is linear with respect to the manifold.Then an indirect control system is built, taking into account that a given manifold is integral to it under certain conditions. In general, the Jacobi matrix is rectangular. The case is investigated when the matrix is quadratic and has zero roots. The manifold is assumed to be linear with respect to the desired variable. A degenerate indirect control system is obtained, unresolved with respect to the highest derivative. Equivalence to a certain system is established, the matrices of which are constant and have a special structure. Lyapunov transformation matrices are found. It is shown that the considered control systems can be reduced to a central canonical form. A brief overview is provided.

Keywords. program manifold, degenerate systems, equivalence of systems, indirect control automatic systems, Lyapunov transformation, canonical forms.

2010 Mathematics Subject Classification: 34K20; 93C15; 34K29.

Funding: This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. BR20281002).

DOI: https://doi.org/10.70474/wcda3590

1 Introduction

We consider an implicit unsolved system with respect to the derivative

$$H((t,x(t))\dot{x}(t) = F(t,x), \quad H \in R^{s\times n},\ x \in R^n,\ F \in R^s,\ t \in I = (-\alpha,\ \beta)\,. \tag{1}$$

Here α, β are finite or infinite numbers. Linear systems of this kind unsolved with respect to high derivative or algebraic differential systems have wide application in everyday practice. These systems have substantial applications in the dynamic of a space vehicle, economic control, robotics, theory of electric chains ets. Here is a brief review of some important properties of these systems. The existence and uniqueness of solutions of degenerate linear systems, and reducibility of systems with variable matrices to the systems with constant matrices were studied by A.M. Samoilenko and V.P. Yacovets [1], V.P. Yacovets [2]. In these works degenerate systems were reduced to different canonical forms, and solution algorithms were constructed for linear systems. In work [3], S.A. Mazanic examined the equivalence problem considering systems to systems with constant and piecewise constant coefficients. Central canonical form and stability of degenerate control systems were considered in [4].

The problem of constructing systems based on a given program manifold is also a subclass for systems of type (1). We investigate the establishment problem of equivalence and reducibility degenerate indirect automatic control systems.

Consider the problem of constructing, for a given smooth program manifold $\Omega(t)$, the following system of differential equations

$$\dot{x} = f\,(t,x)\,, \tag{2}$$

where f, x are n-dimensional vectors, $f \in R^n$ is continuous in all variables and the existence conditions of the solution $x(t) = 0$ are satisfied; and the program manifold $\Omega(t)$ is defined by the following equations

$$\Omega\,(t) \equiv \omega(t,x) = 0, \tag{3}$$

where an s-dimensional vector ω $(s \le n)$ is continuous in the single-connected closed domain including the manifold $\Omega(t)$, together with its partial derivatives.

Definition 1. The set $\Omega(t)$ is called an integral manifold of the equation (2), if the condition $\omega(t_0, x_0) \in \Omega(t_0)$ implies that $\omega(t,x) \in \Omega(t)$ for all $t > t_0$.

Note that the term “program manifold”, used in this paper, is equivalent to the notion of “integral manifold”.

Composing a necessary and sufficient condition that the program manifold $\Omega(t)$ is integral for the system (2) we obtain

$$\dot{\omega} = \frac{\partial \omega}{\partial t} + H\dot{x} = F(t,x,\omega), \tag{4}$$

where $F(t,x,0) \equiv 0$ is some Erugin vector function [5, 6], $H = \dfrac{\partial \omega}{\partial x}$ is the Jacobian matrix and its rank is equal to $rankH = s$ at all points of $\Omega(t)$.

Solving equations with respect to $\dot{x}$ we find

$$H((t,x(t))\dot{x}(t) = F(t,x,\omega) - \frac{\partial \omega}{\partial t}, t \in I = (-\alpha,\ \beta)\,. \tag{5}$$

$$H \in R^{s\times n}, x \in R^n, \omega \in R^s, F \in R^s.$$

At $s < n$ many authors have been studying the construction of equations systems on a given program manifold possessing by the stability properties, optimality, and establishing quality estimates of transition process index in the vicinity of the manifold. A detailed review of these investigations is adduced in [7–9]. We consider the problem of finding transformation matrices for the system (5) allowing us to reduce them to an equivalent system.

2 Transformation of degenerate indirect control systems in the vicinity of program manifold.

Together with Equation (2), we consider the indirect control system with feedback of the following structure [10]:

$$\begin{array}{l} \dot{x} = f\,(t,x) - B_1\varphi(\sigma), \quad t \in I = (-\alpha,\ \beta)\,, \\ \dot{\xi} = \varphi\,(\sigma)\,, \quad \sigma = P^T\omega - Q\xi, \end{array} \tag{6}$$

where $x \in R^n$ is a state vector of the object, $f \in R^n$ is a vector-function, satisfying to conditions of existence of a solution $x(t) = 0$, and $B_1 \in R^{n\times r}$, $P \in R^{s\times r}$ are constant matrices, $Q \in R^{r\times r}$ is a constant matrix of rigid feedback, $\varphi(\sigma)$ is a function differentiable with respect to σ, satisfies the following conditions

$$\varphi(0) = 0 \wedge 0 < \sigma^T\varphi(\sigma) < \sigma^T K\sigma \quad \forall\sigma \neq 0. \tag{7}$$

Here $K = K^T > 0$, $K \in R^{r\times r}$.

For the manifold $\Omega(t)$ to be integral also for the system (6)–(7) on the manifold $\omega = 0$ it is necessary to have the condition $\xi = 0$. This condition is satisfied if and only if $Q \neq 0$.

Taking into account that $\Omega(t)$ is an integral for the system (6)–(7) differentiating program manifold $\Omega(t)$ (3) in time t by virtue of the system (6), we obtain

$$\begin{array}{l} H((t,x(t))\dot{x}(t) = F(t,x,\omega) - q_1(t) - B\xi, \\ \dot{\xi} = \varphi\,(\sigma)\,, \quad \sigma = P^T\omega - Q\xi, \quad t \in I = (\alpha,\ \beta)\,, \end{array} \tag{8}$$

where $H = \dfrac{\partial \omega}{\partial x}$, $q_1 = \dfrac{\partial \omega}{\partial t}$, $B = HB_1$, $H \in R^{s\times n}$, $x \in R^n$, $\omega \in R^s$, $F \in R^s$, nonlinearity $\varphi(\sigma)$ satisfies also to generalized conditions (7).

We consider the case where $s = n$ and the matrix H has k null roots. Choosing the manifold in the following form

$$\omega = A_1(t)x + g(t) = 0, \tag{9}$$

where $A_1(t) \in R^{s\times s}$ is a given continuous matrix, $g(t)$ is a continuous vector function, we present the Erugin function in the form of

$$F(t, x, \omega) = -A_2(t)x. \tag{10}$$

Here $-A_2$ is a Hurwitz matrix, $A_2 \in R^{s\times s}$.

Thus we obtain the following system:

$$\begin{array}{l} H(t)\dot{x}(t) = -A(t)x - q(t) - B\xi, \\ \dot{\xi} = \varphi(\sigma), \quad \sigma = \Pi^T x - P^T g(t) - Q\xi, \quad t \in I = (\alpha,\ \beta), \end{array} \tag{11}$$

where $H(t) = A_1(t)$, $A(t) = -A_2(t)A_1(t) - \dfrac{\partial A_1(t)}{\partial t}$, $q(t) = \dfrac{\partial g(t)}{\partial t} + A_1(t)g(t)$, $\Pi^T = P^T A_1(t)$.

In (11) we select the linear part relatively to x:

$$H(t)\dot{x}(t) = -A(t)x. \tag{12}$$

Definition 2. An absolutely continuous function $x(t)$ is called a *solution of System* (12) if it makes the identity of this system almost everywhere in the interval $t \in I$.

Definition 3. An absolutely continuous matrix $X \in R^{s\times r}$ is called *a fundamental matrix* of System (12) if for all constant vectors $c \in R^r$ a function $x(t) = X(t)c$ is a solution of system (12) and for any solution $x(t)$ of System (12) there exists a unique constant vector c such that $x(t) = X(t)c$.

We consider a system of a similar type together with the system (12):

$$D(t)\dot{y} + G(t)y = 0, t \in I, \tag{13}$$

where D and G are absolutely continuous $(s\times s)$-dimensional matrices bounded on the interval I, for all $t \in I$ the determinant of the matrix D is equal to zero and the automatic control system has the following form

$$\begin{array}{l} D(t)\dot{y} = G(t)y - \overline{q}(t) - \overline{B}\xi, \\ \dot{\xi} = \varphi(\sigma), \quad \sigma = \overline{G}(t)y - \overline{G}g(t) - \overline{Q}\xi, \quad t \in I = (\alpha,\ \beta). \end{array} \tag{14}$$

Here $\overline{B} \in R^{n\times\nu}$, $\overline{G} \in R^{s\times\nu}$, $\overline{Q} \in R^{\nu\times\nu}$ are constant matrices, nonlinearity $\varphi(\sigma)$ satisfies conditions of the type (7).

Definition 4. Systems (12) and (13) are called *asymptotical equivalent* if there exists a Lyapunov matrix L such that for any solution y of System (13) and a function $x = Ly$ is a solution of System (12), and for any solution x of System (12) the function $y = L^{-1}x$ is a solution of System (13).

Theorem 5. *Systems (12) and (13) are* equivalent *if and only if there exists a Lyapunov matrix L such that for the fundamental matrix Y of a solution to System (13) one can find a fundamental matrix X of a solution to System (12) for which the presentation $X = LY$ is valid.*

Theorem 6. *Let $H(t)$ and $A(t)$ are absolute continuous matrices bounded together with their first derivatives in the interval I, $rankH(t) = k$ for all $t \in I$ and for all $k, 1 < k < s$ and there is a sub-matrix $H_0(t) \in R^{k\times k}$ of the matrix $H(t)$ satisfying the following conditions*

$$\inf[\det(H_0(t)] > 0 \quad \forall t \in I, \ \inf[\partial^r/\partial\lambda^r \det(H(t)\lambda + A(t)] > 0 \quad \forall t \in I. \tag{15}$$

Then for all $t \in I$ there exist non-singular matrices T and S such that multiplied by T the left-hand side and replaced by $x = Sz$ System (12) is reduced to the equivalent system (13) and the matrices $D(t)$ and $G(t)$ are of the form:

$$D(t) = \left\| \begin{matrix} O_1 & O_2 \\ O_3 & E_0 \end{matrix} \right\|, \quad G(t) = \left\| \begin{matrix} E_1 & O_2 \\ O_3 & G_0(t) \end{matrix} \right\|, \tag{16}$$

where O_1, O_2, and O_3 are $(k \times k)$, $(k \times r)$, and $(r \times k)$-dimensional null matrices, correspondingly, E_1 and E_0 are $(k \times k)$ and $(r \times r)$-dimensional unique matrices, $G_0(t)$ is a local summable and bounded $(r \times r)$-dimensional matrix.

Proof. Let the submatrix $H_0(t)$ be in the lower right angle of the matrix $H(t)$. We represent the matrix $H(t)$ in the block form.

$$H(t) = \left\| \begin{matrix} H_1(t) & H_2(t) \\ H_3(t) & H_0(t) \end{matrix} \right\|, \tag{17}$$

where $H_1(t)((k \times k)$, $H_2(t)((k \times r)$, and $H_3(t)((r \times k)$ are matrices. Then there exist absolute continuous in the interval I matrices $C_1(t)((k \times r)$ and $C_3(t)((r \times k)$ which are bounded together with their first derivatives

$$C_1(t) = H_2(t)H_0^{-1}(t), \quad C_3(t) = H_0^{-1}(t)H_3(t).$$

Therefore, the matrix $H(t)$ can be represented in the following form:

$$H(t) = \left\| \begin{matrix} C_1(t)H_0(t)C_3(t) & C_1(t)H_0(t) \\ H_0(t)C_3(t) & H_0(t) \end{matrix} \right\|. \tag{18}$$

Choosing matrices $T(t)$ and $S(t)$ in the form of

$$T(t) = \left\| \begin{array}{cc} E_1 & -C_1(t) \\ O_1 & H_0^{-1}(t) \end{array} \right\|, \quad S(t) = \left\| \begin{array}{cc} E_1 & O_2 \\ -C_3(t) & E_0 \end{array} \right\|, \tag{19}$$

and multiplying by T the left hand said of System (11) and replacing by $x = S(t)z$ we obtain

$$\begin{array}{l} D(t)\dot{z} = -F(t)z - q(t) - D(t)B\xi, \\ \dot{\xi} = \varphi(\sigma), \quad \sigma = D^T(t)A_1(t)S(t)z - D^T g(t) - Q\xi, \quad t \in I = (\alpha,\ \beta). \end{array} \tag{19}$$

Here D is the same (16) and

$$F(t) = T(t)H(t)\dot{S}(t) + BS(t). \tag{20}$$

According to the definition of the matrix $C_3(t)$ we conclude that $S(t)$ is a Lyapunov matrix and System (19) is asymptotically equivalent to System (12). Consequently, System (19) is equivalent to System (11).

Now we represent the matrix $F(t)$ in the bloc form.

$$F(t) = \left\| \begin{array}{cc} F_1(t) & F_2(t) \\ F_3(t) & F_0(t) \end{array} \right\|, \tag{21}$$

where $F_1(t)((k \times k), F_2(t)((k \times r), F_3(t)((r \times k), F_0(t)((r \times r)$ are matrices and $z = (z_1^T, z_2^T)^T$. Then the system (19) can be written as follows:

$$\begin{array}{l} F_1(t)z_1 + F_2(t)z_2 = q_1(t), \\ \dot{z}_2 = -F_3(t)z_1 - F_0(t)z_2 - H_0^{-1}q_(t) - B_2\xi, \\ \dot{\xi} = \varphi(\sigma), \quad \sigma = D^T(t)A_1(t)S(t)z - D^T g(t) - Q\xi, \end{array} \tag{22}$$

From Equation (20) it follows that

$$F(t) = T(t)BS(t) + \overline{G}(t), \tag{23}$$

where

$$\overline{G}(t) = D(t)S^{-1}(t)\dot{S}(t) = \left\| \begin{array}{cc} O_1 & O_2 \\ -\dot{C}_3(t) & O_0 \end{array} \right\|, \tag{24}$$

and $O_0(r \times r)$ is the null matrix.

Based on relationships (20), (21) and (23) we derive

$$det\|H(t)\lambda + B(t)\| = detT_{-1}(t)det\|D\lambda + F(t) - \overline{G}\|detS^{-1}(t) =$$

$$= detH_0(t)det\|D\lambda + F(t) - \overline{G}\|. \tag{25}$$

Using Laplace decomposition for computing the determinant from (15), (21), and (25) we obtain

$$det\|D\lambda + F(t) - \overline{G}\| = \lambda^r detF_1(t) + \sum_{i=0}^{r-1} \psi_i(t)\lambda^r, \tag{26}$$

where ψ_i are some functions for $i = 0, \overline{1, \ldots, r-1}$. Therefore, because of (14), (25) and (26) the following inequality is valid:

$$\inf_{t \in I} \det \|F_1(t)\| > 0.$$

Taking into account Expression (22) this inequality implies the equality

$$z_1(t) = -F_1^{-1}(t)F_(t)z_2(t), \tag{27}$$

$$\dot{z}_2(t) = [F_3(t)F_1^{-1}(t)F_2(t) - F_0(t)]z_2(t). \tag{28}$$

Assume that $Z_2(t)$ is the fundamental matrix for solutions of Equation (28). Then from (27) and Definition 2 it follows that the matrix

$$Z(t) = \left\| \begin{array}{c} Z_1(t) \\ Z_2(t) \end{array} \right\| = \left\| \begin{array}{l} F_1^{-1}(t)F_(t)Z_(t) \\ Z_2(t) \end{array} \right\|, \tag{29}$$

is fundamental for System (19).

Now we consider System (13) where D and G are defined by the formula (16) with locally summable and bounded on the interval I matrix

$$G_0(t) = F_0(t) - F_3(t)F_1^{-1}(t)F_2(t).$$

If Y is a fundamental matrix of System (13) then there exists a constant matrix $C_0(r \times r)$ for which the following holds:

$$Y(t) = L(t)C_0 = \left\| \begin{array}{cc} E_1 & F_1^{-1}(t)F_(t) \\ O_3 & E_0 \end{array} \right\| \cdot \left\| \begin{array}{c} Z_1(t) \\ Z_2(t) \end{array} \right\| \cdot C_0. \tag{30}$$

From (19), (21), (23), and (28) it follows that F_1^{-1} and F_2 are absolute continuous matrices that are bounded together with their derivatives in the interval I. Therefore, the matrix L is a Lyapunov matrix. According to Theorem 1, System (22) is asymptotically equivalent to System (11) and, consequently, to System (12).

Now, we note that System (11) may be reduced to the central canonical form. For that, we introduce the operator $L_1(t) = -A(t) - H(t)d/dt$ to System (12).

The following theorem holds.

Theorem 7 (V.P. Yacobets [2]). *Let $A(t)$, $H(t) \in C^{2m}(\alpha, \beta)$, $rankH(t) = k$, and $H(t)$ have the full of Jordan collection with respect to the operator $L_1(t)$ in the interval I which are formed with r cells of degree $l_1, \ldots, l_r$ since $\max_i l_i = m$. Then there exist for all $t \in I$ non-singular $s \times s$-dimensional matrices $\overline{M}G_1(t) \in C^1(\alpha, \beta)$ such that multiplying by $\overline{M}(t)$ and replacing by $x = G_1(t)y$ System (12) is reduced to the following central canonical form*

$$\left\| \begin{array}{ll} E_{s-r} & 0 \\ 0 & J \end{array} \right\| \cdot \dot{y} = \left\| \begin{array}{ll} M(t) & 0 \\ 0(t) & E_l \end{array} \right\| \cdot y + \overline{M}(t)q)(t), \tag{31}$$

where $l = l_1 + l_r$, $J = diag(J_1, \ldots, J_r$ are Jordan cells of degree $l_j, j = 1, ..., r)$.

By Theorem 7 we reduce System (11) to the following system in the central canonical form

$$\begin{array}{rcl} \dot{u} & = & -V(t)u - \overline{M_1}(t)q_1 - \overline{M_1}(t)B_9t)\varphi_1(\sigma_1), \\ \dot{J}v & = & -v - \overline{M_1}(t)q_1 - \overline{M_1}(t)B_9t)\varphi_1(\sigma_1), \\ \sigma_1 & = & Q_1^T u + P_1^T g_1(t) - Q_1\xi_1, \\ \sigma_2 & = & Q_2^T u + P_2^T g_1(t) - Q_1\xi_1, \\ \sigma & = & (\sigma_1^T, \sigma_2^T)^T, \\ y & = & (u^T, v^T)^T. \end{array}$$

This system may be investigated with respect to the stability and other quality characteristics when $Q_1(t)$ and $Q_2(t)$ are bounded external perturbations [11], [12].

Next, we present the results of recent research conducted on various qualitative issues of program manifold for differential system, which can be extended to degenerate control systems [13–19].

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Жұматов С.С. БАҒДАРЛАМАЛЫҚ КӨПБЕЙНЕ МАҢАЙЫНДА АЗЫНҒАН ТУРА ЕМЕС БАСҚАРУ ЖҮЙЕЛЕРІН ТҮРЛЕНДІРУ

Айқын емес дифференциалдық жүйелердің бір класы, жоғарғы туынды бойынша шешілмеген жәй дифференциалдыық теңдеулер жүйесі қарастырылады. Мұндай теңдеулер күнделікті өмірде механика, физика, экономика, биология және т.б. салаларда жиі кездеседі. Мұндай теңдеулерге берілген жатық бағдарламалық көпбейне бойынша автоматты басқару жүйелерін құру есебі де келтіріледі. Бұл құрылып жатқан теңдеулер жүйесінің өлшемі бағдарламалық көпбейненің өлшемінен үлкен болғандағы жағдай. Бұл арада тіктөртбұрышты матрицалы алгебралық теңдеулер жүйесі пайда болады. Біз дискриминанты нөлге тең квадратты матрицалы жүйені қарастырамыз. Берілген көпбейне бойынша дифференциалдыық теңдеулер жүйесін құрудың жалпы есебі қарастырылады. Көпбейненің теңдеулер жүйесі үшін интегралдық болуының қажетті және жеткілікті шарттары құрылады. Еругин функциясы көпбейнеге қатысты сызықты етіп таңдап алынады. Сонан соң белгілі бір шарттар орындалғанда берілген көпбейненің жүйе үшін интегралдық болатынын ескере отырып, тура емес басқару жүйесі тұрғызылады. Жалпы жағдайда Якоби матрицасы тіктөртбұрышты болып табылады. Бізде матрицаның квадраттық болуы және нөлдік түбірлері болу жағдайы қарастырылады. Көпбейне ізделінді айнымалыға қатысты сызықты етіп алынады. Жоғарғы туынды бойынша шешілмеген, азынған тура емес басқару жүйесі алынды. Матрицасы тұрақты және арнайы құрылымды белгілі бір жүйеге эквивалентті болуы тағайындалды. Ляпунов түрлендіру матрицасы табылды. Қарастырылып отырған басқару жүйесінің орталық канондық түрге келтіріле алатындығы көрсетілді. Қысқаша шолу жасалынды.

Түйін сөздер: Бағдарламалық көпбейне, азынған жүйелер, жүйенің эквиваленттілігі, тура емес басқару жүйелері, Ляпунов түрлендіруі, канондық түрлер.

Жуматов С.С. ПРЕОБРАЗОВАНИЯ ВЫРОЖДЕННЫХ СИСТЕМ НЕПРЯМЫХ УПРАВЛЕНИЙ В ОКРЕСТНОСТИ ПРОГРАММНОГО МНОГООБРАЗИЯ

Рассматривается один из классов неявных дифференциальных систем, системы обыкновенных дифференциальных уравнений, не разрешенных относительно старшей производной. Такие уравнения часто встречаются в повседневной жизни в механике, физике, экономике, биологии и т.д. К таким уравнениям приводятся и задачи построения систем автоматических управлений по заданному гладкому программному многообразию. Это случай, когда размерность строящихся систем уравнений больше, чем размерность программного многообразия. Тогда возникают системы алгебраических уравнений с прямоугольной матрицей. Мы рассматриваем систему с квадратной матрицей, дискриминант которой равен нулю. Рассматривается общая задача построения систем дифференциальных уравнений по заданному многообразию. Составляются необходимое и достаточное условия того, что многообразие является интегральным для системы уравнений. Выбранная функция Еругина линейна относительно многообразия. Затем строится система непрямого управления с учетом того, что заданное многообразие является интегральным для нее при выполнении некоторых условий. В общем случае матрица Якоби является прямоугольной. Исследуется случай, когда матрица является квадратичной и имеет нулевые корни. Устанавливается эквивалентность к некоторой системе, матрицы которой постоянны и имеют специальную структуру. Найдены матрицы преобразования Ляпунова. Показано, что рассматриваемые системы управления могут быть приведены к центральной канонической форме. Приведен краткий обзор.

Ключевые слова. Программное многообразие, вырожденные системы, эквивалентность систем, автоматические системы непрямого управления, преобразования Ляпунова, канонические формы.

Kazakh Mathematical Journal ISSN 1682–0525

24:4 (2024) 63–76

Solution of the heat equation with a discontinuous coefficient with nonlocal boundary conditions by the Fourier method

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Communicated by: Makhmud A. Sadybekov

Received: 05.12.2024 ⋆ Accepted/Published Online: 12.01.2025 ⋆ Final Version: 12.01.2025

Abstract. This paper substantiates the solution by the method of separation of variables of the initial-boundary value problem for the heat equation with a discontinuous coefficient, under periodic or anti-periodic boundary conditions. Using the Fourier method, this problem is reduced to the corresponding spectral problem. The eigenvalues and eigenfunctions of this spectral problem are found. It is shown that the spectral problem is non-self-adjoint and a conjugate spectral problem of this original spectral problem is constructed. Further, it is proved that the system of eigenfunctions forms a Riesz basis. For this purpose, a self-adjoint spectral problem is constructed and its eigenvalues and eigenfunctions are found. In conclusion, using biorthogonality, the main theorem on the existence and uniqueness of a classical solution to the problem is proven.

Keywords. Heat equation with discontinuous coefficients, spectral problem, non-self-adjoint problem, Riesz basis, classical solution, Fourier method.

1 Introduction
Problem statement and research methods

We consider an initial boundary value problem for the heat equation with a piecewise constant coefficient

$$\frac{\partial u}{\partial t} = k_i^2 \frac{\partial^2 u}{\partial x^2} + f(x,t) \tag{1}$$

2010 Mathematics Subject Classification: 39B82; 44B20, 46C05.

Funding: This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. BR20281002).

DOI: https://doi.org/10.70474/4sw3g811

in the domain $\Omega = \Omega_1 \cup \Omega_2$, where

$$\Omega_1 = \{(x,t) : 0 < x < x_0,\ 0 < t < T\},\ \Omega_2 = \{(x,t) : x_0 < x < l,\ 0 < t < T\}$$

with the initial condition

$$u(x,0) = \varphi(x), \quad 0 \le x \le l, \tag{2}$$

the boundary conditions of the form

$$\begin{cases} u(0,t) + (-1)^m u(l,t) = 0, \\ k_1 \frac{\partial u(0,t)}{\partial x} + (-1)^m k_2 \frac{\partial u(l,t)}{\partial x} = 0, \end{cases} \qquad 0 \le t \le T, \tag{3}$$

and with the conjugation conditions

$$\begin{cases} u(x_0 - 0,t) = u(x_0 + 0,t), \\ k_1 \frac{\partial u(x_0-0,t)}{\partial x} = k_2 \frac{\partial u(x_0+0,t)}{\partial x}, \end{cases} \tag{4}$$

where the point x_0 is a strictly internal point of the interval $(0,l)$, that is, $0 < x_0 < l$. The coefficients $k_i > 0$, $(i = 1,2)$, $m = 1,2$.

Parabolic type equations with discontinuous coefficients have been studied quite well [1–5]. In these works, the correctness of various initial boundary value problems for a parabolic type equation with discontinuous coefficients was proven using the Green's function and thermal potentials method. In the case without a discontinuity, the spectral theory of these problems is constructed almost completely [6–12]. In [13–14], some properties of the eigenfunctions of the Sturm-Liouville operator with discontinuous coefficients were studied. In the case of a discontinuous coefficient, the spectral theory of such problems is considered in the works [15–17].

First, we consider the case $m = 1$. We look for a solution to Problem (1)–(4) in the form $u(x,t) = v(x,t) + w(x,t)$, where $v(x,t)$ is a solution to the following problem A:

$$\frac{\partial v}{\partial t} = k_i^2 \frac{\partial^2 v}{\partial x^2}$$

$$v(x,0) = \varphi(x), \quad 0 \le x \le l,$$

$$\begin{cases} v(0,t) - v(l,t) = 0, \\ k_1 \frac{\partial v(0,t)}{\partial x} - k_2 \frac{\partial v(l,t)}{\partial x} = 0, \end{cases} \qquad 0 \le t \le T,$$

$$\begin{cases} v(x_0 - 0,t) = v(x_0 + 0,t), \\ k_1 \frac{\partial v(x_0-0,t)}{\partial x} = k_2 \frac{\partial v(x_0+0,t)}{\partial x}, \end{cases}$$

where $w(x,t)$ is a solution to the following problem B:

$$\frac{\partial w}{\partial t} = k_i^2 \frac{\partial^2 w}{\partial x^2} + f(x,t)$$

$$w(x,0) = 0, \quad 0 \le x \le l,$$

$$\begin{cases} w(0,t) - w(l,t) = 0, \\ k_1 \frac{\partial w(0,t)}{\partial x} - k_2 \frac{\partial w(l,t)}{\partial x} = 0, \end{cases} \qquad 0 \le t \le T,$$

$$\begin{cases} w(x_0 - 0, t) = w(x_0 + 0, t), \\ k_1 \frac{\partial w(x_0 - 0,t)}{\partial x} = k_2 \frac{\partial w(x_0+0,t)}{\partial x}, \end{cases}$$

Let W denote the linear variety of functions from the class

$$u(x,t) \in C(\overline{\Omega}) \cup C^{2,1}(\overline{\Omega_1}) \cup C^{2,1}(\overline{\Omega_2})$$

which satisfy all conditions (2)–(4).

We call a function $u(x,t)$ from the class $u(x,t) \in W$ a *classical solution* to Problem (1)–(4), if

1) it is continuous in the domain $\overline{\Omega}$;

2) has in the domain continuous derivatives of the first order with respect to t and continuous derivatives of the second order with respect to x;

3) satisfies Equation (1) and all Conditions (2)–(4) in the usual, continuous sense.

We look for a solution to problem A using the Fourier method: $v_j(x,t) = X_j(x) \cdot T(t) \neq 0$. Substituting the boundary conditions and the pairing conditions into the equations, and separating the variables, we obtain the following spectral problem

$$LX(x) = \begin{Bmatrix} -k_1^2 X''(x), & 0 < x < x_0 \\ -k_2^2 X''(x), & x_0 < x < l \end{Bmatrix} = \lambda X(x) \tag{5}$$

$$\begin{cases} X_1(0) - X_2(l) = 0 \\ k_1 X_1'(0) - k_2 X_2'(l) = 0 \end{cases} \tag{6}$$

$$X_1(x_0 - 0) = X_2(x_0 + 0), \quad k_1 X_1'(x_0 - 0) = k_2 X_2'(x_0 + 0), \tag{7}$$

The function $T(t)$ is a solution to the equation

$$T'(t) + \lambda T(t) = 0.$$

Now we need to find the eigenvalues and eigenfunctions of Problem (5)–(7). The general solution to Equation (5) has the form

$$\begin{cases} X(x) = c_1 \cos \mu_1 x + c_2 \sin \mu_1 x, & 0 < x < x_0, \\ X(x) = d_1 \cos \mu_2 x + d_2 \sin \mu_2 x, & x_0 < x < l, \end{cases} \tag{8}$$

where $\mu_i = \dfrac{\sqrt{\lambda}}{k_i}, \quad (i = 1, 2).$

Substituting the general solution (8) into the boundary conditions (6) and the conjugation conditions (7), and taking into account that $\mu_1 k_1 = \mu_2 k_2 = \sqrt{\lambda}$ we obtain

$$\begin{cases} c_1 = d_1\cos(\mu_2 l) + d_2\sin(\mu_2 l) \\ c_2 = -d_1\sin(\mu_2 l) + d_2\cos(\mu_2 l) \\ c_1\cos(\mu_1 x_0) + c_2\sin(\mu_1 x_0) - d_1\cos(\mu_2 x_0) - d_2\sin(\mu_2 x_0) = 0 \\ -c_1\sin(\mu_1 x_0) + c_2\sin(\mu_1 x_0) + d_1\sin(\mu_2 x_0) - d_2\cos(\mu_2 x_0) = 0 \end{cases} \tag{9}$$

We find the characteristic determinant of the system (9):

$$\Delta(\lambda) = 2\Big(1 - cos\Big(\frac{\sqrt{\lambda}}{r}\Big)\Big) = 4sin^2\Big(\frac{\sqrt{\lambda}}{2r}\Big) = 0, \tag{10}$$

where

$$r = \frac{1}{\frac{x_0}{k_1} + \frac{l-x_0}{k_2}}. \tag{11}$$

From Equation (11) one can find the eigenvalues (they are twofold)

$$\lambda_n = (2\pi n r)^2, \quad \text{where } n = 0,\ 1,\ 2, \ldots \tag{12}$$

These eigenvalues correspond to the eigenfunctions

$$X_n(x) = C\begin{cases} sin(\frac{2\pi n r}{k_1}x), & 0 < x < x_0, \\ sin(\frac{2\pi n r}{k_2}(x-l)), & x_0 < x < l, \end{cases} \tag{13}$$

$$\tilde{X}_n(x) = C\begin{cases} cos(\frac{2\pi n r}{k_1}x), & 0 < x < x_0, \\ cos(\frac{2\pi n r}{k_2}(x-l)), & x_0 < x < l, \end{cases} \tag{14}$$

where r determined by the formula (11).

Lemma 1. *Spectral problem (5)–(7) is non-self-adjoint. The adjoint problem to problem (5)–(7) has the following form:*

$$LY(x) = \begin{Bmatrix} -k_1^2 Y''(x), & 0 < x < x_0 \\ -k_2^2 Y''(x), & x_0 < x < l \end{Bmatrix} = \lambda Y(x) \tag{15}$$

$$\begin{cases} k_1 Y_1(0) - k_2 Y_2(l) = 0 \\ k_1^2 Y_1'(0) - k_2^2 Y_2'(l) = 0 \end{cases} \tag{16}$$

$$k_1 Y_1(x_0 - 0) = k_2 Y_2(x_0 + 0), \quad k_1^2 Y_1'(x_0 - 0) = k_2^2 Y_2'(x_0 + 0), \tag{17}$$

Proof. We find the conjugate problem to Problem (5)–(7). Given the following formula

$$-X''(x)Y(x) = (Y'(x)X(x) - Y(x)X'(x))' - Y''(x)X(x)$$

we obtain

$$\int_0^l Y(x)LX(x)\,dx = -\int_0^{x_0} Y(x)k_1^2X''(x)\,dx - \int_{x_0}^l Y(x)k_2^2X''(x)\,dx =$$

$$= -k_1^2Y(x_0-0)X'(x_0-0) + k_1^2Y(0)X'(0) + k_1^2Y'(x_0-0)X(x_0-0)+$$
$$+k_1^2Y'(0)X(0) - k_2^2Y(l)X'(l) + k_2^2Y(x_0+0)X'(x_0+0) + k_2^2Y'(l)X(l)-$$
$$-k_2^2Y'(x_0+0)X(x_0+0) + \int_0^l X(x)LY(x)\,dx.$$

Using boundary conditions (6) and pairing conditions (7) we have

$$\int_0^l Y(x)LX(x)dx = X(x_0+0)\Big(k_1^2Y'(x_0-0) - k_2^2Y'(x_0+0)\Big)+$$
$$+k_1X'(x_0-0)\Big(k_2Y(x_0+0) - k_1Y(x_0-0)\Big)+$$
$$+k_1X'(0)\Big(k_1Y(0) - k_2Y(l)\Big) + X(0)\Big(k_2^2Y'(l) - k_1^2Y'(0)\Big) + \int_0^l X(x)LY(x)dx.$$

From the last equality it follows that the formula

$$\int_0^l Y(x)LX(x)dx = \int_0^l X(x)LY(x)dx$$

is executed only if Conditions (16)–(17). It follows that Problem (5)–(7) is not self-adjoint. □

Lemma 2. *The following problem is self-adjoint.*

$$LZ(x) = \begin{Bmatrix} -k_1^2Z''(x), & 0<x<x_0 \\ -k_2^2Z''(x), & x_0<x<l \end{Bmatrix} = \lambda Z(x) \tag{18}$$

$$\begin{cases} \sqrt{k_1}Z_1(0) - \sqrt{k_2}Z_2(l) = 0 \\ k_1^{\frac{3}{2}}Z_1'(0) - k_2^{\frac{3}{2}}Z_2'(l) = 0 \end{cases} \tag{19}$$

$$\sqrt{k_1}Z_1(x_0-0) = \sqrt{k_2}Z_2(x_0+0), \quad k_1^{\frac{3}{2}}Z_1'(x_0-0) = k_2^{\frac{3}{2}}Z_2'(x_0+0), \tag{120}$$

The proof of this lemma is similar to the proof of the previous one lemma 1. The eigenvalues of the spectral problem (18)–(20) are equal to $\lambda_n = (2\pi nr)^2$, where $(n = 0, 1, 2, ...)$, and two-fold, i.e. coincide with the eigenvalues of problem (5)–(7). The eigenfunctions are equal

$$Z_n(x) = C \begin{cases} \frac{1}{\sqrt{k_1}} sin(\frac{2\pi nr}{k_1} x), & 0 < x < x_0, \\ \frac{1}{\sqrt{k_2}} sin(\frac{2\pi nr}{k_2} (x - l)), & x_0 < x < l, \end{cases}$$

$$\tilde{Z}_n(x) = C \begin{cases} \frac{1}{\sqrt{k_1}} cos(\frac{2\pi nr}{k_1} x), & 0 < x < x_0, \\ \frac{1}{\sqrt{k_2}} cos(\frac{2\pi nr}{k_2} (x - l)), & x_0 < x < l, \end{cases}$$

From the normalization condition we find $C = \sqrt{2r}$, where r is determined by formula (11). Then finally, the eigenfunctions of Problem (18)-+(20) have the form:

$$Z_n(x) = \sqrt{2r} \begin{cases} \frac{1}{\sqrt{k_1}} sin(\frac{2\pi nr}{k_1} x), & 0 < x < x_0, \\ \frac{1}{\sqrt{k_2}} sin(\frac{2\pi nr}{k_2} (x - l)), & x_0 < x < l, \end{cases} \tag{21}$$

$$\tilde{Z}_n(x) = \sqrt{2r} \begin{cases} \frac{1}{\sqrt{k_1}} cos(\frac{2\pi nr}{k_1} x), & 0 < x < x_0, \\ \frac{1}{\sqrt{k_2}} cos(\frac{2\pi nr}{k_2} (x - l)), & x_0 < x < l, \end{cases} \tag{22}$$

Lemma 3. *The system of the eigenfunctions (13)–(14) forms the Riesz basis.*

Proof. From the formulas (13)–(14) and (21)–(22) it is easy to notice that the eigenvalues of Problem (18)–(20) and Problem (5)–(7) coincide, while the eigenfunctions differ by a piecewise constant factor. From the formulas (13)–(14) and (21)–(22) it is clear that the eigenfunctions of Problem (5)–(7) and (18)–(20) are related by the following equality:

$$\begin{pmatrix} Z_n(x) \\ \tilde{Z}_n(x) \end{pmatrix} = \alpha(x) \begin{pmatrix} X_n(x) \\ \tilde{X}_n(x) \end{pmatrix}, \quad where \quad \alpha(x) = \begin{cases} \frac{1}{\sqrt{k_1}}, & 0 < x < x_0, \\ \frac{1}{\sqrt{k_2}}, & x_0 < x < l \end{cases} \tag{23}$$

Sinse $Z_n(x)$ and $\tilde{Z}_n(x)$ are the eigenfunctions of the self-adjoint problem (18)–(20), the system $Z_n(x)$, $\tilde{Z}_n(x)$ of eigenfunctions forms an $L_2(0, l)$ orthonormal basis [18]. We rewrite formula (23) in the following form:

$$\begin{pmatrix} Z_n(x) \\ \tilde{Z}_n(x) \end{pmatrix} = A \begin{pmatrix} X_n(x) \\ \tilde{X}_n(x) \end{pmatrix}, \quad where \quad A \begin{pmatrix} X_n(x) \\ \tilde{X}_n(x) \end{pmatrix} = \alpha(x) \begin{pmatrix} X_n(x) \\ \tilde{X}_n(x) \end{pmatrix},$$

$A : L_2(0, l) \to L_2(0, l)$ is a bounded operator and there exists A^{-1} that is also bounded. It follows that the system of the eigenfunctions $X_n(x)$, $\tilde{X}_n(x)$ forms a Riesz basis.

Now we find the eigenvalues and the eigenfunctions of the conjugate problem (15)–(17). The eigenvalues of the conjugate problem are not difficult to find, they are equal $\lambda_n = (2\pi nr)^2$, where $(n = 0, 1, 2, ...)$, and they are also twofold and coincide with the eigenvalues of Problem (5)–(7). The eigenfunctions are defined as follows:

$$Y_n(x) = C \begin{cases} \frac{1}{k_1} sin(\frac{2\pi nr}{k_1} x), & 0 < x < x_0, \\ \frac{1}{k_2} sin(\frac{2\pi nr}{k_2}(x - l)), & x_0 < x < l, \end{cases} \tag{24}$$

$$\tilde{Y}_n(x) = C \begin{cases} \frac{1}{k_1} cos(\frac{2\pi nr}{k_1} x), & 0 < x < x_0, \\ \frac{1}{k_2} cos(\frac{2\pi nr}{k_2}(x - l)), & x_0 < x < l, \end{cases} \tag{25}$$

It follows from the general theory that the system of eigenfunctions $X_n(x)$, $\tilde{X}_n(x)$ and $Y_n(x)$, $\tilde{Y}_n(x)$ is biorthogonal, i.e.

$$\int_0^l X_n(x)\tilde{Y}_m(x)dx = 0 \ \text{ and } \int_0^l \tilde{X}_n(x)Y_m(x)dx = 0,$$

for any $(n, m = 1, 2, ...)$, and

$$\int_0^l X_n(x)Y_m(x)dx = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases} \quad \text{and} \int_0^l \tilde{X}_n(x)\tilde{Y}_m(x)dx = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases}$$

From the normalization condition we find $C = \sqrt{2r}$. □

Now we prove the main theorem.

Theorem 4. *Let $\varphi(x)$ be a continuously differentiable function satisfying the conditions $\varphi(0) = \varphi(l)$, $k_1\varphi'(0) = k_2\varphi'(l)$, $\varphi(x_0 - 0) = \varphi(x_0 + 0)$, $k_1\varphi'(x_0 - 0) = k_2\varphi'(x_0 + 0)$.*

Then the function

$$v(x, t) = \sum_{n=1}^{\infty} \left(\varphi_n X_n(x) + \widetilde{\varphi}_n \widetilde{X}_n(x)\right) e^{-\lambda_n t} \tag{26}$$

where the coefficients $\varphi_n, \tilde{\varphi}_n$ are determined by the formulas

$$\varphi_n = \int_0^l \varphi(x)Y_n(x)dx, \tilde{\varphi}_n = \int_0^l \varphi(x)\tilde{Y}_n(x)dx \tag{27}$$

is the only classical solution to problem A.

Proof. First, we prove the existence of solution (26). Since $X_n(x)$, $\tilde{X}_n(x)$ are the eigenfunctions and the eigenvalues of Problem (5)–(7), then it is easy to verify that the function $v(x, t)$ determined by formula (26) satisfies the equation, initial condition, boundary conditions and pairing conditions of problem A. Series (26) is the sum of functions

$$v_n(x, t) = \left(\varphi_n X_n(x) + \widetilde{\varphi}_n \widetilde{X}_n(x)\right) e^{-\lambda_n t} \tag{28}$$

We show that when $t \geq \varepsilon > 0$ (here, ε is an arbitrary positive number) the series

$$\sum_{n=1}^{\infty} v_n(x,t), \quad \sum_{n=1}^{\infty} \frac{\partial v_n}{\partial t}, \quad \sum_{n=1}^{\infty} \frac{\partial^2 v_n}{\partial x^2}$$

converges uniformly. Obviously, $|\varphi| \leq M_1$. Then from the formula (27) it follows that $\left\{|\varphi_n|, |\widetilde{\varphi}_n|\right\} \leq M_2$. Then from Equality (28) and from the following equalities

$$\frac{\partial v_n}{\partial t} = \left(-\lambda_n X_n(x)\varphi_n - \lambda_n \widetilde{X}_n(x)\widetilde{\varphi}_n\right) e^{-\lambda_n t}, \quad \frac{\partial^2 v_n}{\partial x^2} = \frac{\lambda_n}{k_j^2}\left(-X_n(x)\varphi_n - \widetilde{X}_n(x)\widetilde{\varphi}_n\right) e^{-\lambda_n t},$$

we obtain

$$|v_n(x,t)| \leq M_3 e^{-\lambda_n \varepsilon}, \quad \left\{\left|\frac{\partial v_n}{\partial t}\right|, \left|\frac{\partial^2 v_n}{\partial x^2}\right|\right\} \leq M_4 \lambda_n e^{-\lambda_n \varepsilon},$$

where the constants M_3 and M_4 are positive and does not depend on n. Thus

$$\left\{\sum_{n=1}^{\infty}\Big|v_n(x,t)\Big|, \sum_{n=1}^{\infty}\left|\frac{\partial v_n}{\partial t}\right|, \sum_{n=1}^{\infty}\left|\frac{\partial^2 v_n}{\partial x^2}\right|\right\} \leq \sum_{n=1}^{\infty} M n^2 e^{-\left(2\pi n r\right)^2 \varepsilon},$$

where $M > 0$ and does not depend on n. Since the series

$$\sum_{n=1}^{\infty} M n^2 e^{-\left(2\pi n r\right)^2 \varepsilon}$$

is an absolutely convergent series, therefore, according to Weierstrass's test, the series

$$\left\{\sum_{n=1}^{\infty}\Big|v_n(x,t)\Big|, \sum_{n=1}^{\infty}\left|\frac{\partial v_n}{\partial t}\right|, \sum_{n=1}^{\infty}\left|\frac{\partial^2 v_n}{\partial x^2}\right|\right\}$$

converge uniformly for $t \geq \varepsilon$ and the functions $v(x,t)$, $\dfrac{\partial v(x,t)}{\partial t}$, $\dfrac{\partial^2 v(x,t)}{\partial x^2}$ are continuous for $t \geq \varepsilon$.

Now we need to prove that series (26) converges uniformly everywhere in $\overline{\Omega}$. Note that the n-th term of the series (26) is dominated by the sum $|\varphi_n| + |\widetilde{\varphi}_n|$. Integrating by parts the integral in formula (27), we obtain

$$|\varphi_n| \leq \frac{C_1}{2\pi r} \cdot \frac{|\alpha_n|}{n}, \quad |\widetilde{\varphi}_n| \leq \frac{C_1}{2\pi r} \cdot \frac{|\widetilde{\alpha}_n|}{n}, \quad C_1 = \max\left(\sqrt{k_1}, \sqrt{k_2}\right),$$

where

$$\alpha_n = \int_0^l \varphi'(x)\tilde{Z}_n(x)dx \quad \text{and} \quad \widetilde{\alpha}_n = \int_0^l \varphi'(x)Z_n(x)dx$$

are Fourier coefficients of the function $\varphi'(x)$ with respect to the eigenfunctions $Z_n(x)$, $\tilde{Z}_n(x)$ orthonormal on an interval $[0,l]$, determined by the formulas (21)–(22). It is known that the eigenfunctions $Z_n(x)$, $\tilde{Z}_n(x)$ form an orthonormal basis. (See Lemma 2). Taking into account the inequality $ab \le \frac{1}{2}(a^2+b^2)$ we have

$$|\varphi_n| + |\widetilde{\varphi}_n| \le \frac{C_1}{4\pi r} \cdot \left(\alpha_n^2 + \widetilde{\alpha}_n^2 + \frac{2}{n^2}\right).$$

Using the Bessel inequality

$$\sum_{n=1}^{\infty} \left(\alpha_n^2 + \widetilde{\alpha}_n^2\right) \le \|\varphi'\|^2,$$

and the well-known equality $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, we obtain

$$\sum_{n=1}^{\infty} (|\varphi_n| + |\widetilde{\varphi}_n|) \le C.$$

Thus, the majorizing series is absolutely convergent, this means series (26) converges uniformly in $\overline{\Omega}$ and defines a continuous function $v(x,t)$ in $\overline{\Omega}$. Thus, we proved the existence of a solution. Now we prove its uniqueness. We assume that there are two solutions $\widetilde{v}(x,t)$ and $\widehat{v}(x,t)$. Then for the function $v(x,t) = \widetilde{v}(x,t) - \widehat{v}(x,t)$ we have the following *problem* C:

$$\frac{\partial v}{\partial t} = k_j^2 \frac{\partial^2 v}{\partial x^2},$$

$$v(x,0) = 0, \quad 0 \le x \le l,$$

$$\begin{cases} v(0,t) - v(l,t) = 0, \\ k_1 \dfrac{\partial v(0,t)}{\partial x} - k_2 \dfrac{\partial v(l,t)}{\partial x} = 0, \end{cases} \qquad 0 \le t \le T,$$

$$\begin{cases} v(x_0 - 0, t) = v(x_0 + 0, t), \\ k_1 \dfrac{\partial v(x_0 - 0, t)}{\partial x} = k_2 \dfrac{\partial v(x_0 + 0, t)}{\partial x} \end{cases}.$$

The solution to this *problem* C can be represented in the form of an expansion in terms of the basis $\{X_n(x), \widetilde{X}_n(x)\}$ and it has the form:

$$v(x,t) = \sum_{n=1}^{\infty} \big(A_n(t) X_n(x) + \widetilde{A}_n(t) \widetilde{X}_n(x)\big). \tag{29}$$

The coefficients $A_n(t)$ and $\widetilde{A}_n(t)$ are easy to find if we multiply both sides of equality (29) respectively by the functions $Y_n(x)$ and $\widetilde{Y}_n(x)$, and integrate the resulting relationship from 0 to l and take into account the biorthogonality of the sequences $\left\{X_n(x), \widetilde{X}_n(x)\right\}$ and $\left\{Y_n(x), \widetilde{Y}_n(x)\right\}$ Then we obtain

$$A_n(t) = \int_0^l v(x,t)Y_n(x)dx, \quad \widetilde{A}_n = \int_0^l v(x,t)\widetilde{Y}_n(x)dx. \tag{30}$$

First we transform the first equality in formula (30). Differentiating with respect to the variable t we obtain

$$A_n'(t) = \int_0^l \frac{\partial v(x,t)}{\partial t} Y_n(x)dx =$$

$$= k_1 \int_0^{x_0} \frac{\partial^2 v(x,t)}{\partial x^2} \sin\left(\frac{2\pi nr}{k_1}x\right) dx + k_2 \int_{x_0}^l \frac{\partial^2 v(x,t)}{\partial x^2} \sin\left(\frac{2\pi nr}{k_2}(x-l)\right) dx$$

Integrating by parts twice and using the boundary conditions and conjugation conditions, we have

$$A_n'(t) = -\frac{(2\pi nr)^2}{k_1} \int_0^{x_0} v(x,t) \sin\left(\frac{2\pi nr}{k_1}x\right) dx -$$

$$-\frac{(2\pi nr)^2}{k_2} \int_{x_0}^l v(x,t) \sin\left(\frac{2\pi nr}{k_2}(x-l)\right) dx = -\lambda_n \int_0^l v(x,t)Y_n(x)dx = -\lambda_n A_n(t),$$

Therefore $A_n(t) = c_n e^{-\lambda_n t}, (n = 1, 2, ...)$. Transforming in a similar way we obtain for the coefficient $\widetilde{A}_n(t)$ the following:

$$\widetilde{A}_n'(t) = -\lambda_n \widetilde{A}_n(t) \quad \Rightarrow \quad \widetilde{A}_n(t) = \widetilde{c}_n e^{-\lambda_n t}.$$

Substituting the found $A_n(t)$ and $\widetilde{A}_n(t)$ into formula (30) we obtain

$$\int_0^l v(x,t)Y_n(x)dx = c_n e^{-\lambda_n t}, \quad \int_0^l v(x,t)\widetilde{Y}_n(x)dx = \widetilde{c}_n e^{-\lambda_n t}. \tag{31}$$

Passing to the limit $t \to 0$ in equality (31) what is possible due to continuity $v(x,t)$ in $\overline{\Omega}$, we obtain

$$\lim_{t\to 0} \int_0^l v(x,t)Y_n(x)dx = 0 = A_n(0), \quad \lim_{t\to 0} \int_0^l v(x,t)\widetilde{Y}_n(x)dx = 0 = \widetilde{A}_n(0),$$

therefore $c_n = 0, \quad \widetilde{c}_n = 0, \quad (n = 1, 2, ...)$.

Then from Formula (29) we obtain $v(x,t) = 0$. It follows from this that $\widetilde{v}(x,t) = \widehat{v}(x,t)$. The theorem is proved. $\square$

Knowing the solution to problem A, it is not difficult to obtain a solution to problem B. This solution is given by the formula

$$w(x,t) = \sum_{n=1}^{\infty} \left(\int_0^t f_n e^{-\lambda_n(t-\tau)} X_n(x) + \int_0^t \widetilde{f}_n e^{-\lambda_n(t-\tau)} \widetilde{X}_n(x) \right), \tag{32}$$

where

$$f_n(\tau) = \int_0^l f(x,\tau) Y_n(x) dx, \quad \widetilde{f}_n(\tau) = \int_0^l f(x,\tau) \widetilde{Y}_n(x) dx.$$

Adding (26) and (32) we obtain a solution to Problem (1)–(4).

Now consider the case $m = 2$. Then, after applying the method of separation of variables, we obtain the following spectral problem

$$LX(x) = \begin{Bmatrix} -k_1^2 X''(x), & 0 < x < x_0 \\ -k_2^2 X''(x), & x_0 < x < l \end{Bmatrix} = \lambda X(x) \tag{33}$$

$$\begin{cases} X_1(0) + X_2(l) = 0 \\ k_1 X_1'(0) + k_2 X_2'(l) = 0 \end{cases} \tag{34}$$

$$X_1(x_0 - 0) = X_2(x_0 + 0), \quad k_1 X_1'(x_0 - 0) = k_2 X_2'(x_0 + 0), \tag{35}$$

The eigenvalues of Problem (33)–(35) have the form: $\lambda_n = ((2n+1)\pi r)^2, (n = 0, 1, 2, ...)$. The following eigenfunctions correspond to these eigenvalues.

$$X_n(x) = C \begin{cases} sin(\frac{(2n+1)\pi r}{k_1} x), & 0 < x < x_0, \\ sin(\frac{(2n+1)\pi r}{k_2}(l - x)), & x_0 < x < l, \end{cases}$$

$$\tilde{X}_n(x) = C \begin{cases} cos(\frac{(2n+1)\pi r}{k_1} x), & 0 < x < x_0, \\ -cos(\frac{(2n+1)\pi r}{k_2}(l - x)), & x_0 < x < l, \end{cases}$$

where r is determined by the formula (11).

All other calculations, including the proof of the theorem, are carried out in a similar way.

2 Conclusion

The method proposed in this article can be used in the case of n break points, where $n \geq 3$, and for the more general case of the conjugation condition (in this work, the ideal contact condition is considered).

3 Author Contributions and Conflict of Interest

All authors contributed equally to this work. The authors declare no conflict of interest.

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Қойлышов Ү.Қ., Абдырахимов Н.Т. КОЭФФИЦИЕНТІ ҮЗІЛІСТІ БЕЙЛОКАЛЬДЫ ШЕКАРАЛЫҚ ШАРТТАРМЕН БЕРІЛГЕН ЖЫЛУӨТКІЗГІШТІК ТЕҢДЕУДІ ФУРЬЕ ӘДІСІМЕН ШЕШУ

Мақалада коэффициенті үзілісті жылуөткізгіштік теңдеу үшін периодтық немесе антипериодтық шарттармен берілген бастапқы-шеттік есепті айнымалыларды ажырату әдісімен шешу негізделген Фурье әдісін қолдану арқылы бұл есеп сәйкес спектрлік есепке келтірілген. Берілген спектрлік есептің меншікті мәндері мен меншікті функциялары табылған. Спектрлік есептің өзіне-өзі түйіндес емес екені көрсетілген және берілген спектрлік есепке түйіндес есеп құрылған. Берілген есептің меншікті функциялар жүйесі Рисс базисін құрайтыны дәлелденген. Өзіне-өзі түйіндес спектрлік есеп құрылған және оның меншікті мәндері мен меншікті функциялары табылған. Қорытындылай келе, биортогональдықты пайдалана отырып, қойылған есептің классикалық шешімінің бар және жалғыздығы туралы негізгі теорема дәлелденді.

Түйін сөздер: коэффициенті үзілісті жылуөткізгіштік теңдеу, спектрлік есеп, өзіне-өзі түйіндес емес есеп, Рисс базисі, классикалық шешім, Фурье әдісі.

Койлышов У.К., Абдырахимов Н.Т. РЕШЕНИЕ УРАВНЕНИЯ ТЕПЛОПРОВОДНОСТИ С РАЗРЫВНЫМ КОЭФФИЦИЕНТОМ С НЕЛОКАЛЬНЫМИ КРАЕВЫМИ УСЛОВИЯМИ МЕТОДОМ ФУРЬЕ

В данной работе обосновано решение методом разделения переменных начально-краевой задачи для уравнения теплопроводности с разрывным коэффициентом, при периодических или антипериодических граничных условиях. Методом Фурье данная задача сведена к соответствующей спектральной задаче. Найдены собственные значения и собственные функции данной спектральной задачи. Показана, что спектральная задача несамосопряженная и построена сопряженная спектральная задача данной первоначальной спектральной задачи. Далее, доказывается, что система собственных функций образует базис Рисса. Для этого построена самосопряженная спектральная задача и найдены ее собственные значения и собственные функции. В заключении, используя биортогональность доказана основная теорема о существовании и единственности классического решения поставленной задачи.

Ключевые слова: Уравнение теплопроводности с разрывными коэффициентами, спектральная задача, несамосопряженная задача, базис Рисса, классическое решение, метод Фурье.

KAZAKH MATHEMATICAL JOURNAL

24:4 (2024)

Собственник "Kazakh Mathematical Journal":
Институт математики и математического моделирования

Журнал подписан в печать
и выставлен на сайте http://MathJournal.kz
Института математики и математического моделирования
12.01.2025 г.

Тираж 300 экз. Объем 77 стр.
Формат 70×100 1/16. Бумага офсетная № 1

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