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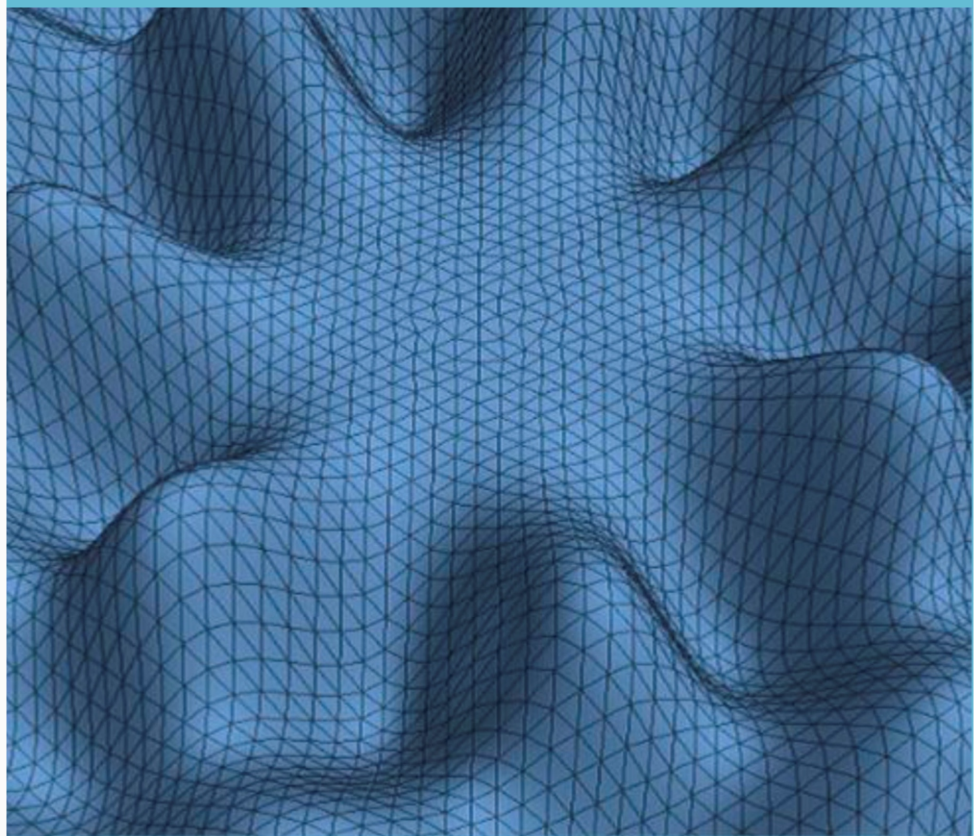

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Application of the Method of Decomposition into Exponential Series by the Spectral Parameter in Eigenvalue Problems

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Abstract. The Sturm-Liouville operator plays a central role in the theory of differential equations, mathematical physics, and applied mathematics. This operator arises in the Sturm-Liouville problem, which is an eigenvalue problem for a differential equation under consideration. The Sturm-Liouville operator generates a spectrum of eigenvalues and corresponding eigenfunctions. This is essential for solving partial differential equations through the separation of variables. The Sturm-Liouville theory is fundamental in understanding and solving linear differential equations with boundary conditions and serves as a bridge between pure and applied mathematics. The article explores the application of exponential series based on the spectral parameter to solve eigenvalue problems of Sturm-Liouville operators. A novel approach for decomposing the characteristic determinant into exponential series is proposed, demonstrating effectiveness in computing large eigenvalues. The asymptotic formulas for eigenvalues and eigenfunctions support the theoretical framework. Practical methods for achieving higher computational precision are also discussed. The work is based on an extension of earlier methods and offers new perspectives for numerical analysis in mathematical physics.

Keywords. Sturm-Liouville operator, spectral analysis, exponential series.

1 Introduction

In the paper [6], a method for decomposition into power series by the spectral parameter was proposed, which turned out to be effective for the numerical determination of the eigenvalues of the Sturm-Liouville operator. The problem of computing the eigenvalues of the

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Sturm-Liouville operator reduces to finding the zeros of the so-called characteristic determinant $\Delta(\lambda)$. The characteristic determinant of the Sturm-Liouville operator represents an entire function of the spectral parameter λ. Thus, the characteristic determinant $\Delta(\lambda)$ is decomposed into a power series by the spectral parameter λ with an infinite radius of convergence. In the paper [6], a simple method for finding the Taylor coefficients was provided. It turned out that the recurrence formulas for determining the Taylor coefficients give a simple and powerful method for numerically computing the eigenvalues. However, this approach is effective for calculating relatively small eigenvalues. For very large eigenvalues, the method proposed in [6] is not exactly ineffective, but for finding such eigenvalues, it is advisable to use exponential series by the spectral parameter. The exponential series we propose are effective for calculating sufficiently large eigenvalues in terms of magnitude. Exponential Series by the Spectral Parameter for the Sturm-Liouville Equation on a Segment.

The spectral properties of Sturm-Liouville operators have been analyzed in numerous studies. In particular, the works of Bondarenko [2], [3] investigate inverse problems for Sturm-Liouville operators, analyzing the sufficiency of information regarding the potential coefficients of the operator. Additionally, the studies by Law and Pivovarchik [4] on characteristic functions in quantum graphs are closely related to Sturm-Liouville theory. In this context, the works of Carlson and Pivovarchik [5] examine the spectral asymptotics of quantum graphs, investigating the fundamental conditions and regularities affecting the distribution of the operator's eigenvalues. Furthermore, a comprehensive review of quantum graphs and their applications is provided in the works of Berkolaiko, Carlson, Fulling, and Kuchment [7]. These studies contribute to the improvement of spectral analysis methods and enhance the understanding of their application in various operator systems. In this regard, the effectiveness of the proposed method is examined in comparison with the spectral characteristics of graphs, aiming to improve its capability in computing sufficiently large eigenvalues and to expand its applicability to other spectral problems.

2 Exponential Series by the Spectral Parameter for the Sturm-Liouville Equation on a Segment

Let us consider a second-order linear ordinary differential equation on a segment

$$-y'' + q(x)y = \lambda y, \quad 0 \le x \le b. \tag{1}$$

Such equations are called Sturm-Liouville equations. The coefficient $q(x)$ is often referred to as the potential. The conditions that the potential satisfies will be specified depending on the problem under study. The complex number λ plays the role of a spectral parameter. Often, instead of the parameter λ, it is convenient to use the parameter ρ, such that $\rho^2 = \lambda$.

Assume that λ is a complex number. Let $\varphi(x, \rho)$ denote the solution of the homogeneous equation (1), subjected to the Cauchy conditions at $x = b$.

$$\varphi(b,\rho) = 1, \quad \varphi'(b,\rho) = h, \tag{2}$$

where h is some complex constant.

According to the results of the monograph [1], the solution $\varphi(b,\rho)$ is the solution of the integral equation:

$$\varphi(x,\rho) = \cos\rho\,(x-b)\ + h\frac{\sin\rho\,(x-b)}{\rho} + \int\limits_x^b \frac{\sin\rho(x-b)}{\rho} q(t)\varphi(t,\rho)dt.$$

We define:

$$\varphi_0(x,\rho) = \cos\rho(x-b) + h\frac{\sin\rho(x-b)}{\rho}.$$

Let us assume:

$$\varphi_n(x,\rho) = \int\limits_x^b \frac{\sin\rho(x-b)}{\rho} q(t)\varphi_{n-1}(t,\rho)dt. \tag{3}$$

In the monograph [1], it was proven that the series $\sum_{k=0}^{\infty}\varphi_k(x,\rho)$ converges uniformly in λ for $|\lambda| \le N$ and uniformly for $x \in [0;b]$. Here, N is an arbitrary positive number. Thus, the function $\varphi(x,\rho)$ is an entire function of the parameter ρ^2.

For further purposes, it is convenient to obtain the exponential representation for $\varphi_n(x,\rho)$. From relation (3), for a fixed natural n, we have the following equality:

$$\varphi_n(t_{n+1},\lambda) = \frac{1}{(\sqrt{\lambda})^n}\int\limits_b^{t_{n+1}} dt_n \cdots \int\limits_b^{t_3} dt_2 \int\limits_b^{t_2} dt_1 \prod_{i=1}^{n} q(t_i)\cdot$$
$$\cdot\prod_{i=1}^{n}\sin\sqrt{\lambda}(t_{i+1}-t_i)\left[\cos\sqrt{\lambda}(t_1-b) + \frac{h}{\lambda}\sin(t_1-b)\right], \quad n = 1,2,\ldots$$

where $t_{n+1} = x$.

Let I_n denote an n-dimensional unit parallelepiped, whose vertices are of the form $\varepsilon = (0,\ \varepsilon_2,\ \varepsilon_3,\ \ldots,\ \varepsilon_n)$. Here, the values of ε_i (for $i = \overline{1,n}$) can be either zero or one. Let a fixed vertex $\varepsilon \in I_n$, then let M_0 denote the number of equal neighboring coordinate pairs. In other words,

$$M_0 = \text{card}\left\{\exists i \in [1,n] \cap \mathbb{N} : \varepsilon_i = \varepsilon_{i+1}\right\}.$$

Using the identity

$$\prod_{i=1}^{n} \sin\sqrt{\lambda}\,(t_{i+1}-t_i) =$$
$$= \begin{cases} \dfrac{4}{4^k} \displaystyle\sum_{(0\varepsilon_1\varepsilon_2\ldots\varepsilon_{2k-1})\epsilon I_{2k-1}} (-1)^{k-1+\sum_{S=1}^{2k-1}\varepsilon_s} \sin\sqrt{\lambda}\sum_{i=1}^{2k-1}(-1)^{\varepsilon_i}(t_{i+1}-t_i), & \neg n = 2k-1, \\ \dfrac{2}{4^k} \displaystyle\sum_{(0\varepsilon_1\varepsilon_2\ldots\varepsilon I_{2k})\epsilon I_{2k}} (-1)^{k+\sum_{S=1}^{2k}\varepsilon_s} \cos\sqrt{\lambda}\sum_{i=1}^{2k}(-1)^{\varepsilon_i}(t_{i+1}-t_i), & \neg n = 2k. \end{cases} \tag{4}$$

For $n = 2k-1$ and $n = 2k$, from equation (4), we derive the required exponential representations. To do this, we introduce the quantities for $n = 2k-1$:

$$\begin{aligned}
\min\tau_{1(2k-1)} &= \left((-1)^{\varepsilon_{2k-1}} + 2\left(k-1+A\left[\frac{M_0}{2}\right]-M_0\right)\right)x - \left(2\left(k-A\left[\frac{M_0}{2}\right]\right)-1\right)b,\\
\max\tau_{1(2k-1)} &= \left((-1)^{\varepsilon_{2k-1}} - 2\left(k-1-A\left[\frac{M_0}{2}\right]\right)\right)x + \left(2\left(k+A\left[\frac{M_0}{2}\right]-M_0\right)-3\right)b,\\
\min\tau_{2(2k-1)} &= \left(-(-1)^{\varepsilon_{2k-1}} + 2\left(k+A\left[\frac{M_0}{2}\right]-M_0\right)\right)x - \left(2\left(k-A\left[\frac{M_0}{2}\right]\right)-1\right)b,\\
\max\tau_{2(2k-1)} &= \left(-(-1)^{\varepsilon_{2k-1}} - 2\left(k-1-A\left[\frac{M_0}{2}\right]\right)\right)x + \left(2\left(k+A\left[\frac{M_0}{2}\right]-M_0\right)+1\right)b,
\end{aligned}$$

for $n = 2k$:

$$\begin{aligned}
\min\tau_{1(2k)} &= \left((-1)^{\varepsilon_{2k-1}} + 2\left(k-A\left[\frac{M_0}{2}\right]\right)\right)x - \left(2\left(k-A\left[\frac{M_0}{2}\right]-M_0\right)-1\right)b,\\
\max\tau_{1(2k)} &= \left((-1)^{\varepsilon_{2k-1}} + 2\left(k-1-A\left[\frac{M_0}{2}\right]-M_0\right)\right)x + \left(2\left(k+A\left[\frac{M_0}{2}\right]\right)-1\right)b,\\
\min\tau_{2(2k)} &= \left(2-(-1)^{\varepsilon_{2k-1}} + 2\left(k-A\left[\frac{M_0}{2}\right]\right)\right)x - \left(2\left(k+A\left[\frac{M_0}{2}\right]-M_0\right)-1\right)b,\\
\max\tau_{2(2k)} &= \left(-(-1)^{\varepsilon_{2k-1}} - 2\left(k-1+A\left[\frac{M_0}{2}\right]-M_0\right)\right)x - \left(2\left(k-A\left[\frac{M_0}{2}\right]\right)+1\right)b.
\end{aligned}$$

Lemma 1. *Let k be a fixed natural number. Then there exist continuous functions with respect to τ, $A^{M_0}_{1(2k-1)}(x,\tau_{1(2k-1)},b)$, $A^{M_0}_{2(2k-1)}\left(x,\tau_{2(2k-1)},b\right)$, $B_{2k-1}(x)$ such that the following* *exponential representation holds:*

$$\varphi_{2k-1}(x,\rho) = \sum_{\epsilon_{2k-1}=0}^{1} \sum_{M_0=0}^{2k-1} C_{2k-2}^{M_0} \Bigg[\frac{1}{\rho^{2k-1}} \int_{\min \tau_{1(2k-1)}}^{\max \tau_{1(2k-1)}} A_{1(2k-1)}(x, \tau_{1(2k-1)}, b) \sin \rho\tau_{1(2k-1)} d\tau_{1(2k-1)}$$
$$- \frac{1}{\rho^{2k-1}} \int_{\min \tau_{2(2k-1)}}^{\max \tau_{2(2k-1)}} A_{2(2k-1)}^{M_0}(x, \tau_{2(2k-1)}, b) \sin \rho\tau_{2(2k-1)} d\tau_{2(2k-1)}$$
$$+ \frac{h}{\rho^{2k}} \int_{\min \tau_{2(2k-1)}}^{\max \tau_{2(2k-1)}} A_{2(2k-1)}^{M_0}(x, \tau_{2(2k-1)}, b) \cos \rho\tau_{2(2k-1)} d\tau_{2(2k-1)}$$
$$- \frac{h}{\rho^{2k}} \int_{\min \tau_{1(2k-1)}}^{\max \tau_{1(2k-1)}} A_{1(2k-1)}^{M_0}(x, \tau_{1(2k-1)}, b) \cos \rho\tau_{1(2k-1)} d\tau_{1(2k-1)} \Bigg]$$
$$+ \frac{1}{\rho^{2k-1}} B_{2k-1}(x) \sin \rho(x-b) - \frac{1}{\rho^{2k-1}} \int_{b-x}^{x-b} A_{2(2k-1)}^{2k-1}(x, \tau_{2(2k-1)}, b) \sin \rho\tau_{2(2k-1)} d\tau_{2(2k-1)}$$
$$- \frac{h}{\rho^{2k}} B_{2k-1}(x) \cos \rho(x-b) + \frac{h}{\rho^{2k-1}} \int_{b-x}^{x-b} A_{2(2k-1)}^{2k-1}(x, \tau_{2(2k-1)}, b) \cos \rho\tau_{2(2k-1)} d\tau_{2(2k-1)}.$$

Lemma 2. *Let k be a fixed natural number. Then there exist continuous functions with respect to τ, $A_{1(2k)}^{M_0}(x, \tau_{1(2k)}, b), A_{2(2k)}^{M_0}\left(x, \tau_{2(2k)}, b\right), B_{2k}(x)$, such that the following exponential representation holds:*

$$\varphi_{2k}(x,\rho) = \sum_{\varepsilon_{2k-1}=0}^{1} \sum_{M_0=0}^{2k-3} C_{2k-1}^{M_0} \Bigg[\frac{1}{\rho^{2k}} \int_{min\tau_{1(2k)}}^{max\tau_{1(2k)}} A_{1(2k)}(x, \tau_{1(2k)}, b) \cos \rho\tau_{1(2k)} d\tau_{1(2k)}$$
$$+ \frac{1}{\rho^{2k}} \int_{min\tau_{2(2k)}}^{max\tau_{2(2k)}} A_{2(2k)}^{M_0}(x, \tau_{2(2k)}, b) \cos \rho\tau_{2(2k)} d\tau_{2(2k)}$$
$$+ \frac{h}{\rho^{2k+1}} \int_{min\tau_{2(2k)}}^{max\tau_{2(2k)}} A_{2(2k)}^{M_0}(x, \tau_{2(2k)}, b) \sin \rho\tau_{2(2k)} d\tau_{2(2k)}$$
$$- \frac{h}{\rho^{2k+1}} \int_{min\tau_{1(2k)}}^{max\tau_{1(2k)}} A_{1(2k)}^{M_0}(x, \tau_{1(2k)}, b) \sin \rho\tau_{1(2k)} d\tau_{1(2k)} \Bigg]$$
$$+ \frac{1}{\rho^{2k}} B_{2k}(x) \cos \rho(x-b) - \frac{1}{\rho^{2k}} \int_{b-x}^{x-b} A_{2(2k-1)}^{2k}(x, \tau_{2(2k)}, b) \cos \rho\tau_{2(2k)} d\tau_{2(2k)}$$
$$+ \frac{h}{\rho^{2k+1}} B_{2k}(x) \sin \rho(x-b) + \frac{h}{\rho^{2k+1}} \int_{b-x}^{x-b} A_{2(2k-1)}^{2k}(x, \tau_{2(2k)}, b) \sin \rho\tau_{2(2k)} d\tau_{2(2k)}.$$

Using Lemmas 1 and 2, the required solution of Problem (1)–(2) can be written as an exponential series in terms of the spectral parameter.

Theorem 3. *Let $q \in C[a,b]$. Then, for any complex λ, the solution to the problem (1)–(2) exists and is unique, and moreover, the following representation is valid:*

$$\begin{aligned}
\varphi(x,\rho) = &\cos\rho(x-b) + \frac{1}{\rho}(h + B_1(x))\sin(x-b)\\
&- \frac{1}{\rho}\int_{b-x}^{x-b} A_{23}^{1}(x,\tau_{23},b)\sin\rho\tau_{23}d\tau_{23}\\
&- \sum_{k=1}^{\infty}\Bigg\{\frac{1}{\rho^{2k}}(B_{2k}(x) - hB_{2k+1}(x))\cos\rho(x-b)\\
&+ \frac{1}{\rho^{2k}}\sum_{\varepsilon_{2k-1}=0}^{1}\Bigg(\sum_{M_0=0}^{2k-3} C_{2k-1}^{M_0}\int_{min\tau_{1(2k-1)}}^{max\tau_{1(2k-1)}} A_{2(2k+1)}(x,\tau_{2(2k+1)},b)\cos\rho\tau_{1(2k-1)}d\tau_{1(2k+1)}\\
&+ \sum_{M_0=0}^{2k-2} C_{2k-1}^{M_0}\int_{min\tau_{2(2k)}}^{max\tau_{2(2k)}} A_{2(2k)}^{M_0}(x,\tau_{2(2k)},b)\cos\rho\tau_{2(2k)}d\tau_{2(2k)}\Bigg)\\
&+ h\int_{b-x}^{x-b} A_{2(2k-1)}^{2k+1}(x,\tau_{2(2k-1)},b)\cos\rho\tau_{2(2k-1)}d\tau_{2(2k-1)}\\
&+ \int_{b-x}^{x-b} A_{2(2k-1)}^{2k}(x,\tau_{2(2k)},b)\cos\rho\tau_{2(2k)}d\tau_{2(2k)}\\
&+ \frac{1}{\rho^{2k+1}}(B_{2k+1}(x) + hB_{2k}(x))\sin\rho(x-b)\\
&+ \frac{1}{\rho^{2k+1}}\sum_{\varepsilon_{2k-1}=0}^{1}\Bigg(\sum_{M_0=0}^{2k-3} C_{2k-2}^{M_0}\int_{min\tau_{1(2k-1)}}^{max\tau_{1(2k-1)}} A_{1(2k+1)}(x,\tau_{1(2k+1)},b)\sin\rho\tau_{1(2k+1)}d\tau_{1(2k+1)}\\
&- \sum_{M_0=0}^{2k-3} C_{2k-2}^{M_0}\int_{min\tau_{2(2k+1)}}^{max\tau_{2(2k+1)}} A_{2(2k+1)}^{M_0}(x,\tau_{2(2k+1)},b)\sin\rho\tau_{2(2k+1)}d\tau_{2(2k+1)}\\
&+ \sum_{M_0=0}^{2k-2} C_{2k-1}^{M_0}\int_{min\tau_{2(2k)}}^{max\tau_{2(2k)}} A_{2(2k)}^{M_0}(x,\tau_{2(2k)},b)\sin\rho\tau_{2(2k)}d\tau_{2(2k)}\\
&+ h\sum_{M_0=0}^{2k-2} C_{2k-1}^{M_0}\int_{min\tau_{1(2k)}}^{max\tau_{1(2k)}} A_{1(2k)}^{M_0}(x,\tau_{1(2k)},b)\sin\rho\tau_{1(2k)}d\tau_{1(2k)}\Bigg)\\
&- \int_{b-x}^{x-b} A_{2(2k+1)}^{2k-1}(x,\tau_{2(2k+1)},b)\sin\rho\tau_{2(2k+1)}d\tau_{2(2k+1)}\\
&+ h\int_{b-x}^{x-b} A_{2(2k-1)}^{2k}(x,\tau_{2(2k)},b)\sin\rho\tau_{2(2k)}d\tau_{2(2k)}\Bigg\}.
\end{aligned} \quad (5)$$

3 Asymptotic formulas for eigenvalues and eigenfunctions

Now we derive the asymptotic formulas for eigenvalues and eigenfunctions. From these formulas, in particular, it follows that an infinite set of eigenvalues exists.

We still assume at the beginning that $h \neq \infty$ and $H \neq \infty$. For any λ, the function $\varphi(x, \lambda)$ obviously satisfies the first boundary condition (1)–(2). Therefore, we will determine the eigenvalues if we substitute the function $\varphi(x, \lambda)$ into the second boundary condition.

According to Lemma 1-2 from [1], the eigenvalues are real, i.e. $Im\,\rho = 0$. Therefore, we estimate the series (5) as follows:

$$\varphi(x,\rho) = cos\rho\,(x-b) + \frac{1}{\rho}\,(h + B_1(x))\,sin\rho\,(x-b) - \frac{1}{\rho}\int_{b-x}^{x-b} A_{23}^1(x,\tau_{23},b)\,sin\rho\tau d\tau + \xi_1. \quad (5')$$

Next, differentiating equation (5') with respect to x and using the estimate (5'), it is easy to obtain the following estimate:

$$\begin{aligned}\varphi'_x(x,\rho) =& \rho\sin\rho(x-b) + (h + B_1(x))\cos\rho(x-b) - A_{23}^1(x,\tau_{23},b)\frac{\sin\rho\tau}{\rho}\\ &+ \frac{1}{\rho}\int_b^x A_{23}^1(x,\tau_{23},b)_x\frac{\sin\rho\tau}{\rho}d\tau + \xi_1.\end{aligned} \quad (6)$$

Now, substituting the values of the functions $\varphi(x,\rho)$ and $\varphi'_x(x,\rho)$ from estimates (5') and (6) into the second boundary condition (2), we obtain the following equation for determining the eigenvalues:

$$\begin{gathered}\cos\rho b - (h + B_1(x))\,\frac{\sin\rho\tau}{\rho} + \int_{-b}^{b} A_{23}^1(0,\tau_{23},b)\frac{\sin\rho\tau}{\rho}d\tau + \xi_1 = 0\\ \rho\to\infty:\ \cos\rho b = 0,\ \rho_0 = \frac{\pi}{2b}(2m+1),\ m\in\mathbb{Z}.\end{gathered} \quad (7)$$

We look for the root in the form $\rho = \frac{\pi}{2b}(2m+1) + \delta(m),\ m\in\mathbb{Z}$. Then from equation (7), we have the following relationship:

$$\begin{aligned}&\cos\left(\frac{\pi}{2}(2m+1) + b\delta\right) - (h + B_1(0))\left(\frac{\pi}{2b}(2m+1) + \delta\right)^{-1}\sin\left(\frac{\pi}{2}(2m+1) + b\delta\right)\\ &+ \int_{-b}^{b} A_{23}^1(0,\tau_{23},b)\left(\frac{\pi}{2b}(2m+1) + \delta\right)^{-1}\sin\left(\frac{\pi}{2}(2m+1) + b\delta\right)d\tau + \xi_1 = 0\end{aligned}$$

or

$$\begin{aligned}&(-1)^{m+1}\sin b\delta + (-1)^{m+1}(h + B_1(0))\left(\frac{\pi}{2b}(2m+1) + \delta\right)^{-1}\cos b\delta\\ &+ (-1)^m\int_{-b}^{b} A_{23}^1(0,\tau_{23},b)\left(\frac{\pi}{2b}(2m+1) + \delta\right)^{-1}\cos b\delta d\tau + \xi_1 = 0\end{aligned} \quad (8)$$

From this, as $m \to \infty$, we get the limiting relation

$$\lim_{m\to\infty} \sin b\delta(m) = 0,$$

which is equivalent to the following equality:

$$\lim_{m\to\infty} \delta(m) = 0 \tag{9}$$

From the relation (8), taking into account the limiting equality (9), we have

$$(-1)^{m+1} \sin b\delta + (-1)^{m+1}(h + B_1(0)) \left(\frac{\pi}{2b}(2m+1) + \delta\right)^{-1} \cos b\delta$$

$$+ (-1)^m \int_{-b}^{b} A_{23}^1(0, \tau_{23}, b) \left(\frac{\pi}{2b}(2m+1) + \delta\right)^{-1} \cos b\delta d\tau + \xi_1 = 0$$

Thus, we find an approximate value of $\delta\,(m)$. This process of refining the root computation can continue to the desired level of accuracy. Therefore, the obtained approximate value can be used as a solution with the required accuracy. Continuing the process of refining the root computation, we can achieve even greater accuracy.

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Қангужин Б.Е., Хужахметов Ж.Ж. СПЕКТРАЛЬДЫҚ ПАРАМЕТР БОЙЫНША ЭКСПОНЕНЦИАЛДЫ ҚАТАРЛАРҒА ЖІКТЕУ ӘДІСІН МЕНШІКТІ МӘНДЕР ЕСЕПТЕРІНДЕ ҚОЛДАНУ

Штурм-Лиувилл операторы дифференциалдық теңдеулер теориясында, математикалық физикада және қолданбалы математикада орталық рөл атқарады. Бұл оператор Штурм-Лиувилл есебінде туындайды, ол қарастырылып отырған дифференциалдық теңдеу үшін меншікті мән есебі болып табылады. Штурм-Лиувилль операторы меншікті мәндер мен сәйкес меншікті функциялардың спектрін жасайды. Бұл айнымалыларды бөлу арқылы дербес дифференциалдық теңдеулерді шешу үшін өте қажет. Штурм-Лиувилл теориясы шекаралық шарттары бар сызықтық дифференциалдық теңдеулерді түсіну және шешу үшін негіз болып табылады және таза математика мен қолданбалы математика арасындағы көпір қызметін атқарады.

Бұл мақалада Штурм-Лиувилль операторларының меншікті мәндер есебін шешу үшін спектрлік параметр бойынша экспоненциалды қатарларды қолдану қарастырылады. Сипаттамалық анықтауышы экспоненциалды қатарларға жіктеудің жаңа тәсілі ұсынылып, үлкен меншікті мәндерді есептеуде тиімділігі көрсетіледі. Меншікті мәндер мен меншікті функциялар үшін асимптотикалық формулалар теориялық негізді растайды. Сондай-ақ, есептеу дәлдігін арттырудың практикалық әдістері талқыланады. Жұмыс бұрынғы әдістердің кеңейтілуіне негізделген және математикалық физикадағы сандық талдау үшін жаңа көзқарастар ұсынады.

Түйін сөздер: Штурм-Лиувилль операторы, спектрлік талдау, экспоненциалды қатарлар.

Кангужин Б.Е., Хужахметов Ж.Ж. ПРИМЕНЕНИЕ МЕТОДА РАЗЛОЖЕНИЯ В ЭКСПОНЕНЦИАЛЬНЫЕ РЯДЫ ПО СПЕКТРАЛЬНОМУ ПАРАМЕТРУ В ЗАДАЧАХ НА СОБСТВЕННЫЕ ЗНАЧЕНИЯ

Оператор Штурма-Лиувилля играет центральную роль в теории дифференциальных уравнений, математической физике и прикладной математике. Этот оператор возникает в задаче Штурма-Лиувилля, которая является задачей на собственные значения для рассматриваемого дифференциального уравнения. Оператор Штурма-Лиувилля генерирует спектр собственных значений и соответствующих собственных функций. Это необходимо для решения уравнений в частных производных путем разделения переменных. Теория Штурма-Лиувилля является основополагающей для понимания и решения линейных дифференциальных уравнений с граничными условиями и служит мостом между чистой и прикладной математикой.

В данной статье исследуется применение экспоненциальных рядов по спектральному параметру для решения задач на собственные значения операторов Штурма-Лиувилля.

Предлагается новый подход к разложению характеристического определителя в экспоненциальные ряды, демонстрирующий эффективность при вычислении больших собственных значений. Асимптотические формулы для собственных значений и собственных функций подтверждают теоретическую основу. Также обсуждаются практические методы достижения более высокой вычислительной точности. Работа основана на расширении более ранних методов и предлагает новые перспективы для численного анализа в математической физике.

Ключевые слова: оператор Штурма-Лиувилля, спектральный анализ, экспоненциальные ряды.

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Solution to the periodic problem for the impulsive hyperbolic equation with discrete memory

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Abstract. In this article, we consider the periodic problem for the impulsive hyperbolic equation with discrete memory. Impulsive hyperbolic equations with discrete memory arise as a mathematical model for describing physical processes in the neural networks, discontinuous dynamical systems, hybrid systems, and etc. Questions of the existence and construction of solutions to periodic problems for impulsive hyperbolic equations with discrete memory remain important issues in the theory of discontinuous partial differential equations. To find the solvability conditions of this problem we apply Dzhumabaev's parametrization method. The coefficient conditions for the existence and uniqueness of the periodic problem for the impulsive hyperbolic equation with discrete memory are established. We offer an algorithm for determining the approximate solution to this problem and show its convergence to the exact solution of the periodic problem for the impulsive hyperbolic equation with discrete memory.

Keywords. hyperbolic equation, impulse effects, periodic condition, discrete memory, partition of domain, problem with parameters, solvability conditions.

1 Introduction

On the domain $\Omega = [0,T]\times[0,\omega]$ we consider the periodic problem for the impulsive hyperbolic equation with discrete memory in the following form

$$\frac{\partial^2 u}{\partial t\partial x} = A(t,x)\frac{\partial u(t,x)}{\partial x} + A_0(t,x)\frac{\partial u(\gamma(t),x)}{\partial x} + B(t,x)\frac{\partial u(t,x)}{\partial t} + C(t,x)u(t,x) + f(t,x), \quad (1)$$

$$t \neq \theta_j, \qquad j = \overline{1, N-1},$$

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$$u(0,x) = u(T,x), \qquad x \in [0,\omega], \tag{2}$$

$$\lim_{t\to\theta_p+0} u(t,x) - \lim_{t\to\theta_p-0} u(t,x) = \varphi_p(x), \qquad x \in [0,\omega], \quad p = \overline{1,N-1}, \tag{3}$$

$$u(t,0) = \psi(t), \qquad t \in [0,T], \tag{4}$$

where $u(t,x)$ is unknown function, the functions $A(t,x)$, $B(t,x)$, $C(t,x)$, $A_0(t,x)$ and n vector function $f(t,x)$ are continuous on Ω;

$$\gamma(t) = \zeta_s \text{ if } t \in [\theta_{s-1},\theta_s), \ s = \overline{1,N};$$

$$\theta_{s-1} < \zeta_s < \theta_s \text{ for all } s = 1,2,...,N; \ 0 = \theta_0 < \theta_1 < ... < \theta_{N-1} < \theta_N = T;$$

the functions $\varphi_p(x)$ are continuously differentiable on $[0,\omega]$, $p = \overline{1,N-1}$; the function $\psi(t)$ is continuously differentiable on $[0,T]$ and satisfies the compatibility condition: $\psi(0) = \psi(T)$.

We introduce the notation

$$\Omega_s = [\theta_{s-1},\theta_s) \times [0,\omega], \ s = \overline{1,N}, \ \text{i.e. } \Omega = \bigcup_{s=1}^{N} \Omega_s.$$

Let $PC(\Omega, \{\theta_j\}_{j=1}^{N-1}, \mathbb{R})$ be the space of piecewise continuous on Ω functions $u(t,x)$ with possible discontinuities on lines $t = \theta_j$, $j = \overline{1,N-1}$, and the norm

$$||u||_1 = \max_{s=\overline{1,N}} \sup_{(t,x)\in\Omega_s} |u(t,x)|.$$

A function $u(t,x) \in PC(\Omega, \{\theta_j\}_{j=1}^{N-1}, \mathbb{R})$ is a solution to problem (1)–(4) if:

(i) $u(t,x)$ has partial derivatives

$$\frac{\partial u(t,x)}{\partial x} \in PC(\Omega, \{\theta_j\}_{j=1}^{N-1}, \mathbb{R}), \ \frac{\partial u(t,x)}{\partial t} \in PC(\Omega, \{\theta_j\}_{j=1}^{N-1}, \mathbb{R});$$

(ii) the mixed partial derivative $\frac{\partial^2 u(t,x)}{\partial t \partial x}$ exists at each point $(t,x) \in \Omega$ with the possible exception of the points (θ_{s-1}, x), $s = \overline{1,N}$, for all $x \in [0,\omega]$, where the one-sided mixed partial derivatives exist;

(iii) hyperbolic equation (1) is satisfied for $u(t,x)$ on each subdomain $(\theta_{s-1},\theta_s) \times [0,\omega]$, $s = \overline{1,N}$, and it holds for the right mixed partial derivative of $u(t,x)$ at the points (θ_{s-1}, x), $s = \overline{1,N}$, $x \in [0,\omega]$;

(iv) periodic condition (2) and initial condition (4) are satisfied for $u(t,x)$ at the lines $t = 0$, $t = T$, and $x = 0$, respectively;

(v) the conditions of the impulse effects (3) are satisfied for $u(t,x)$ at the lines $t=\theta_p$, $p=\overline{1,N-1}$, $x\in[0,\omega]$.

Differential equations with discrete memory (or generalized piecewise constant argument) are more suitable for modeling and solving various application problems, including areas of neural networks, discontinuous dynamical systems, biological and medical models, etc. [1, 2, 3, 4, 5, 6, 7].

Questions of solvability and construction of solutions to boundary value problems for differential and hyperbolic equations with generalized piecewise constant argument on a finite interval were studied in [8, 9, 10, 11].

For impulsive partial differential equations with discrete memory, however, the questions of solvability of boundary value problems on a finite interval still remain open [12].

This issue can be resolved by developing constructive methods.

The non-local problem for a system of hyperbolic equations with impulse discrete memory were considered in [13]. Conditions for the existence and uniqueness solution to the non-local problem for a system of hyperbolic equations with impulse discrete memory were established in the term of special matrix composed by coefficient matrices and boundary matrices.

In the present paper, we propose a new approach for solving periodic problems for the impulsive hyperbolic equation with discrete memory (1)–(4) based on the introduction of new functions and on Dzhumabaev's parametrization method [14].

2 Introduction of new functions and algorithm of Dzhumabaev's parametrization method

First, we introduce new functions $v(t,x)=\frac{\partial u(t,x)}{\partial x}$, $w(t,x)=\frac{\partial u(t,x)}{\partial t}$.

We have a periodic problem for a family of impulsive differential equations with discrete memory in the next form

$$\frac{\partial v}{\partial t}=A(t,x)v(t,x)+A_0(t,x)v(\gamma(t),x)+f(t,x)+B(t,x)w(t,x)+C(t,x)u(t,x), \quad (5)$$

$$t\neq\theta_j, \quad j=\overline{1,N-1},$$

$$v(0,x)=v(T,x), \qquad x\in[0,\omega], \quad (6)$$

$$\lim_{t\to\theta_p+0}v(t,x)-\lim_{t\to\theta_p-0}v(t,x)=\dot{\varphi}_p(x), \qquad x\in[0,\omega], \quad p=\overline{1,N-1}, \quad (7)$$

$$u(t,x)=\psi(t)+\int_0^x v(t,\xi)d\xi, \qquad w(t,x)=\dot{\psi}(t)+\int_0^x \frac{\partial v(t,\xi)}{\partial t}d\xi, \quad (8)$$

$$(t,x)\in\Omega_s, \qquad s=\overline{1,N}.$$

A triple of functions $\{v(t,x),u(t,x),w(t,x)\}$ is a solution to the problem for the family of impulsive differential equations with discrete memory (5)–(8) if:

(i) the function $v(t,x) \in PC(\Omega, \{\theta_j\}_{j=1}^{N-1}, \mathbb{R})$ has partial derivative

$$\frac{\partial v(t,x)}{\partial t} \in PC(\Omega, \{\theta_j\}_{j=1}^{N-1}, \mathbb{R});$$

(ii) the family of the differential equations (5) is satisfied for $v(t,x)$, $u(t,x)$ and $w(t,x)$ on each subdomain $(\theta_{s-1}, \theta_s) \times [0,\omega]$, $s = \overline{1,N}$, and it holds for the right partial derivative of $v(t,x)$ by t at the points (θ_{s-1}, x), $s = \overline{1,N}$, $x \in [0,\omega]$;

(iii) the periodic condition (6) is satisfied for $v(t,x)$ at the lines $t = 0$ and $t = T$.

(iv) the functions $u(t,x)$ and $w(t,x)$ are connected with $v(t,x)$ and $\frac{\partial v(t,x)}{\partial t}$ by the integral equations (8).

(v) Denote by $\Delta_N(\omega)$ a partition of the domain Ω by lines $t = \theta_s$:

$$\Omega_s = [\theta_{s-1}, \theta_s) \times [0,\omega], \quad s = \overline{1,N}.$$

Let $C(\Omega, \Delta_N(\omega), \mathbb{R}^N)$ be the space of functions systems

$$v([t],x) = (v_1(t,x), v_2(t,x), \ldots, v_N(t,x))',$$

where $v_s : \Omega_s \to \mathbb{R}$ are continuous and have finite left-hand side limits $\lim\limits_{t\to\theta_s-0} v_s(t,x)$ for all $s = \overline{1,N}$, and $x \in [0,\omega]$ with the norm

$$\|v([\cdot],x)\|_2 = \max_{s=\overline{1,N}} \sup_{t\in[\theta_{s-1},\widetilde{\theta}_s)} ||v_s(t,x)||.$$

We denote by $v_s(t,x)$ the restriction of a function $v(t,x)$ on the s-th subdomain $\widetilde{\Omega}_s$, i.e.

$$v_s(t,x) = v(t,x) \text{ for } (t,x) \in \Omega_s,\ s = \overline{1,N}.$$

Then the function system $v([t],x) = (v_1(t,x), v_2(t,x), \ldots, v_N(t,x))$ belongs to the space $C(\Omega, \Delta_N(\omega), \mathbb{R}^N)$, and its elements $v_s(t,x)$, $s = \overline{1,N}$, satisfy the following family of differential equations with discrete memory

$$\frac{\partial v_s}{\partial t} = A(t,x)v_s(t,x) + A_0(t,x)v_s(\zeta_s,x) + f(t,x) + B(t,x)w(t,x) + C(t,x)u(t,x), \tag{9}$$

$$(t,x) \in \Omega_s, \qquad s = \overline{1,N},$$

$$v_1(0,x) = \lim_{t\to T-0} v_N(t,x), \qquad x \in [0,\omega], \tag{10}$$

$$v_{p+1}(\theta_p,x) - \lim_{t\to\theta_p-0} v_p(t,x) = \dot{\varphi}_p(x), \qquad x \in [0,\omega], \qquad p = \overline{1,N-1}, \tag{11}$$

$$u(t,x)=\psi(t)+\int\limits_0^x v_s(t,\xi)d\xi,\ w(t,x)=\dot{\psi}(t)+\int\limits_0^x \frac{\partial v_s(t,\xi)}{\partial t}d\xi,\quad (t,x)\in\Omega_s,\ s=\overline{1,N}. \quad (12)$$

In (9) we take into account that $\gamma(t)=\zeta_s$ for all $t\in[\theta_{s-1},\theta_s)$, $s=\overline{1,N}$.

We introduce functional parameters in the next form: $\mu_s(x)=v_s(\zeta_s,x)$ for all $s=\overline{1,N}$ and $x\in[0,\omega]$.

Making the substitution $\widetilde{v}_s(t,x)=v_s(t,x)-\mu_s(x)$, $(t,x)\in\Omega_s$, $s=\overline{1,N}$, we obtain a problem with parameters for the family of differential equations in the following form

$$\frac{\partial\widetilde{v}_s}{\partial t}=A(t,x)\widetilde{v}_s(t,x)+[A(t,x)+A_0(t,x)]\mu_s(x)+f(t,x)+$$
$$+B(t,x)w(t,x)+C(t,x)u(t,x),\quad (t,x)\in\Omega_s,\ s=\overline{1,N}, \quad (13)$$

the initial conditions are

$$\widetilde{v}_s(\zeta_s,x)=0,\qquad x\in[0,\omega],\qquad s=\overline{1,N}, \quad (14)$$

the periodic condition is

$$\widetilde{v}_1(0,x)+\mu_1(x)=\lim_{t\to T-0}\widetilde{v}_N(t,x)+\mu_N(x),\qquad x\in[0,\omega], \quad (15)$$

the conditions with impulse effects

$$\widetilde{v}_{p+1}(\theta_p,x)+\mu_{p+1}(x)-\lim_{t\to\theta_p-0}\widetilde{v}_p(t,x)-\mu_p(x)=\dot{\varphi}_p(x),\quad x\in[0,\omega],\quad p=\overline{1,N-1}, \quad (16)$$

and the integral equations

$$u(t,x)=\psi(t)+\int\limits_0^x[\widetilde{v}_s(t,\xi)+\mu_s(\xi)]d\xi,\quad w(t,x)=\dot{\psi}(t)+\int\limits_0^x\frac{\partial\widetilde{v}_s(t,\xi)}{\partial t}d\xi,$$
$$(t,x)\in\Omega_s,\ s=\overline{1,N}. \quad (17)$$

A solution to the problem with parameters (13)–(17) is called a quadruple

$$\{\widetilde{v}([t],x),\mu(x),u(t,x),w(t,x)\},$$

with elements $\{\widetilde{v}_s(t,x),\mu_s(x),\,u(t,x),w(t,x)\}$, where the functions $\widetilde{v}_s(t,x)\in C(\Omega,\Delta_N(\omega),\mathbb{R})$ have the derivative $\frac{\partial\widetilde{v}_s(t,x)}{\partial t}\in C(\Omega,\Delta_N(\omega),\mathbb{R})$, the functional parameters $\mu_s(x)\in C([0,\omega],\mathbb{R})$, $s=\overline{1,N}$, the functions $u(t,x),w(t,x)\in C(\Omega,\Delta_N(\omega),\mathbb{R})$, and satisfies to the family of the differential equations (13) for all $(t,x)\in\Omega_s$, $s=\overline{1,N}$, the initial conditions (14), the boundary condition (15), the conditions with impulse effects (16) for all $x\in[0,\omega]$. The functions $u(t,x)$

and $w(t,x)$ are connected with $\widetilde{v}_s(t,x)$ and $\frac{\partial \widetilde{v}_s(t,x)}{\partial t}$ by the integral equations (17) for all $(t,x) \in \Omega_s,\ s=\overline{1,N}$.

At fixed $\mu_s(x)$, $w(t,x)$, $u(t,x)$ the problem (13)–(14) is a family of Cauchy problems for differential equations.

Let $\alpha(t,x) = \int\limits_{\zeta_s}^{t} A(\tau,x)d\tau, \quad (t,x) \in \Omega_s, \quad s=\overline{1,N}.$

A solution of the family Cauchy problems (13)–(14) is unique and has the next form

$$v\widetilde{v}_s(t,x) = e^{\alpha(t,x)} \int\limits_{\zeta_s}^{t} e^{-\alpha(\tau,x)}[A(\tau,x)+A_0(\tau,x)]\mu_s(x)d\tau + e^{\alpha(t,x)} \int\limits_{\zeta_s}^{t} e^{-\alpha(\tau,x)} f(\tau,x)d\tau +$$
$$+ e^{\alpha(t,x)} \int\limits_{\zeta_s}^{t} e^{-\alpha(\tau,x)} \Big[B(\tau,x)w(\tau,x) + C(\tau,x)u(\tau,x)\Big] d\tau, \quad (t,x) \in \Omega_s,\ s=\overline{1,N}. \quad (18)$$

We introduce the following notations:

$$D_s(t,x) = e^{\alpha(t,x)} \int\limits_{\zeta_s}^{t} e^{-\alpha(\tau,x)}[A(\tau,x)+A_0(\tau,x)]d\tau,$$
$$H_s(t,x,w,u) = e^{\alpha(t,x)} \int\limits_{\zeta_s}^{t} e^{-\alpha(\tau,x)} \Big[B(\tau,x)w(\tau,x) + C(\tau,x)u(\tau,x)\Big] d\tau,$$
$$F_s(t,x) = e^{\alpha(t,x)} \int\limits_{\zeta_s}^{t} e^{-\alpha(\tau,x)} f(\tau,x)d\tau, \quad (t,x) \in \Omega_r,\ r=\overline{1,N}.$$

From the integral representation (18) we find

$$\widetilde{v}_1(0,x),\ \lim_{t\to T-0} \widetilde{v}_N(t,x),\ \widetilde{v}_{p+1}(\theta_p,x),\ \lim_{t\to\theta_p-0} \widetilde{v}_p(t,x), \quad p=\overline{1,N-1}.$$

Substituting the found expressions into Relations (15) and (16), we obtain

$$\big[1+D_1(0,x)\big]\mu_1(x) - \big[1+D_N(T,x)\big]\mu_N(x) = -F_1(0,x) + F_N(T,x) -$$
$$- H_1(0,x,w,u) + H_N(T,x,w,u), \quad x \in [0,\omega], \quad (19)$$

$$\big[1+D_{p+1}(\theta_p,x)\big]\mu_{p+1}(x) - \big[1+D_p(\theta_p,x)\big]\mu_p(x) = \varphi_p(x) + F_p(\theta_p,x) - F_{p+1}(\theta_p,x) +$$
$$+ H_p(\theta_p,x,w,u) - H_{p+1}(\theta_p,x,w,u), \quad p=\overline{1,N-1},\ x \in [0,\omega]. \quad (20)$$

Using the coefficients for $\mu_s(x)$, $s=\overline{1,N}$, on the left-hand sides of the system of the equations (19), (20), we compose an $N\times N$ matrix $Q(x)$ in the following form:

$$Q(x)=\begin{bmatrix} 1+D_1(0,x) & 0 & 0 & . & 0 & -1-D_N(T,x) \\ -1-D_1(\theta_1,x) & 1+D_2(\theta_1,x) & 0 & . & 0 & 0 \\ 0 & -1-D_2(\theta_2,x) & 1+D_3(\theta_2,x) & . & 0 & 0 \\ 0 & 0 & -1-D_3(\theta_3,x) & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & -1-D_{N-1}(\theta_{N-1},x) & 1+D_N(\theta_{N-1},x) \end{bmatrix}$$

Let us write down the system of the equations (19)–(20) in the next form:

$$Q(x)\mu(x)=-F_*(x)-H_*(x,w,u), \qquad x\in[0,\omega], \tag{21}$$

where the N vector functions $F_*(\Delta_N(\omega),x)$, $H_*(\Delta_N(\omega),x,w,u)$ have the forms

$$F_*(x)=\Big(F_1(0,x)-F_N(T,x),-\varphi_1(x)-F_1(\theta_1,x)+F_2(\theta_1,x),\dots,$$
$$-\varphi_{N-1}(x)-F_{N-1}(\theta_{N-1},x)+F_N(\theta_{N-1},x)\Big),$$

$$H_*(x,w,u)=\Big(H_1(0,x,w,u)-H_N(T,x,w,u),-H_1(\theta_1,x,w,u)+H_2(\theta_1,x,w,u),\dots$$
$$-H_{N-1}(\theta_{N-1},x,w,u)+H_N(\theta_{N-1},x,w,u)\Big).$$

3 Algorithm and Main result

If the functions $w(t,x)$ and $u(t,x)$ are known for all $(t,x)\in\Omega_s$, $s=\overline{1,N}$, then from the system of functional equations (21) we find $\mu(x)$ with components $\mu_s(x)\in C([0,\omega],\mathbb{R})$, $s=\overline{1,N}$. Then from the integral representation (18) and the differential equations (13), we define $\widetilde{v}_s(t,x)$ and its derivative $\frac{\partial\widetilde{v}_s}{\partial t}$ for all $(t,x)\in\Omega_s$, $\quad s=\overline{1,N}$.

Conversely, if $\widetilde{v}_s(t,x)$, $\frac{\partial\widetilde{v}_s}{\partial t}$, and $\mu_s(x)\in C([0,\omega],\mathbb{R})$ are known for all $(t,x)\in\Omega_s$, where $s=\overline{1,N}$, then from the integral equations (17) we can find the functions $u(t,x)$, $w(t,x)$ for all $(t,x)\in\Omega_s$, $s=\overline{1,N}$.

Since the function $\widetilde{v}_s(t,x)$ and its derivative $\frac{\partial\widetilde{v}_s}{\partial t}$, the functions $u(t,x)$, $w(t,x)$ and the functional parameters $\mu_s(x)$, $s=\overline{1,N}$, are unknown together, we use the iteration method to find a solution to the problem (13)–(17).

We determine a quadruple

$$\{\widetilde{v}^*([t],x),\mu^*(x),\ u^*(t,x),w^*(t,x)\},$$

with elements $\{\widetilde{v}_s^*(t,x),\ \mu_s^*(x),\ u^*(t,x),w^*(t,x)\}$, as a limit of sequence of quadruples

$$\{\widetilde{v}^{(k)}([t],x),\ \mu^{(k)}(x),\ u^{(k)}(t,x),\ w^{(k)}(t,x)\},$$

with elements $\{\widetilde{v}_s^{(k)}(t,x), \mu_s^{(k)}(x), u^{(k)}(t,x), w^{(k)}(t,x)\}$, $s=\overline{1,N}$, $k=0,1,2,...$ by the following *algorithm*:

Step 0. Assume that the $(N\times N)$ matrix $Q(x)$ is invertible for all $x\in[0,\omega]$.

1) Putting $u(t,x)=\psi(t)$, $w(t,x)=\dot{\psi}(t)$ on the right-hand side of system (21), we define initial approximation of functional parameter $\mu^{(0)}(x)=(\mu_1^{(0)}(x),\mu_2^{(0)}(x),...,\mu_N^{(0)}(x))$ with components $\mu_s^{(0)}(x)\in C([0,\omega],\mathbb{R})$ from the system of functional equations

$$Q(x)\mu(x)=-F_*(x)-H_*(x,\dot{\psi},\psi),\quad x\in[0,\omega].$$

2) Assuming on the right-hand side of the family of the differential equations (13) that $u(t,x)=\psi(t)$, $w(t,x)=\dot{\psi}(t)$, $\mu_s(x)=\mu_s^{(0)}(x)$, $s=\overline{1,N}$, and solving the family of Cauchy problems (13)–(14), we find $\widetilde{v}_s^{(0)}(t,x)$

$$\widetilde{v}_s^{(0)}(t,x)=D_s(t,x)\mu_s^{(0)}(x)+F_s(t,x)+H_s(t,x,\dot{\psi},\psi),\quad (t,x)\in\Omega_s,\ s=\overline{1,N},\tag{22}$$

and we determine its derivative

$$\frac{\partial\widetilde{v}_s^{(0)}}{\partial t}=A(t,x)\widetilde{v}_s^{(0)}(t,x)+[A(t,x)+A_0(t,x)]\mu_s^{(0)}(x)+f(t,x)+\\ +B(t,x)\dot{\psi}(t)+C(t,x)\psi(t),\quad (t,x)\in\Omega_s,\ s=\overline{1,N}.\tag{23}$$

3) From the integral equations (17) we define $u^{(0)}(t,x)$ and $w^{(0)}(t,x)$ as follows:

$$u^{(0)}(t,x)=\psi(t)+\int_0^x[\widetilde{v}_s^{(0)}(t,\xi)+\mu_s^{(0)}(\xi)]d\xi,\quad w^{(0)}(t,x)=\dot{\psi}(t)+\int_0^x\frac{\partial\widetilde{v}_s^{(0)}(t,\xi)}{\partial t}d\xi,\\ (t,x)\in\Omega_s,\ s=\overline{1,N}.\tag{24}$$

Step 1. 1) Putting $u(t,x)=u^{(0)}(t,x)$ and $w(t,x)=w^{(0)}(t,x)$ on the right-hand side of the system (21), we define the first approximation of the functional parameter $\mu^{(1)}(x)=(\mu_1^{(1)}(x),\mu_2^{(1)}(x),\ldots,\mu_N^{(1)}(x))$ with components $\mu_s^{(1)}(x)\in C([0,\omega],\mathbb{R})$ from the system of the functional equations

$$Q(x)\mu(x)=-F_*(x)-H_*(x,w^{(0)},u^{(0)}),\qquad x\in[0,\omega].$$

2) Assuming on the right-hand side of the family of the differential equations (13)

$$u(t,x)=u^{(0)}(t,x),\ w(t,x)=w^{(0)}(t,x),\ \mu_s(x)=\mu_s^{(1)}(x),\ s=\overline{1,N},$$

and solving the family of Cauchy problems (13)–(14), we find $\widetilde{v}_s^{(1)}(t,x)$:

$$\widetilde{v}_s^{(1)}(t,x)=D_s(t,x)\mu_s^{(1)}(x)+F_s(t,x)+H_s(t,x,w^{(0)},u^{(0)}),\quad (t,x)\in\Omega_s,\ s=\overline{1,N},\tag{25}$$

and we determine its derivative

$$\frac{\partial\widetilde{v}_s^{(1)}}{\partial t} = A(t,x)\widetilde{v}_s^{(1)}(t,x) + [A(t,x) + A_0(t,x)]\mu_s^{(1)}(x) + f(t,x)+ \\ + B(t,x)w^{(0)}(t,x) + C(t,x)u^{(0)}(t,x), \quad (t,x) \in \Omega_s,\ s = \overline{1,N}. \quad (26)$$

3) From the integral equations (17) we define $u^{(1)}(t,x)$ and $w^{(1)}(t,x)$ as follows:

$$u^{(1)}(t,x) = \psi(t) + \int\limits_0^x [\widetilde{v}_s^{(1)}(t,\xi) + \mu_s^{(1)}(\xi)]d\xi, \quad w^{(1)}(t,x) = \dot{\psi}(t) + \int\limits_0^x \frac{\partial\widetilde{v}_s^{(1)}(t,\xi)}{\partial t}d\xi, \\ (t,x) \in \Omega_s,\ s = \overline{1,N}. \quad (27)$$

And so on.

Step k. 1) Putting $u(t,x) = u^{(k-1)}(t,x)$ and $w(t,x) = w^{(k-1)}(t,x)$ on the right-hand side of the system (21), we define the kth approximation of the functional parameter $\mu^{(k)}(x) = (\mu_1^{(k)}(x), \mu_2^{(k)}(x), ..., \mu_N^{(k)}(x))$ with the components $\mu_s^{(k)}(x) \in C([0,\omega],\mathbb{R})$ from the system of the functional equations

$$Q(x)\mu(x) = -F_*(x) - H_*(x, w^{(k-1)}, u^{(k-1)}), \quad x \in [0,\omega].$$

2) Assuming on the right-hand side of the family of the differential equations (13) that

$$u(t,x) = u^{(k-1)}(t,x),\ w(t,x) = w^{(k-1)}(t,x),\ \mu_s(x) = \mu_s^{(k)}(x),\ s = \overline{1,N},$$

and solving the family of Cauchy problems (13)–(14), we find $\widetilde{v}_s^{(1)}(t,x)$

$$\widetilde{v}_s^{(k)}(t,x) = D_s(t,x)\mu_s^{(k)}(x) + F_s(t,x) + H_s(t,x,w^{(k-1)},u^{(k-1)}), \quad (t,x) \in \Omega_s,\ s = \overline{1,N}, \quad (28)$$

and determine its derivative

$$\frac{\partial\widetilde{v}_s^{(k)}}{\partial t} = A(t,x)\widetilde{v}_s^{(k)}(t,x) + [A(t,x) + A_0(t,x)]\mu_s^{(k)}(x) + f(t,x)+ \\ + B(t,x)w^{(k-1)}(t,x) + C(t,x)u^{(k-1)}(t,x), \quad (t,x) \in \Omega_s,\ s = \overline{1,N}. \quad (29)$$

3) From the integral equations (17) we define $u^{(k)}(t,x)$ and $w^{(k)}(t,x)$ as follows:

$$u^{(k)}(t,x) = \psi(t) + \int\limits_0^x [\widetilde{v}_s^{(k)}(t,\xi) + \mu_s^{(k)}(\xi)]d\xi, \quad w^{(k)}(t,x) = \dot{\psi}(t) + \int\limits_0^x \frac{\partial\widetilde{v}_s^{(k)}(t,\xi)}{\partial t}d\xi, \\ (t,x) \in \Omega_s,\ s = \overline{1,N}. \quad (30)$$

Here $k = 1, 2, \ldots$

The developed method allows us to find unknown functions in three stages:

1) From the system of the functional equations (21) we determine the introduced functional parameters $\mu_s(x)$ for all $x \in [0, \omega]$, $s = \overline{1, N}$.

2) From the family of Cauchy problems (13), (14) we find unknown functions $\widetilde{v}_s(t, x)$ and its derivative $\frac{\partial \widetilde{v}_s(t,x)}{\partial t}$ for all $(t, x) \in \Omega_s$, $s = \overline{1, N}$.

3) From the integral equations (17) we define $u(t, x)$ and $w(t, x)$ for all $(t, x) \in \Omega_s$, $s = \overline{1, N}$.

Each of the problems has a unique solution under assumptions about the initial data. To implement the algorithm, it is necessary to establish the convergence of approximate solutions to the exact solution of the problem with parameters (13)–(17).

We use the following notations:

$$\begin{aligned} \alpha(x) &= \max_{t \in [0,T]} ||A(t, x)||, \\ \alpha_0(x) &= \max_{t \in [0,T]} ||A_0(t, x)||, \\ \theta &= \max\Big\{ \max_{r=\overline{1,N}} (\theta_r - \zeta_{r-1}), \max_{r=\overline{1,N}} (\zeta_{r-1} - \theta_{r-1}) \Big\}. \end{aligned}$$

The following theorem establishes conditions for the convergence of the proposed algorithm and the existence of a unique solution to the problem with parameters (13)–(17).

Theorem 1. *Assume that the $N \times N$ matrix $Q(x)$ is invertible for all $x \in [0, \omega]$. Then the problem with parameters (13)–(17) has a unique solution.*

From the equivalent problems (1)–(4) and (13)–(17) we have the following.

Theorem 2. *Assume that the $N \times N$ matrix $Q(x)$ is invertible for all $x \in [0, \omega]$. Then the periodic problem for the impulsive hyperbolic equation with discrete memory (1)–(4) has a unique solution.*

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Асанова А. Т., Молыбайқызы А. ДИСКРЕТ ЖАДЫЛЫ ИМПУЛЬСТІК ГИПЕРБОЛАЛЫҚ ТЕҢДЕУ ҮШІН ПЕРИОДТЫ ЕСЕПТІҢ ШЕШІМІ

Мақалада дискрет жадылы импульстік гиперболалық теңдеу үшін периодты есеп қарастырылады. Дискрет жадылы импульстік гиперболалық теңдеулер нейрондық желілердегі, үзілісті динамикалық жүйелердегі, гибрид жүйелердегі және т.т. физикалық үдерістерді сипаттауға арналған математикалық моделдер ретінде туындайды. Дискрет жадылы импульстік гиперболалық теңдеулер үшін периодты есептердің шешімдерінің бар болуы мен құру мәселелері үзілісті дербес туындылы дифференциалдық теңдеулер теориясының маңызды мәселесі болып қалып отыр. Осы есептің шешілімділік шарттарын табу үшін Джумабаевтың параметрлеу әдісі пайдаланылады. Дискрет жадылы импульстік гиперболалық теңдеу үшін периодты есептің шешімінің бар болуы мен жалғыздығының коэффициенттік шарттары орнатылған. Осы есептің жуық шешімін анықтау алгоритмі ұсынылған және дискрет жадылы импульстік гиперболалық теңдеу үшін периодты есептің дәл шешіміне жинақтылығы көрсетілген.

Түйін сөздер: гиперболалық теңдеу, импульс әсерлері, периодты шарт, дискрет жады, облысты бөліктеу, параметрлері бар есеп, шешілімділік шарттары.

Асанова А. Т., Молыбайкызы А. РЕШЕНИЕ ПЕРИОДИЧЕСКОЙ ЗАДАЧИ ДЛЯ ИМПУЛЬСНОГО ГИПЕРБОЛИЧЕСКОГО УРАВНЕНИЯ С ДИСКРЕТНОЙ ПАМЯТЬЮ

В статье рассматривается периодическая задача для импульсного гиперболического уравнения с дискретной памятью. Импульсные гиперболические уравнения с дискретной памятью возникают как математические модели, описывающие физические процессы в нейронных сетях, в разрывных динамических системах, в гибридных системах и т.д. Вопросы существования и построения решения периодических задач для импульсных гиперболических уравнений с дискретной памятью остаются важными проблемами теории разрывных дифференциальных уравнений в частных производных. Для нахождения условий разрешимости этой задачи используется метод параметризации Джумабаева. Установлены коэффициентные условия существования и единственности решения периодической для импульсного гиперболического уравнения с дискретной памятью. Предложен алгоритм для определения приближенного решения данной задачи и показана сходимость к точному решению периодической задачи для импульсного гиперболического уравнения с дискретной памятью.

Ключевые слова: гиперболическое уравнение, импульсные воздействия, периодическое условие, дскретная память, деление области, задача с параметрами, условия разрешимости.

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Dirichlet and Neumann problems for the heat equation on linear multilink thermal graphs and their solutions

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Abstract. We consider boundary value problems of thermal conductivity on a linear thermal graph, which can be used to study various structures under conditions of thermal heating (cooling). Here, based on the generalized function method, a unified technique has been developed for solving boundary value problems of thermal conductivity, typical for engineering applications. Generalized solutions to nonstationary and stationary boundary value problems of heat conduction on an edge and on a thermal linear graph are constructed under various boundary conditions at the ends of the graph and generalized Kirchhoff conditions at its node. Using the properties of the Fourier transformant of the fundamental solution, regular integral representations of solutions to boundary value problems are obtained in analytical form. The solutions obtained make it possible to simulate heat sources of various types, including using singular generalized functions. The method of generalized functions presented here makes it possible to solve a wide class of boundary value problems with local and connected boundary conditions at the ends of the edges of the graph and different transmission conditions at its nodes.

Keywords. Thermal conductivity, generalized functions, fundamental and generalized solution, Fourier transform, resolving boundary equations, linear graph.

1 Introduction

Graph theory has wide applications in subjects such as economics, logistics, sociology, optimal control, and navigation [1–2]. The properties of graphs are also actively used to solve

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boundary value problems (BVPs) on network-like structures, e.g., oil pipelines, gas pipelines, and electrical networks [3–10]. With the development of mechanical engineering, complex multi-link rod structures operating under various thermal conditions began to be actively used. They are widely used in structural mechanics, mechanical engineering, robotics, and many other fields.

Mathematical modeling of the thermodynamics of rod structures and the creation of information technologies based on it is one of the more effective and inexpensive methods for researching and designing such systems. An urgent scientific and technical task is to study the thermal state of network systems for various purposes under thermal influences, taking into account their construction and thermal influences, including impact types. This is necessary to analyze the strength and reliability of such objects, determine safe operating modes, and prevent disasters.

Here boundary value problems are considered on the linear multilink thermal graph (Fig. 1), which can be used to study various mesh structures under conditions of volume and thermal heating (cooling).

The novelty of the present work lies in the fact that a generalized function method is used to solve boundary value problems, leading to a differential equation solution with a singular right-hand side [11]. The solution is constructed as the convolution of the Green's function of the equation with the appropriate right-hand side. To determine the unknown boundary values of the solution and its derivatives on each segment, resolving boundary equations are constructed at the ends, employing the asymptotic properties of Green's function and its derivative at zero. To construct a closed system of equations, the obtained algebraic equations for each edge of the graph are supplemented with transmission conditions at the node and linear boundary conditions at its ends. These conditions can be either locally or not locally connected.

A resolving system of equations in the space of Fourier transforms over time and Fourier transforms of temperature on each link of the graph are constructed, which give a solution to stationary boundary value problems with oscillations with a fixed frequency. The inverse Fourier transform is used to construct the original. The obtained solutions give analytical formulas for calculating the temperature of such structures under thermal heating conditions, and can be used in the design of heating networks, as well as for solving boundary value problems in environments stratified by thermal graphs.

2 Statement of the boundary value problem on thermal linear graphs

We consider a thermal linear graph which contains N edges (A_{j-1}, A_j) of the length L_j, where $j = 1, 2, \ldots, N$ (Fig. 1). On each edge $S_j = \left\{x \in R^1 : 0 \le x \le L_j\right\}$ there is its own coordinate system (x_j, t) with the origin at the point A_{j-1}, that is, $x_j = 0$ at A_{j-1} and $x_j = L_j$ at A_j.

The temperature $\theta_j(x,t)$ satisfies the heat conduction equation at S_j:

$$\frac{\partial \theta_j}{\partial t} - \kappa_j \frac{\partial^2 \theta_j}{\partial x^2} = F_j(x,t). \tag{1}$$

Here κ_j is the thermal diffusivity coefficient on the j-th segment, $F_j(x,t)$ describes the action of the heat source, $\theta_1^j(t)$ and $\theta_2^j(t)$ are the temperatures at the ends of the j-th edge.

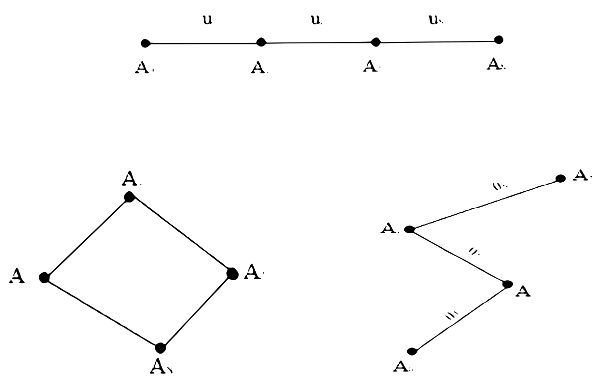

Figure 1. Linear graphs

The initial conditions at $t = 0$ for the temperature of a graph are known:
(Cauchy conditions)

$$\theta_j(x,0) = {\theta_0}^j(x), \quad 0 \le x \le L_j, \quad t = 0, \tag{2}$$

$$\theta_j(0) = \theta_0, \tag{3}$$

where ${\theta_0}^j(x) \in C^2(R_+)$ for each j. Here we consider the two boundary value problems (BVP), $R_+^1 = \{t \in [0,\infty)\}$

Dirichlet conditions (BVP1). Temperature values are known at the ends of the graph:

$$\begin{array}{ll} \theta_1^1(t) \;=\; \theta_1(0,t) = \vartheta_1(t), & t \ge 0, \quad \vartheta_1(t) \in C(R_+^1), \\ \theta_2^N(t) = \theta_N(L_N,t) = \vartheta_2(t), & t \ge 0, \quad \vartheta_2(t) \in C(R_+^1). \end{array} \tag{4}$$

Here and further

$$\theta_j^1(t) = \theta_j(0,t),\ q_j^1(t) = \partial_x \theta_j(0,t),\ \theta_j^2(t) = \theta_j(L_J,t),\ q_j^2(t) = \partial_x \theta_j(L_J,t).$$

Neumann conditions (BVP 2). The values of the heat flows are known at the ends of the graph:

$$\begin{array}{ll} \kappa_1\, q_1^1\,(t) = \kappa_1 q_1(0,t) = \chi_1(t), & t \ge 0, \quad \chi_1(t) \in C(R_+^1), \\ \kappa_N q_2^N(t) = \kappa_N q_N(L_N,t) = \chi_2(t), & t \ge 0, \quad \chi_2(t) \in C(R_+^1). \end{array} \tag{5}$$

The following continuity conditions and generalized Kirchhoff conditions are specified in the common node A_0 of the graph.

Transmission conditions:

$$\begin{array}{l} \theta_2^j(t) = \theta_1^{j+1}(t), \quad j = 1, ..., N-1, \quad t \geq 0, \\ \theta_1^1(0) = \vartheta_1(0), \\ \theta_2^N(0) = \vartheta_2(0), \end{array} \tag{6}$$

$$\kappa_j q_2^j(t) = \kappa_{j+1} q_1^{j+1}(t) + Q_j(t), \quad j = 1, ..., N-1, \quad t \geq 0. \tag{7}$$

Here

$$\theta_1^j(t) = \theta_j(0,t), \; q_1^j(t) = \left.\frac{\partial \theta_j}{\partial x}\right|_{x=0}, \; q_2^j(t) = \left.\frac{\partial \theta_j}{\partial x}\right|_{x=L_j},$$

θ_0 is the initial temperature at the common node A_0.

We need to find the solutions of these two BVP on the heat linear graph by known $Q_j(t)$, where $j = 1, \ldots, N$, $\vartheta_1(t)$ and $\vartheta_N(t)$ (Dirichlet problem) or $\chi_1(t)$ and $\chi_N(t)$ (Neumann problem).

3 Statement of boundary value problem on a segment of a graph

At first we construct a solution of some boundary value on one graph segment. Let consider $\theta(x,t)$ on $[0, L]$, which is the solution of heat equation:

$$\frac{\partial \theta}{\partial t} - \kappa \frac{\partial^2 \theta}{\partial x^2} = F(x,t). \tag{8}$$

Initial conditions: the temperature is known at $t = 0$:

$$\theta(x,0) = \theta_0(x), \quad \theta_0(x) \in C\{0 \leq x \leq L\} \tag{9}$$

Here we consider solutions to BVPs with local and associated boundary conditions.

Local boundary conditions:

$$\begin{cases} (\alpha_1 \theta_1 + \beta_1 \Pi_1(t))|_{x=0} = G_1(t), \\ (\alpha_2 \theta_2 + \beta_2 \Pi_2(t))|_{x=L} = G_2(t). \end{cases} \tag{10}$$

where α_j, β_j arbitrary constants, $\theta_j(t)$, $\Pi_j(t) = -k \frac{\partial \theta}{\partial x}\big|_{x=x_j}$ $(j = 1, 2)$ are the temperature and heat flow at ends of the segment in points: $x = x_1 = 0$, $x = x_2 = L$. $G_j(t)$ are known functions which are integrated functions on $R_+^1 : G_j(t) \in L_1(R_+^1)$.

Connected boundary conditions:

$$\alpha_{1j}\theta_1(t) + \beta_{1j}\Pi_1(t) + \alpha_{2j}\theta_1(t) + \beta_{2j}\Pi_2(t) = D_j(t), \quad j = 1, 2. \tag{11}$$

Conditions for matching initial and boundary conditions:

$$\theta_1(t) = \theta(0,t), \quad \theta_2(t) = \theta(L,t), \quad \theta_j(t) \in C(R_1^+).$$

It is assumed that all functions defining boundary conditions also belong to Lebesgue space L_1. Relations (11) contain all classical formulations of heat BVPs if we take some $\alpha_{ij} = 0$, $\beta_{ij} = 0$. We find solutions to BVPs using the Generalized Function Method [14].

4 Generalized solution of boundary value problems on an graph segment. Generalized function method

To determine the solution on the graph, at first, we consider the BVP on the graph segment by using the general function method. For this, we consider the BVP for the heat equation on the segment $[0, L]$ in the space $S'(R^2) = \{\hat{f}(x,t), (x,t) \in R^2\}$ of generalized functions of slow growth [15]. To do this, we introduce a regular generalized function (we mark it with a cap):

$$\hat{\theta}(x,t) = \begin{cases} \theta(x,t), & (x,t) \in D^- \\ 0, & x \notin D^- \end{cases},$$

where $\theta(x,t)$ is the solution of BVP, $D^- = [0,L] \times [0,\infty)$. It can be represented in the form

$$\hat{\theta}(x,t) = \theta(x,t)H(L-x)H(x)H(t).$$

Here $H(x)$ is the Heaviside step function.

To construct the equation for $\hat{\theta}(x,t)$ in $S'(R^2)$, we calculate the generalized derivatives of $\hat{\theta}(x,t)$:

$$\frac{\partial\hat{\theta}}{\partial x} = \frac{\partial\theta}{\partial x}H(L-x)H(x)H(t) - \theta_2(t)\delta(L-x)H(t) + \theta_1(t)\delta(x)H(t),$$

$$\frac{\partial^2\hat{\theta}}{\partial x^2} = \frac{\partial^2\theta}{\partial x^2}H(L-x)H(x)H(t) - q_2(t)\delta(L-x)H(t) + q_1(t)\delta(x)H(t) + \\ +\theta_2(t)\delta'(L-x)H(t) + \theta_1(t)H(t)\delta'(x),$$

$$\frac{\partial\hat{\theta}}{\partial t} = \frac{\partial\theta}{\partial t}H(L-x)H(t) + \theta_0(x)H(L-x)\delta(t),$$

where $\delta(x)$ is a singular generalized δ-function, $q_j(t) = \frac{\partial\theta}{\partial x}\big|_{x=x_j}$, $j = 1,2$.

The equation (7) in $S'(R^2)$ has the following form for $\hat{\theta}(x,t)$:

$$\frac{\partial\hat{\theta}}{\partial t} - \kappa\frac{\partial^2\hat{\theta}}{\partial x^2} = \hat{F}_2(x,t) + \kappa q_2(t)\delta(L-x)H(t) - \kappa q_1(t)\delta(x)H(t) - \\ - \kappa\theta_2(t)\delta'(L-x)H(t) - \kappa\theta_1(t)\delta'(x)H(t) + \theta_0(x)H(L-x)H(x)\delta(t). \quad (12)$$

Note that the right side of this equation includes all initial and boundary temperatures $\theta_j(t)$ and heat flows $\Pi_j(t) = \kappa q_j(t)$ (j=1, 2).

Throughout the paper, we denote the partial derivative $\frac{\partial U}{\partial x}$ by $U_{,x}(x,t)$.

According to the theory of generalized functions [15], the solution of (12) can be represented as a convolution of the fundamental solution of the heat equation (8) with the right-hand side of this equation:

$$\hat{\theta}(x,t) = \hat{F}_2(x,t) * U(x,t) + \kappa q_2(t)H(t) \underset{t}{*} U(L-x,t) - \\ - \kappa q_1(t)H(t) \underset{t}{*} U(x,t) - \kappa\theta_2(t)H(t) \underset{t}{*} U_{,x}(L-x,t) - \\ - \kappa\theta_1(t)H(t) \underset{t}{*} U_{,x}(x,t) + \theta_0(x)H(L-x)H(x) \underset{x}{*} U(x,t). \quad (13)$$

Here, $U(x,t)$ is the fundamental solution of the heat equation (1) by $F(x,t) = \delta(x,t) = \delta(x)\delta(t)$. It decays at ∞ and has the form [15]:

$$U(x,t) = \frac{1}{\sqrt{2\pi\kappa t}} \exp(-x^2/4\kappa t)H(t). \quad (14)$$

We denote $\hat{F}(x,t) = F(x,t)H(x)H(L-x)H(t)$. If it is a regular function, then relation (13) can be represented in the next integral form:

$$\theta(x,t)H(L-x)H(x)H(t) = \\ = H(t)\int_0^t d\tau \int_{-\infty}^{+\infty} U(x-y,t-\tau)\,F_2(y,\tau)dy + \kappa H(x)H(t)\int_0^t q_2(t-\tau)U(L-x,\tau)d\tau - \\ - \kappa H(L-x)H(t)\int_0^t U(x-y,t-\tau)q_1(\tau)d\tau - \kappa H(x)H(t)\int_0^t \theta_2(t-\tau)U_{,x}(L-x,\tau)d\tau - \\ - \kappa H(L-x)H(t)\int_0^t U_{,x}(x,t-\tau)\theta_1(\tau)d\tau + \int_0^L U(x-y,t)\theta_0(y)H(L-y)H(y)dy. \quad (15)$$

Formula (15) determines the temperature inside a segment by known temperature and heat flows at its ends and is very useful for engineering applications. However, for correctly posed boundary value problems, out of 4 boundary functions on the right side of formula (15), only 2 are known. To determine two unknown boundary functions, resolving boundary equations should be constructed using boundary conditions at the ends of the segment.

5 Solving boundary value problem in Fourier transformation space in time

To construct the resolving system of equations, we use Fourier transformation in time:

$$\bar{\theta}(x,\omega) = \mathrm{F}\left[\hat{\theta}(x,t)\right] = H(x)H(L-x)\int_0^\infty \theta(x,t)e^{i\omega t}dt,$$
$$\hat{\theta}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\theta}(x,\omega)e^{-i\omega t}d\omega. \tag{16}$$

To define the Fourier transform of the generalized solution (11) we use the property of Fourier transform of convolution [15]:

$$\begin{aligned}\hat{\theta}(x,\omega) = \bar{F}_2(x,\omega) \underset{x}{*} \bar{U}(x,\omega) + \theta_0(x)H(L-x)H(x) \underset{x}{*} \bar{U}(x,\omega)+\\ + \kappa\bar{q}_2(\omega)H(x)\bar{U}(L-x,\omega) - \kappa\bar{q}_1(\omega)H(L-x)\bar{U}(x,\omega)-\\ - \kappa\bar{\theta}_2(\omega)H(x)\bar{U}_{,x}(L-x,\omega) - \kappa\bar{\theta}_1(\omega)H(L-x)\bar{U}_{,x}(x,\omega).\end{aligned} \tag{17}$$

Here, a variable under the sign of convolution $\left(\underset{x}{*}\right)$ shows the convolution is applied only over the variable x. The integral representation of Equation (17) has the form:

$$\begin{aligned}\bar{\theta}(x,\omega)H(L-x)H(x)H(\omega) =\\ = H(x)\int_0^L \bar{U}\left(x-y,\omega\right)F_2(y,\omega)dy + \kappa H(x)\int_0^L \bar{U}(x-y,\omega)\theta_0(y)dy+\\ + \kappa\bar{q}_2(\omega)H(x)\bar{U}(L-x,\omega) - \kappa\bar{q}_1(\omega)H(L-x)\bar{U}(x,\omega) -\\ - \kappa\bar{\theta}_2(\omega)H(x)\bar{U}_{,x}(L-x,\omega) - \kappa\bar{\theta}_1(\omega)H(L-x)\bar{U}_{,x}(x,\omega).\end{aligned} \tag{18}$$

Fourier transform of Green's function of the heat equation is equal to

$$\bar{U}(x,\omega) = -\frac{\sin\left(k\left|x\right|\right)}{2k\,\kappa}, \tag{19}$$

where $k = \sqrt{i\omega\,\kappa^{-1}} = e^{i\pi/4}\sqrt{\omega\,\kappa^{-1}} = (1+i)\sqrt{\frac{\omega}{2\kappa}}$. It satisfies the equation:

$$\frac{d^2\bar{U}}{dx^2} + i\,\omega\,\kappa^{-1}\bar{U} = \delta(x).$$

Its derivative has the gap in point $x=0$ and equal to

$$\bar{U}_{,x}(x,\omega) = -\frac{\mathrm{sgn}x}{2\kappa}\cos(\kappa\left|x\right|),\ \mathrm{sgn}x = \begin{cases} 1,\ x>0,\\ -1,\ x<0.\end{cases}$$

There are the following symmetry conditions:

$$\bar{U}(x,\omega) = \bar{U}(-x,\omega),\ \bar{U}_{,x}(\pm 0,\omega) = \mp\frac{1}{2\kappa}. \tag{20}$$

We use these properties for solving BVP.

6 Resolving equations of boundary value problems

To find unknown boundary functions, we pass in relation (18) to the limit at $x \to 0+\varepsilon$, where $\varepsilon > 0$:

$$\bar{\theta}_1(\omega) = \lim_{\varepsilon\to 0}\bar{\theta}(0+\varepsilon,\omega) = \bar{F}(x,\omega) \underset{x}{*} \bar{U}(x,\omega)\big|_{x=0} + \theta_0(x)H(L-x)H(x) \underset{x}{*} \bar{U}(x,\omega)\big|_{x=0}+$$

$$+\kappa\bar{q}_2(\omega)H(x)\bar{U}(L-0-\varepsilon,\omega) - \kappa\bar{q}_1(\omega)H(L-x)\bar{U}(0+\varepsilon,\omega)-$$

$$-\kappa\bar{\theta}_2(\omega)H(x)\bar{U}_x(L-0-\varepsilon,\omega) - \kappa\bar{\theta}_1(\omega)H(L-x)\bar{U}_{,x}(0+\varepsilon,\omega).$$

Next, we move the last term to the left side and take into account the right limit of $\bar{U}_{,x}(x,\omega)$ at zero (20). We obtain the next equation on the left end of the segment:

$$\frac{1}{2}\bar{\theta}_1(\omega) = \bar{F}(x,\omega) \underset{x}{*} \bar{U}(x,\omega)\big|_{x=0} + \theta_0(x)H(L-x)H(x) \underset{x}{*} \bar{U}(x,\omega)\big|_{x=0}+ \\ + \kappa\bar{q}_2(\omega)H(x)\bar{U}(L,\omega) - \kappa\bar{q}_1(\omega)\bar{U}(0,\omega) - \kappa\bar{\theta}_2(\omega)H(x)\bar{U}_{,x}(L,\omega) \tag{21}$$

Similarly, we consider the limit at $x \to L-\varepsilon,\ \varepsilon > 0$.

$$\bar{\theta}_2(\omega) = \lim_{\varepsilon\to 0}\bar{\theta}(L-\varepsilon,\omega) = \bar{F}(x,\omega) \underset{x}{*} \bar{U}(x,\omega)\big|_{x=L} + \theta_0(x)H(L-x)H(x) \underset{x}{*} \bar{U}(x,\omega)\big|_{x=L}- \\ - \kappa\bar{q}_1(\omega)\bar{U}(L-\varepsilon,\omega) - \kappa\bar{\theta}_1(\omega)\bar{U}_{,x}(L-\varepsilon,\omega) - \kappa\bar{\theta}_2(\omega)H(x)\bar{U}_{,x}(\varepsilon,\omega) \tag{22}$$

We move the last term to the left side, and obtain the second boundary equation:

$$\frac{1}{2}\bar{\theta}_2(\omega) = \bar{F}(x,\omega) \underset{x}{*} \bar{U}(x,\omega)\big|_{x=L} + \theta_0(x)H(L-x)H(x) \underset{x}{*} \bar{U}(x,\omega)\big|_{x=L}- \\ - \kappa\bar{q}_1(\omega)\bar{U}(L,\omega) - \kappa\bar{\theta}_1(\omega)\bar{U}_{,x}(L,\omega) \tag{23}$$

We formulate the obtained results in the form of this theorem.

Theorem 1. *The Fourier time transformants of boundary functions of boundary value problems* (7)–(10) *satisfy the system of linear algebraic equations of the form:*

$$\begin{bmatrix} 0,5 & 0 \\ \kappa\bar{U}_{,x}(L,\omega) & \kappa\bar{U}(L,\omega) \end{bmatrix} \begin{bmatrix} \bar{\theta}_1(\omega) \\ \bar{q}_1(\omega) \end{bmatrix} + \\ + \begin{bmatrix} \kappa\bar{U}_{,x}(L,\omega) & -\kappa\bar{U}(L,\omega) \\ 0,5 & 0 \end{bmatrix} \begin{bmatrix} \bar{\theta}_2(\omega) \\ \bar{q}_2(\omega) \end{bmatrix} = \begin{bmatrix} \bar{Q}_1(0,\omega) \\ \bar{Q}_2(L,\omega) \end{bmatrix}, \quad (24)$$

where

$$\bar{Q}_1(0,\omega) = \bar{F}(x,\omega) \underset{x}{*} \bar{U}(x,\omega)\big|_{x=0} + \theta_0(x)H(L-x)H(x) \underset{x}{*} \bar{U}(x,\omega)\big|_{x=0}, \quad (25)$$

$$\bar{Q}_2(L,\omega) = \bar{F}(x,\omega) \underset{x}{*} \bar{U}(x,\omega)\big|_{x=L} + \theta_0(x)H(L-x)H(x) \underset{x}{*} \bar{U}(x,\omega)\big|_{x=L}. \quad (26)$$

The resulting system (20) makes it possible to solve BVP for any given two boundary functions of temperature and heat flow at the ends of a segment of four boundary functions. To solve all temperature BVPs, it is convenient to consider the extended system of equations in the form of a matrix equation:

$$\mathrm{A}(\omega) \cdot \mathrm{B}(\omega) = \mathrm{C}(\omega), \quad (27)$$

where

$$\mathrm{A}(\omega) = \begin{pmatrix} 0,5 & 0 & \kappa\bar{U}_{,x}(L,\omega) & -\kappa\bar{U}(L,\omega) \\ \kappa\bar{U}_{,x}(L,\omega) & \kappa\bar{U}(L,\omega) & 0,5 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix},$$

$$B(\omega) = \left(\bar{\theta}_1(\omega), \bar{q}_1(\omega), \bar{\theta}_2(\omega), \bar{q}_2(\omega)\right),$$

$$\mathrm{C}(\omega) = (\bar{Q}_1(0,\omega), \bar{Q}_2(L,\omega), \bar{b}_3(\omega), \bar{b}_4(\omega)).$$

The last two equations in the system (27) are determined by boundary conditions at the ends of the segment, which are known for BVP:

$$\begin{bmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \begin{bmatrix} \bar{\theta}_1(\omega) \\ \bar{q}_1(\omega) \end{bmatrix} + \begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} \bar{\theta}_2(\omega) \\ \bar{q}_2(\omega) \end{bmatrix} = \begin{bmatrix} \bar{b}_3(\omega) \\ \bar{b}_4(\omega) \end{bmatrix}. \quad (28)$$

By given coefficients a_{ij} and right-hand side $b_i(\omega)$, we have four equations (27) for definition of four boundary functions. The solution of Eqs (27) has the form:

$$B(\omega) = \mathrm{A}^{-1}(\omega) \times C(\omega), \quad (29)$$

where $\mathbf{A}^{-1}(\omega)$ is the inverse matrix of $\mathrm{A}(\omega)$.

So, all boundary functions are defined; therefore, the Fourier transform (17) for solving the boundary value problem is constructed. Using the inverse Fourier transform (16), we obtain the original $\theta(x,t)$ on the segment [0, L].

We use the solution (17) and Eqs (24) for constructing the solution of BVP on the linear graph.

7 Algebraic equations for determining unknown boundary functions on a heat linear graph

We return to the consideration of BVP for the heat equation on a heat linear graph (Fig. 1). On each segment L_j of the graph, we have the system of linear algebraic equations for determining four boundary functions:

$$\begin{pmatrix} 1 & 0 & -\cos(k_j L_j) & \frac{\sin(k_j L_j)}{k_j(\omega)} \\ -\cos(k_j L_j) & -\frac{\sin(k_j L_j)}{k_j(\omega)} & 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{\theta}_1^j(\omega) \\ \bar{q}_1^j(\omega) \\ \bar{\theta}_2^j(\omega) \\ \bar{q}_2^j(\omega) \end{pmatrix} = \begin{pmatrix} \bar{F}_1^j(\omega) \\ \bar{F}_2^j(\omega) \end{pmatrix}, \tag{30}$$

$$k_j(\omega) = (1+i)\sqrt{\frac{\omega}{2\kappa_j}}, \quad j = 1, ..., N.$$

Here, j denotes the number of the corresponding graph segment, and $\bar{F}_1^j(\omega) = 2\bar{Q}_1^j(0,\omega)$, $\bar{F}_2^j(\omega) = 2\bar{Q}_2^j(L,\omega)$. So, we have $2N$ equation for the determination of $4N$ boundary functions at every edge: $B(\omega) = \left(\bar{\theta}_1^1, \bar{q}_1^1, \bar{\theta}_2^1, \bar{q}_2^1, ..., \bar{\theta}_1^N, \bar{q}_1^N, \bar{\theta}_2^N, \bar{q}_2^N\right)$. Also, we have 2 conditions on the ends of the graph (4) or (5) and $2(N-1)$ transmission conditions at the node points of this graph (6). So we have the full system $4N$ equations for determination of $4N$ boundary functions at every edge.

Theorem 2. *Resolving system of equations of Dirichlet boundary value problem* (2), (4), (6) *on a heat linear graph with N different segments has the form:*

$$\mathbf{\Lambda}1(\omega) \times B(\omega) = C(\omega), \tag{31}$$

Resolving system of equations of Neumann boundary value problem (2), (4), (6) *on a heat linear graph with N different segments has the form:*

$$\mathbf{\Lambda}2(\omega) \times B(\omega) = C(\omega), \tag{32}$$

Here the matrices $\mathbf{\Lambda}1(\omega)$, $\mathbf{\Lambda}2(\omega)$ *have the following dimensions* $4N \times 4N$.

The first $2N$ *lines along the diagonal* $\mathbf{\Lambda}1(\omega)$, $\mathbf{\Lambda}2(\omega)$ *contain the connection matrices* (30) *of unknown boundary functions of edges. The remaining elements are zero:*

$$\{\Lambda_{ij}\} = \begin{pmatrix} \Lambda_1(\omega) & O_{2\times4} & O_{2\times4} & ... & & ... & ... & ... & O_{2\times4} \\ O_{2\times4} & \Lambda_2(\omega) & O_{2\times4} & ... & & ... & ... & ... & O_{2\times4} \\ O_{2\times4} & ... & ... & ... & . & ... & ... & ... & O_{2\times4} \\ O_{2\times4} & O_{2\times4} & O_{2\times4} & ... & & ... & ... & ... & \Lambda_N(\omega) \end{pmatrix},$$

$$i = 1, ..., 2N, \quad j = 1, ..., 4N.$$

Here

$$\Lambda_j(\omega) = \begin{pmatrix} 1 & 0 & -\cos(k_j L_j) & \frac{\sin(k_j L_j)}{k_j(\omega)} \\ -\cos(k_j L_j) & -\frac{\sin(k_j L_j)}{k_j(\omega)} & 1 & 0 \end{pmatrix},$$

$$O_{2\times 4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The next $N-1$ *rows of the matrices* $\mathbf{\Lambda}1(\omega)$, $\mathbf{\Lambda}2(\omega)$ *contain the continuity conditions* (6) *at node points*

$$\{\Lambda_{ij}\} = \begin{pmatrix} 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & ... & ... & ... & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & ... & ... & ... & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ... & ... & ... & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ... & ... & ... & ... & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$i = 2N+1, ..., 3N, \quad j = 1, ..., 4N$$

The next $N-1$ *rows of these matrices contain the conditions* (7) *at node points:*

$$\{\Lambda_{ij}\} = \begin{pmatrix} 0 & 0 & 0 & \kappa_1 & 0 & -\kappa_2 & 0 & 0 & ... & ... & ... & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \kappa_2 & ... & ... & ... & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ... & ... & ... & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ... & ... & ... & ... & 0 & -\kappa_N & 0 & 0 \end{pmatrix},$$

$$i = 3N-1, ..., 4N-2, \quad j = 1, ..., 4N.$$

The last two rows of the matrix are the boundary conditions at the ends of the graph. For the Dirichlet problem, this is condition (4)*:*

$$\{\Lambda_{ij}\} = \left\{ \begin{matrix} 1 & 0 & 0 & 0 & ... & ... & ... & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & ... & ... & ... & 0 & 0 & 1 & 0 \end{matrix} \right\},$$

$$i = 4N-1, 4N; \quad j = 1, ..., 4N$$

For the Neumann problem, this is condition (5)*:*

$$\{\Lambda_{ij}\} = \left\{ \begin{matrix} 0 & \kappa_1 & 0 & 0 & ... & ... & ... & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & ... & ... & ... & 0 & 0 & 0 & \kappa_N \end{matrix} \right\},$$

$$i = 4N-1, 4N; \quad j = 1, ..., 4N$$

Theorem 4 follows from Theorem 3.

Theorem 3. *The solution to boundary value problems* (1)–(6) *on the thermal graph has the form:*

$$\bar{\theta}_j(x_j,\omega)H(L-x_j)H(x_j) =$$
$$= H(x_j)\int_0^{L_j} \bar{U}\left(x_j-y,\omega\right) F_2^j(y,\omega)dy + \kappa_j H(x_j)\int_0^{L} \bar{U}_j(x_j-y,\omega)\theta_0^j(y)dy+$$
$$+\kappa_j\bar{q}_2^j(\omega)H(x)\bar{U}_j(L_j-x_j,\omega) - \kappa_j\bar{q}_1^j(\omega)H(L_j-x_j)\bar{U}_j(x_j,\omega)-$$
$$-\kappa_j\bar{\theta}_2^j(\omega)H(x_j)\bar{U}_{j,x}(L_j-x_j,\omega) - \kappa_j\bar{\theta}_{j,j}^j(\omega)H(L_j-x_j)\bar{U}_{j,x_j}(x_j,\omega). \quad (33)$$

Here $\left(\bar{\theta}_1^1(\omega),\bar{q}_1^1(\omega),\bar{\theta}_2^1(\omega),\bar{q}_2^1(\omega),......,\bar{\theta}_1^N(\omega),\bar{q}_1^N(\omega),\bar{\theta}_2^N(\omega),\bar{q}_2^N(\omega)\right) = B(\omega)$, *where* $B(\omega)$ *are the solution of resolving system equations:*

for Dirichlet problem: $B(\omega) = \mathbf{\Lambda}1^{-1}(\omega)\times C(\omega)$,

for Neumann problem: $B(\omega) = \mathbf{\Lambda}2^{-1}(\omega)\times C(\omega)$.

So we defined the Fourier transformant of the solution of BVPs on the thermal graph. Then by using the formula of inverse Fourier transformations (16) we calculate the original solution—-the temperature at every point of the graph. So, both BVPs have been solved.

Conclusion

Using the method of generalized functions, we solved the boundary value problems of thermal conductivity on the thermal linear graph, which can be used to study various network-like structures under conditions of thermal heating (cooling). A unified technique has been developed for solving various boundary value problems typical for practical applications.

The action of heat sources can be modeled by both regular and singular generalized functions under various boundary conditions at the ends of the graph. The obtained regular integral representations of generalized solutions make it possible to determine the temperature and heat flows on each element of the graph, at any point of it, for stationary oscillations with a constant frequency and in the case of periodic oscillations.

For nonstationary processes, performing the inverse Fourier transform in time, we obtain the original solution in the original space-time. The construction of the original depends on the boundary conditions and the type of functions that determine them and should be considered separately for a specific boundary value problem. The generalized function method presented here makes it possible to solve a wide class of boundary value problems with local and connected boundary conditions at the ends of the edges of the graph and various transmission conditions at its nodes and can be extended to network structures of very different types. It distinguishes this method from all others that are used to solve similar problems.

It should be noted that if we change the transmission condition (6), setting $Q_j(t) = 0$ and $q_j^2(t) = q_{j+1}^1(t)$, i.e. introduce the continuity condition of the derivatives with respect to

x at the nodes of the linear graph, then the solution to this problem for the heat equation with discontinuous coefficients is also constructed by this method. Only in the rows of the matrix of the resolving system of equations that contain this transmission condition, should we put 1 instead of κ_j. The issues of the correctness of setting such problems for parabolic equations with discontinuous coefficients on a certain class of functions were considered in [16], [17], [18].

In [19], a boundary value problem for the heat equation with a piecewise constant thermal conductivity coefficient with one discontinuity point under homogeneous boundary conditions with the condition of equality of heat fluxes at the discontinuity point was considered.

The generalized function method presented here makes it possible to solve a wide class of boundary value problems with local and connected boundary conditions at the ends of the edges of the graph and various transmission conditions at its node and can be extended to network structures of very different types. It distinguishes this method from all others that are used to solve similar problems.

The proposed method applies to a wide range of BVPs, including those on mesh structures.

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Айнакеева Н.Ж., Алексеева Л.А., Приказчиков Д.А. Жылулық теңдеу үшін Дирихле және Нейман есептері сызықтық көп буынды жылу графтары және олардың шешімдері

Жылулық қыздыру (салқындату) жағдайында әртүрлі құрылымдарды зерттеу үшін пайдаланылуы мүмкін сызықтық жылулық граф бойынша жылу өткізгіштіктің шекаралық есептері қарастырылады. Мұнда жалпыланған функциялық әдіс негізінде инженерлік қолданбаларға тән жылу өткізгіштіктің шекаралық есептерін шешудің бірыңғай әдістемесі әзірленді. Жиекте және жылу сызығының графында жылу өткізгіштіктің стационарлы емес және стационар шекаралық есептерінің жалпыланған шешімдері графтың шеттерінде әртүрлі шекаралық шарттарда және оның түйінінде жалпыланған Кирхгоф шарттарында құрастырылады. Негізгі шешімнің Фурье түрлендірушісінің қасиеттерін пайдалана отырып, аналитикалық түрде шекаралық есептердің шешімдерінің тұрақты интегралдық бейнелері алынады. Алынған шешімдер әртүрлі типтегі жылу көздерін модельдеуге мүмкіндік береді, соның ішінде сингулярлы жалпыланған функцияларды пайдаланады. Мұнда келтірілген жалпыланған функциялар әдісі графтың шеттерінің шеттеріндегі жергілікті және байланысқан шекаралық шарттармен және оның түйіндеріндегі әртүрлі берілу жағдайларымен шекаралық есептердің кең класын шешуге мүмкіндік береді.

Түйін сөздер: жылу өткізгіштік, жалпыланған функциялар, іргелі және жалпылама шешім, Фурье түрлендіруі, шекаралық теңдеулерді шешу, сызықтық граф.

Айнакеева Н.Ж., Алексеева Л.А., Приказчиков Д.А. Задача Дирихле и Неймана для уравнения теплопроводности линейных многозвенных тепловых графов и их решения

Рассматриваются краевые задачи теплопроводности на линейном тепловом графе, которые могут быть использованы для исследования различных конструкций в условиях теплового нагрева (охлаждения). Здесь на основе метода обобщенных функций разработана единая методика решения краевых задач теплопроводности, типичная для инженерных приложений. Построены обобщенные решения нестационарных и стационарных краевых задач теплопроводности на ребре и на линейном тепловом графе при различных граничных условиях на концах графа и обобщенных условиях Кирхгофа в его узле. Используя свойства трансформанты Фурье фундаментального решения, получены регулярные интегральные представления решений краевых задач в аналитическом виде. Полученные решения позволяют моделировать источники тепла различных типов, в том числе с использованием сингулярных обобщенных функций. Представленный здесь метод обобщенных функций позволяет решать широкий класс краевых задач с локальными и связанными граничными условиями на концах ребер графа и различными условиями пропускания в его узлах.

Ключевые слова: теплопроводность, обобщенные функции, фундаментальное и обобщенное решение, преобразование Фурье, разрешение граничных уравнений, линейный граф.

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Some Hardy-type inequalities with sharp constants via the divergence theorem

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Abstract. Hardy's inequality originated in the early twentieth century when G.H. Hardy introduced this fundamental result in real analysis to bound integral operators. Its elegant formulation and optimal constants spurred widespread interest, leading to numerous refinements. These developments laid the groundwork for further exploration and multidimensional extensions, deeply influencing harmonic analysis, partial differential equations, and mathematical physics. This historical evolution continues to inspire modern advancements in research. We discuss multidimensional generalizations of some improved Hardy inequalities based on the divergence theorem. The obtained Hardy-type inequalities extend a recent version of the one-dimensional Hardy inequality with the best constant to multidimensional cases.

Keywords. Hardy inequality, sharp constant, non-increasing rearrangement, divergence theorem.

1 Introduction

This paper is motivated by recent advancements in the Hardy inequality, as discussed in [1]. For readers interested in further exploration of this topic, we recommend [2] and [3], along with the references therein.

In this section, we discuss some preliminary concepts to set the groundwork for the proofs of the main theorems in the following section. While we provide the results specifically for the three-dimensional case, our techniques are applicable in any dimension.

Let μ be the 3-dimensional Lebesgue measure given in $\mathbb{R}^3$. Let f be a measurable function defined on $\Omega \subset \mathbb{R}^3$.

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A function

$$f^*(t) = \inf\{\sigma : \mu\{x \in \Omega : |f(x)| > \sigma\} \leq t\}$$

is called a non-increasing rearrangement of the function f.

Let D be an open set of $\mathbb{R}^3$ satisfying:

1. $0 \in D$,
2. ∂D is a smooth manifold,
3. $\mu D = 1$.

We set

$$D_t = \left\{ \left(t^{1/3}x_1, t^{1/3}x_2, t^{1/3}x_3 \right) : (x_1, x_2, x_3) \in D \right\}, \ t > 0.$$

Thus, $\mu D_t = t$. We have

Lemma 1. *Let $1 < p < \infty$. For any $\varepsilon > 0$ there exists a non-increasing function ϕ_ε defined on $(0, \infty)$ such that*

$$\frac{\left(\int_0^\infty \left(\frac{1}{t} \int_0^t \phi_\varepsilon(s) ds \right)^p dt \right)^{1/p}}{\left(\int_0^\infty (\phi_\varepsilon(t))^p \, dt \right)^{1/p}} \geq \frac{p}{p-1} - \varepsilon.$$

Proof. A key inspiration for this construction comes from the classical Hardy inequality:

$$\left(\int_0^\infty \left(\frac{1}{t} \int_0^t f(s) ds \right)^p dt \right)^{\frac{1}{p}} \leq \frac{p}{p-1} \left(\int_0^\infty (f(t))^p dt \right)^{\frac{1}{p}}$$

which holds for any nonnegative measurable function f. The best constant in this inequality is precisely $\frac{p}{p-1}$, so our goal is to construct a function ϕ_ε that gets arbitrarily close to achieving equality in this inequality.

To achieve this, we consider the choice:

$$\phi_\varepsilon(t) = (t + \delta)^{-1/p}, \tag{1}$$

for a small parameter $\delta > 0$, ensuring smooth truncation and controlling behavior at small values of t.

For this choice of ϕ_ε, we approximate:

$$H\phi_\varepsilon(t) := \frac{1}{t} \int_0^t (s + \delta)^{-1/p} ds. \tag{2}$$

Using asymptotic expansion,

$$H\phi_\varepsilon(t) \approx \frac{p}{p-1} (t + \delta)^{-1/p}. \tag{3}$$

Taking L^p-norms on both sides, we obtain:

$$\frac{\|H\phi_\varepsilon\|_{L^p}}{\|\phi_\varepsilon\|_{L^p}} \geq \frac{p}{p-1} - \varepsilon. \tag{4}$$

Thus, for sufficiently small δ, the desired inequality holds. $\square$

Lemma 2. *Let $\{D_t\}_{t>0}$ be a set defined above. Then for any nonincreasing function ϕ on $(0,\infty)$ there exists a measurable function $u(x)$ on $\mathbb{R}^3$ such that*

$$u^*(t) = \phi(t), \;\; a.e.$$

and

$$\int_{D_t} u(x)dx = \int_0^t \phi(s)ds.$$

Proof. Let $m \in \mathbb{N},\; A_k = \left\{s : \frac{k-1}{2^m} \leq \phi(s) < \frac{k}{2^m}\right\},\; k \in \mathbb{N}$. Let

$$\phi_m(s) = \frac{k}{2^m}, \quad \text{if } s \in A_k,\; k \in \mathbb{N}.$$

Then $\phi_m \rightrightarrows \phi$ and $\phi_1 \geq \phi_2 \geq \ldots$. Since ϕ is nonincreasing, there exists a sequence $\{t_k\}_{k\in\mathbb{N}}$ with $(t_{k-1}, t_k) \subset A_k \subset [t_{k-1}, t_k]$. Now we define

$$u_m(x) = \frac{k}{2^m}, \quad \text{if } x \in D_{t_k}\backslash D_{t_{k-1}}, \quad k \in \mathbb{N}.$$

Then

$$u_m^*(s) = \phi_m^*(s).$$

Since

$$|u_m(x) - u_{m+r}(x)| \leq \frac{1}{2^m},$$

there exists a limit $\lim_{m\to\infty} u_m(x) = u(x)$ and $u_m \rightrightarrows u$. $\square$

2 Main results

In this section, we present the main results of this paper.

Theorem 3. *Let $1 < p < \infty$. For any locally absolutely continuous function f on $\mathbb{R}^3$ we have*

$$\int_0^\infty \max\left\{\sup_{0<t\leq r} \frac{|F_t|^p}{r^p}, \sup_{r<t} \frac{|F_t|^p}{t^p}\right\} dr \leq \left(\frac{p}{p-1}\right)^p \int_{\mathbb{R}^3} |divf|^p dx. \tag{5}$$

The constant $\left(\frac{p}{p-1}\right)^p$ is sharp. Here $F_t := \int_{\partial D_t} f\,(x_1, x_2, x_3)\,(dx_1dx_2 + dx_2dx_3 + dx_3dx_1)$.

Proof. By using the divergence theorem, we obtain

$$\int_{\partial D_t} f(x_1,x_2,x_3)(dx_1dx_2+dx_2dx_3+dx_3dx_1)=\int_{D_t}\left(\frac{\partial f}{\partial x_1}+\frac{\partial f}{\partial x_2}+\frac{\partial f}{\partial x_3}\right)dx_1dx_2dx_3$$
$$\le \sup_{|e|=t}\int_e\left|\frac{\partial f}{\partial x_1}+\frac{\partial f}{\partial x_2}+\frac{\partial f}{\partial x_3}\right|dx_1dx_2dx_3=\int_0^t u^*(s)ds,$$

where

$$u=\frac{\partial f}{\partial x_1}+\frac{\partial f}{\partial x_2}+\frac{\partial f}{\partial x_3},$$

and

$$\sup_{t\ge r}\frac{1}{t}\left|\int_{D_t}u(y_1,y_2,y_3)\,dy_1dy_2dy_3\right|\le$$
$$\le \sup_{|e|\ge t}\frac{1}{|e|}\int_e|u(y_1,y_2,y_3)|\,dy_1dy_2dy_3=\frac{1}{t}\int_0^t u^*(s)ds.$$

Then

$$\max\left\{\sup_{0<t\le r}\frac{\left|\int_{\partial D_t}f(x_1,x_2,x_3)(dx_1dx_2+dx_2dx_3+dx_3dx_1)\right|}{r},\right.$$
$$\left.\sup_{r<t}\frac{\left|\int_{\partial D_t}f(x_1,x_2,x_3)(dx_1dx_2+dx_2dx_3+dx_3dx_1)\right|}{t}\right\}$$
$$\le\frac{1}{t}\int_0^t u^*(s)ds.$$

Thus, we have

$$\int_0^\infty\max\left\{\sup_{0<t\le r}\frac{\left|\int_{\partial D_t}f(x_1,x_2,x_3)(dx_1dx_2+dx_2dx_3+dx_3dx_1)\right|^p}{r^p},\right.$$
$$\left.\sup_{r<t}\frac{\left|\int_{\partial D_t}f(x_1,x_2,x_3)(dx_1dx_2+dx_2dx_3+dx_3dx_1)\right|^p\}}{t^p}\right\}dr$$
$$\le\int_0^\infty(u^{**}(t))^p\,dt\le\left(\frac{p}{p-1}\right)^p\int_0^\infty(u^*(t))^p\,dt=\left(\frac{p}{p-1}\right)^p\int_{\mathbb{R}^3}|u(x)|^pdx$$
$$=\left(\frac{p}{p-1}\right)^p\int_{\mathbb{R}^3}\left|\frac{\partial f}{\partial x_1}+\frac{\partial f}{\partial x_2}+\frac{\partial f}{\partial x_3}\right|^pdx_1dx_2dx_3.$$

Now, we show that the constant is sharp.

Let $\varepsilon > 0$ and ϕ_ε be a function given in Lemma 1. According to Lemma 2, there exists u_ε with

$$u_\varepsilon(s) = \phi_\varepsilon(s).$$

Let f_ε be a solution to the equation

$$\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} + \frac{\partial f}{\partial x_3} = u_\varepsilon\left(x_1, x_2, x_3\right).$$

By using the divergence theorem, we have

$$\begin{aligned}
&\left(\int_0^\infty \left|\frac{1}{t}\int_{\partial D_t} f_\varepsilon\left(x_1, x_2, x_3\right)\left(dx_1dx_2 + dx_2dx_3 + dx_3dx_1\right)\right|^p dt\right)^{1/p} \\
&= \left(\int_0^\infty \left|\frac{1}{t}\int_{D_t} \frac{\partial f_\varepsilon}{\partial x_1} + \frac{\partial f_\varepsilon}{\partial x_2} + \frac{\partial f_\varepsilon}{\partial x_3} dx\right|^p dt\right)^{1/p} \\
&= \left(\int_0^\infty \left|\frac{1}{t}\int_{D_t} u_\varepsilon(x)dx\right|^p dt\right)^{1/p} = \left(\int_0^\infty \left|\frac{1}{t}\int_0^t \phi_\varepsilon(s)ds\right|^p dt\right)^{1/p} \\
&\geq \left(\frac{p}{p-1} - \varepsilon\right)\left(\int_0^\infty \left(\phi_\varepsilon(t)\right)^p dt\right)^{1/p} = \left(\frac{p}{p-1} - \varepsilon\right)\left(\int_0^\infty \left(u_\varepsilon^*(t)\right)^p dt\right)^{1/p} \\
&= \left(\frac{p}{p-1} - \varepsilon\right)\left(\int_{\mathbb{R}^3} \left|\frac{\partial f_\varepsilon}{\partial x_1} + \frac{\partial f_\varepsilon}{\partial x_2} + \frac{\partial f_\varepsilon}{\partial x_3}\right|^p dx_1dx_2dx_3\right)^{1/p}.
\end{aligned}$$

Hence, the constant $\frac{p}{p-1}$ is sharp for the inequality

$$\begin{aligned}
&\left(\int_0^\infty \left|\frac{1}{t}\int_{\partial D_t} f\left(x_1, x_2, x_3\right)\left(dx_1dx_2 + dx_2dx_3 + dx_3dx_1\right)\right|^p dt\right)^{1/p} \\
&\leq \frac{p}{p-1}\left(\int_{\mathbb{R}^3} \left|\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} + \frac{\partial f}{\partial x_3}\right|^p dx_1dx_2dx_3\right)^{1/p}.
\end{aligned}$$

This implies that the constant is also sharp for inequality (1). □

Below, to make the formulas shorter, we write $\bar{x}$ for (x_1, x_2, x_3), so $f(\bar{x})$ stands for $f(x_1, x_2, x_3)$.

Theorem 4. *Let $1 < p < \infty$. Then, for any locally absolutely continuous function f on $\mathbb{R}^3$*

$$\int_0^\infty \frac{\left(\left(\int\limits_{\partial D_t} f\left(\bar{x}\right)dx_1dx_2\right)^2 + \left(\int\limits_{\partial D_t} f\left(\bar{x}\right)dx_2dx_3\right)^2 + \left(\int\limits_{\partial D_t} f\left(\bar{x}\right)dx_3dx_1\right)^2\right)^{p/2}}{t^p} dt \leq$$

$$\leq \left(\frac{p}{p-1}\right)^p \int\limits_{\mathbb{R}^3} |\nabla f(x)|^p dx.$$

Here the constant $\left(\frac{p}{p-1}\right)^p$ is sharp.

Proof. By using the divergence theorem, we obtain

$$\sup_{\alpha_1^2+\alpha_2^2+\alpha_3^2=1} \int_{\partial D_t} f\left(x_1, x_2, x_3\right)\left(\alpha_1 dx_1 dx_2 + \alpha_2 dx_2 dx_3 + \alpha_3 dx_3 dx_1\right)$$
$$= \sup_{\alpha_1^2+\alpha_2^2+\alpha_3^2=1} \int_{D_t} \left(\alpha_1 \frac{\partial f}{\partial x_1} + \alpha_2 \frac{\partial f}{\partial x_2} + \alpha_3 \frac{\partial f}{\partial x_3}\right) dx_1 dx_2 dx_3$$
$$\leq \sup_{\alpha_1^2+\alpha_2^2+\alpha_3^2=1} \sup_{|e|=t} \int_e \left|\alpha_1 \frac{\partial f}{\partial x_1} + \alpha_2 \frac{\partial f}{\partial x_2} + \alpha_3 \frac{\partial f}{\partial x_3}\right| dx_1 dx_2 dx_3$$
$$\leq \sup_{|e|=t} \int_e \left|\nabla f\left(x_1, x_2, x_3\right)\right| dx_1 dx_2 dx_3 = \int_0^t (|\nabla f|)^*(s) ds.$$

Then

$$\left(\left(\int_{\partial D_t} f\left(\bar{x}\right) dx_1 dx_2\right)^2 + \left(\int_{\partial D_t} f\left(\bar{x}\right) dx_2 dx_3\right)^2 + \left(\int_{\partial D_t} f\left(\bar{x}\right) dx_3 dx_1\right)^2\right)^{1/2} \leq$$
$$\leq \int_0^t (|\nabla f|)^*(s) ds.$$

Thus, we obtain

$$\int_0^\infty \frac{\left(\left(\int_{\partial D_t} f\left(\bar{x}\right) dx_1 dx_2\right)^2 + \left(\int_{\partial D_t} f\left(\bar{x}\right) dx_2 dx_3\right)^2 + \left(\int_{\partial D_t} f\left(\bar{x}\right) dx_3 dx_1\right)^2\right)^{p/2}}{t^p} dt \leq$$
$$\leq \int_0^\infty \left((|\nabla f|)^{**}(t)\right)^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty \left((|\nabla f|)^*(t)\right)^p dt = \left(\frac{p}{p-1}\right)^p \int_{\mathbb{R}^3} |\nabla f(x)|^p dx.$$

The sharpness is proved as in the case of Theorem 3. □

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Нұрсұлтанов Е. Д., Сұраған Д. Дивергенция теоремасы арқылы алынған оптималды тұрақтылары бар кейбір Харди типті теңсіздіктер

Харди теңсіздігі XX ғасырдың басында Г.Х. Харди нақты талдау саласында интеграл операторларды бағалау үшін осы негізгі нәтижені енгізген. Оның әдемі сипаттамасы және оңтайлы тұрақтысы кеңінен қызығушылық тудырды, кейін бұл көптеген жетілдірулерге әкелді. Бұл жетістіктер әрі қарай зерттеулер мен көпөлшемді кеңейтулердің негізін қалады, гармоникалық талдау, бөлшектік теңдеулер және математикалық физика салаларына терең ықпал етті. Бұл тарихи даму қазіргі зерттеулерге шабыт беруді жалғастыруда. Дивергенция теоремасы негізінде кейбір жетілдірілген Харди теңсіздіктерінің көпөлшемді жалпылаулары келтірілген. Алынған Харди типті теңсіздіктер жуықта жарияланған бір өлшемді Харди теңсіздігін оптималды тұрақтымен көпөлшемді жағдайларға кеңейтеді.

Түйін сөздер: Харди теңсіздігі, оптималды тұрақты, өспейтін ауыстыру, дивергенция теоремасы.

Нурсултанов Ерлан Даутбекович, Сураган Дурвудхан. Некоторые неравенства типа Харди с оптимальными константами, полученные с помощью теоремы о дивергенции

Неравенство Харди возникло в начале XX века, когда Г.Х. Харди представил этот фундаментальный результат в вещественном анализе для оценки интегральных операторов. Его элегантная формулировка и оптимальные константы вызвали широкий интерес, что привело к многочисленным усовершенствованиям. Эти разработки заложили основу для дальнейших исследований и многомерных обобщений, оказав глубокое влияние на гармонический анализ, дифференциальные уравнения и математическую физику. Эта историческая эволюция продолжает вдохновлять современные исследования. Даны многомерные обобщения некоторых улучшенных неравенств Харди, основанные на теореме о дивергенции. Полученные неравенства типа Харди расширяют недавнюю версию одномерного неравенства Харди с наилучшей константой на многомерные случаи.

Ключевые слова: Неравенство Харди, оптимальная константа, невозрастающая перестановка, теорема о дивергенции.

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Weighted and Logarithmic Caffarelli-Kohn-Nirenberg type inequalities on stratified groups and applications

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Abstract. The classical Caffarelli–Kohn–Nirenberg inequalities, originally established in Euclidean space in the 1980s, provide a unified framework for interpolation between Sobolev and Hardy inequalities. Their extension to stratified (or homogeneous Carnot) Lie groups began in the early 2000s, motivated by subelliptic analysis and geometric measure theory, revealing rich interactions between group structure, dilation symmetry, and functional inequalities. In this paper, we establish the weighted and logarithmic Caffarelli-Kohn-Nirenberg type inequalities on a stratified Lie group. As a consequence, we can apply it to prove the weighted ultracontractivity of positive strong solutions to

$$d^{\alpha}\frac{\partial u}{\partial t} = \mathcal{L}_p (d^{\alpha} u)^m,$$

where $\mathcal{L}_p f = \nabla_H(|\nabla_H f|^{p-2}\nabla_H f)$ is a p-sub-Laplacian, d is a homogeneous norm associated with a fundamental solution for sub-Laplacian and $\alpha \in \mathbb{R}$, $1 < p < Q$.

Keywords. Caffarelli-Kohn-Nirenberg inequality; logarithmic Caffarelli-Kohn-Nirenberg inequality; stratified Lie group.

1 Introduction

In 1984, Caffarelli, Kohn and Nirenberg published a celebrated work [4] where they derived the general case of inequalities such as Gagliardo-Nirenberg inequalities, Sobolev inequalities, Hardy-Sobolev inequalities, Nash's inequalities and Hardy's inequalities in the following form:

Theorem 1 (Caffarelli-Kohn-Nirenberg inequality [4]). *Let p, q, r, α, β, σ and a be fixed pa-*

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rameters in $\mathbb{R}$ *satisfying*

$$p, q \geq 1,\ r > 0,\ 0 \leq a \leq 1, \tag{1}$$

$$\frac{1}{p} + \frac{\alpha}{n},\ \frac{1}{q} + \frac{\beta}{n},\ \frac{1}{r} + \frac{\gamma}{n} > 0, \tag{2}$$

where

$$\gamma = a\sigma + (1-a)\beta. \tag{3}$$

There exists a positive constant C *such that the following inequality holds for all* $f \in C_0^\infty(\mathbb{R}^n)$

$$|||x|^\gamma f||_{L^r(\mathbb{R}^n)} \leq C|||x|^\alpha |\nabla f|||^a_{L^p(\mathbb{R}^n)} |||x|^\beta f||^{1-a}_{L^q(\mathbb{R}^n)}, \tag{4}$$

if and only if the following relations hold:

$$\frac{1}{r} + \frac{\gamma}{n} = a\left(\frac{1}{p} + \frac{\alpha - 1}{n}\right) + (1-a)\left(\frac{1}{q} + \frac{\beta}{n}\right), \tag{5}$$

(this is dimensional balance),

$$0 \leq \alpha - \sigma \ \textit{if}\ a > 0, \tag{6}$$

$$\alpha - \sigma \leq 1 \ \textit{if}\ a > 0 \ \textit{and}\ \frac{1}{p} + \frac{\alpha - 1}{n} = \frac{1}{r} + \frac{\gamma}{n}. \tag{7}$$

Note that when $a = 1$, $\alpha = 0$, $\gamma = -s/r$, $0 \leq s \leq p < n$ and $p^*(s) = r = \frac{p(n-s)}{n-p}$, Theorem 1 gives the Hardy-Sobolev inequality that is the interpolation of Sobolev's inequality ($s = 0$) and Hardy's inequality ($s = p$) such as

$$\left(\int_{\mathbb{R}^n} \frac{|f|^{p^*(s)}}{|x|^s} dx\right)^{\frac{1}{p^*(s)}} \leq C(n,p,s)\left(\int_{\mathbb{R}^n} |\nabla f|^p dx\right)^{\frac{1}{p}}. \tag{8}$$

Since then, the Caffarelli-Kohn-Nirenberg (CKN) inequality has been extended in different directions. For instance, in the Euclidean setting, the sharpness of constants and extremal functions in the CKN inequality was investigated by many authors such as Bouchez-Willem [3], Catrina-Wang [5], Chou-Chu [6], Del Pino-Dolbeault [7], Dolbeault-Esteban-Laptev-Loss [8], Lin-Wang [17], and Liu-Zhao [18]. In recent years, CKN inequality has been actively investigated in the setting of the Heisenberg group, stratified Lie group, and homogeneous groups. For example, Garofalo-Lanconelli [9], Feng-Niu-Qiao [13], Han [11], Zhang-Han-Dou [28], Han-Niu-Zhang [12] on Heisenberg group, Ruzhansky-Suragan [23, 26], Ruzhansky-Suagan-Yessirkegenov [24, 25], S.-Suragan [27] on stratified groups, Ozawa-Ruzhansky-Suragan [20] on homogeneous groups.

Motivated by results in [11], [19] and [13], in this paper, we investigate the weighted and logarithmic Caffarelli-Kohn-Nirenberg type inequalities on a stratified Lie group. As a consequence, we can apply it to prove the weighted ultracontractivity of positive strong solutions to the equation of the form

$$d^{\alpha}\frac{\partial u}{\partial t} = \mathcal{L}_p(d^{\alpha}u)^m, \tag{9}$$

where $\mathcal{L}_p f = \nabla_H(|\nabla_H f|^{p-2}\nabla_H f)$ is a p-sub-Laplacian, d is a homogeneous norm associated with a fundamental solution for sub-Laplacian and $\alpha \in \mathbb{R}$, $1 < p < Q$. The outline of this work has the following form:

Section 2 is devoted to obtaining a weighted Caffarelli-Kohn-Nirenberg type inequality with respect to the homogeneous norm associated with a fundamental solution for sub-Laplacian. First, we introduce the propositions of Hardy and Sobolev inequalities on a stratified group. Combining these inequalities, we prove a weighted Hardy-Sobolev-type inequality with a homogeneous norm associated with a fundamental solution for sub-Laplacian. As a result, we could derive a weighted Caffarelli-Kohn-Nirenberg-type inequality.

Section 3 presents the logarithmic and parametric logarithmic Caffarelli-Kohn-Nirenberg inequalities with a homogeneous norm associated with a fundamental solution for sub-Laplacian on a stratified group. There are obtained the weighted and logarithmic Hölder inequalities. Using those inequalities and the weighted Caffarelli-Kohn-Nirenberg type inequalities in the case $\gamma = \alpha = \beta$, we prove the logarithmic and parametric-logarithmic Caffarelli-Kohn-Nirenberg inequalities.

Section 4 is dedicated to proving the weighted ultracontractivity of positive strong solutions to a kind of evolution equation (9) by using the parametric logarithmic Caffarelli-Kohn-Nirenberg inequality.

1.1. Preliminaries

Let $\mathbb{G}$ be a stratified Lie group (or a homogeneous Carnot group), with dilation structure δ_λ and Jacobian generators $X_1, \ldots, X_N$, so that N is the dimension of the first stratum of $\mathbb{G}$. We refer to [15] and [2], or to a recent book [22] for extensive discussions of stratified Lie groups and their properties. Let Q be the homogeneous dimension of $\mathbb{G}$. The sub-Laplacian on $\mathbb{G}$ is given by

$$\mathcal{L} = \sum_{k=1}^{N} X_k^2. \tag{10}$$

It was shown by Folland [15] that the sub-Laplacian has a unique fundamental solution ε,

$$\mathcal{L}\varepsilon = \delta,$$

where δ denotes the Dirac distribution with singularity at the neutral element 0 of $\mathbb{G}$. The fundamental solution $\varepsilon(x, y) = \varepsilon(y^{-1}x)$ is homogeneous of degree $-Q + 2$ and can be written in the form

$$\varepsilon(x, y) = [d(y^{-1}x)]^{2-Q}, \tag{11}$$

for some homogeneous d which is called the $\mathcal{L}$-gauge. Thus, the $\mathcal{L}$-gauge is a symmetric homogeneous (quasi-) norm on the stratified group $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$, that is,

- $d(x) > 0$ if and only if $x \neq 0$,
- $d(\delta_\lambda(x)) = \lambda d(x)$ for all $\lambda > 0$ and $x \in \mathbb{G}$,
- $d(x^{-1}) = d(x)$ for all $x \in \mathbb{G}$.

We also recall that the standard Lebesgue measure dx on $\mathbb{R}^n$ is the Haar measure for $\mathbb{G}$ (see, e.g. [16, Proposition 1.6.6]). The left-invariant vector field X_j has an explicit form and satisfies the divergence theorem, see e.g. [16] for the derivation of the exact formula: more precisely, we can write

$$X_k = \frac{\partial}{\partial x'_k} + \sum_{l=2}^{r} \sum_{m=1}^{N_l} a_{k,m}^{(l)}(x', ..., x^{(l-1)}) \frac{\partial}{\partial x_m^{(l)}}, \tag{12}$$

with $x = (x', x^{(2)}, \ldots, x^{(r)})$, where r is the step of $\mathbb{G}$ and $x^{(l)} = (x_1^{(l)}, \ldots, x_{N_l}^{(l)})$ are the variables in the l^{th} stratum, see also [16, Section 3.1.5] for a general presentation. The horizontal gradient is given by

$$\nabla_H := (X_1, \ldots, X_N),$$

and the horizontal divergence is defined by

$$\mathrm{div}_H f := \nabla_H \cdot f.$$

The p-sub-Laplacian is defined by

$$\mathcal{L}_p f := \mathrm{div}_H(|\nabla_H f|^{p-2} \nabla_H f), \quad 1 < p < \infty. \tag{13}$$

2 Weighted Caffarelli-Kohn-Nirenberg-type inequalities

Proposition 2 ([11]). *For all vectors $v_1, v_2 \in \mathbb{R}^n$, we have the following expressions such as*

- *For $p \leq 2$ we have*

$$|v_1 + v_2|^p - |v_1|^p - p|v_1|^{p-2}\langle v_1, v_2\rangle \leq C(p)|v_2|^p.$$

- *For $p > 2$ we have*

$$|v_1 + v_2|^p - |v_1|^p - p|v_1|^{p-2}\langle v_1, v_2\rangle \leq \frac{p(p-1)}{2}(|v_1| + |v_2|)^{p-2}|v_2|^2,$$

where $\langle v_1, v_2\rangle$ is the inner product.

Proposition 3 (Hardy type inequality)**.** *Let $\mathbb{G}$ be a stratified Lie group and let $d = \varepsilon^{\frac{1}{2-Q}}$, where ε is the fundamental solution to the sub-Laplacian $\mathcal{L}$. Suppose that $Q \geq 3$ then for every $u \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have the following Hardy type inequality*

$$\int_{\mathbb{G}} \frac{|\nabla_H d|^p}{d^p}|u|^p dx \geq \left(\frac{p}{Q-p}\right)^p \int_{\mathbb{G}} |\nabla_H u|^p dx. \tag{14}$$

Note that the Hardy type inequality (14) was obtained by Garofalo-Lanconelli [9], D'Ambrosio [1], Goldstein-Kombe-Yener [10], and authors [21].

Proposition 4 (Sobolev inequality)**.** *Let $\mathbb{G}$ be a stratified Lie group, and let C be a positive constant. Then for every function $u \in C_0^\infty(\mathbb{G})$ we have*

$$\left(\int_{\mathbb{G}} |u|^{p^*} dx\right)^{\frac{1}{p^*}} \leq C \left(\int_{\mathbb{G}} |\nabla_H u|^p dx\right)^{\frac{1}{p}}, \tag{15}$$

where $p^* = \frac{pQ}{Q-p}$ with $1 < p < Q$.

Note that in the setting of the Heisenberg group and stratified Lie group, Sobolev inequalities (15) were obtained by Folland-Stein [14] and Folland [15], respectively.

Lemma 5 (Hardy-Sobolev type inequality)**.** *Let $\mathbb{G}$ be a stratified Lie group and let $d = \varepsilon^{\frac{1}{2-Q}}$, where ε is the fundamental solution to the sub-Laplacian $\mathcal{L}$. Then there exists a positive constant $C_1(s, p, Q)$ such that for all functions $u \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have*

$$\int_{\mathbb{G}} \frac{|\nabla_H d|^s}{d^s}|u|^{p_*(s)} dx \leq C_1 \left(\int_{\mathbb{G}} |\nabla_H u|^p dx\right)^{\frac{Q-s}{Q-p}}, \tag{16}$$

where $p_*(s) = \frac{p(Q-s)}{Q-p}$, $0 \leq s \leq p$ and $1 < p < Q$.

Proof of Lemma 5. The outline of the proof is to apply the Hölder inequality with $p_*(s) = \left(1 - \frac{s}{p}\right)p^* + \frac{s}{p}p$, the Hardy type inequality (14), and the Sobolev inequality (15), respectively.

Then we have

$$\int_{\mathbb{G}} \frac{|\nabla_H d|^s}{d^s}|u|^{p_*(s)}dx \le \left(\int_{\mathbb{G}} |u|^{p^*}dx\right)^{1-\frac{s}{p}} \left(\int_{\mathbb{G}} \frac{|\nabla_H d|^p}{d^p}|u|^p dx\right)^{\frac{s}{p}}$$
$$\le \left(C^{p^*}\left(\int_{\mathbb{G}} |\nabla_H u|^p dx\right)^{\frac{p^*}{p}}\right)^{1-\frac{s}{p}} \left(\left(\frac{p}{Q-p}\right)^p \int_{\mathbb{G}} |\nabla_H u|^p dx\right)^{\frac{s}{p}}$$
$$= C_1 \left(\int_{\mathbb{G}} |\nabla_H u|^p dx\right)^{\frac{Q-s}{Q-p}},$$

where $C_1 = C^{p^*(1-s/p)}\left(\frac{p}{Q-p}\right)^p$. This proves Lemma 5. □

Now, we prove the following weighted Hardy-Sobolev inequality on a stratified Lie group.

Theorem 6 (Weighted Hardy-Sobolev inequality)**.** *Let $\mathbb{G}$ be a stratified Lie group and let $d = \varepsilon^{\frac{1}{2-Q}}$, where ε is the fundamental solution to the sub-Laplacian $\mathcal{L}$. Then*

$$p_*(p,s,Q) = \frac{p(Q-s)}{Q-p}, \textit{ and } 1 < p < Q \textit{ with } Q \ge 3,$$
$$0 \le s \le p \textit{ and } \alpha \ge \frac{p-Q}{p},$$

there exists a positive constant $C(p,s,\alpha,Q)$ such that for all functions $u \in W^{1,p}_{\alpha}(\mathbb{G}\backslash\{0\})$ we have

$$\int_{\mathbb{G}} \frac{|\nabla_H d|^s}{d^s}|d^{\alpha}u|^{p_*(p,s,Q)}dx \le C\left(\int_{\mathbb{G}} |d^{\alpha}\nabla_H u|^p dx\right)^{\frac{Q-s}{Q-p}}, \tag{17}$$

where $W^{1,p}_{\alpha}(\mathbb{G}\backslash\{0\})$ is the closure of $C_0^{\infty}(\mathbb{G}\backslash\{0\})$ with respect to the norm

$$||u||_{W^{1,p}_{\alpha}(\mathbb{G})} := \left(\int_{\mathbb{G}} |d^{\alpha}\nabla_H u|^p dx\right)^{\frac{1}{p}}.$$

Remark 7. *Note that Lemma 5 is inferred from Theorem 6 when $\alpha = 0$.*

Proof of Theorem 6. The proof is divided in the cases such as $1 < p \le 2$ and $2 < p < Q$. In each case, we make use of Proposition 2 and Lemma 5.

Case $1 < p \le 2$. The outline of proof consists of the following steps: we take $v = d^{\alpha}u$, then apply Proposition 2, the integral by parts, the divergence theorem, and inequality (14), respectively. Then for $v_1 = \alpha v d^{-1}\nabla_H d$ and $v_2 = \nabla_H v - \alpha v d^{-1}\nabla_H d$ in Proposition 2, we have

$$
\begin{aligned}
&C(p)\int_{\mathbb{G}} |d^{\alpha}\nabla_H u|^p dx = C(p)\int_{\mathbb{G}} \left|\nabla_H v - \frac{\alpha v \nabla_H d}{d}\right|^p dx \\
&\geq \int_{\mathbb{G}} \left[|\nabla_H v|^p - |\alpha|^p \frac{|v|^p|\nabla_H d|^p}{d^p}\right] dx \\
&\quad - p\alpha|\alpha|^{p-2}\int_{\mathbb{G}} \frac{v|v|^{p-2}}{d^{p-1}}|\nabla_H d|^{p-2}\langle \nabla_H d, \nabla_H v - \alpha d^{-1} v \nabla_H d\rangle dx \\
&= \int_{\mathbb{G}} \left[|\nabla_H v|^p + (p-1)|\alpha|^p \frac{|\nabla_H d|^p}{d^p}|v|^p\right] dx - p\alpha|\alpha|^{p-2}\int_{\mathbb{G}} \frac{v|v|^{p-2}}{d^{p-1}}|\nabla_H d|^{p-2}\langle \nabla_H d, \nabla_H v\rangle dx \\
&= \int_{\mathbb{G}} \left[|\nabla_H v|^p + (p-1)|\alpha|^p \frac{|\nabla_H d|^p}{d^p}|v|^p\right] dx - \alpha|\alpha|^{p-2}\int_{\mathbb{G}} \frac{|\nabla_H d|^{p-2}}{d^{p-1}}\langle \nabla_H d, \nabla_H |v|^p\rangle dx \\
&= \int_{\mathbb{G}} \left[|\nabla_H v|^p + (p-1)|\alpha|^p \frac{|\nabla_H d|^p}{d^p}|v|^p\right] dx + \alpha|\alpha|^{p-2}\int_{\mathbb{G}} |v|^p \mathrm{div}_H(|\nabla_H d|^{p-2} d^{1-p}\nabla_H d) dx \\
&\geq \int_{\mathbb{G}} \left[|\nabla_H v|^p + (p-1)|\alpha|^p \frac{|\nabla_H d|^p}{d^p}|v|^p\right] dx + \alpha|\alpha|^{p-2}(Q-p)\int_{\mathbb{G}} \frac{|\nabla_H d|^p}{d^p}|v|^p dx \\
&= \int_{\mathbb{G}} |\nabla_H v|^p dx + \alpha|\alpha|^{p-2}(Q-p+\alpha(p-1))\int_{\mathbb{G}} \frac{|\nabla_H d|^p}{d^p}|v|^p dx \\
&\geq \left(1 + \alpha|\alpha|^{p-2}(Q-p+\alpha(p-1))\left(\frac{p}{Q-p}\right)^p\right)\int_{\mathbb{G}} |\nabla_H v|^p dx.
\end{aligned}
$$

So we arrive at

$$
C(p)\int_{\mathbb{G}} |d^{\alpha}\nabla_H u|^p dx \geq C_1(\alpha, Q, p)\int_{\mathbb{G}} |\nabla_H v|^p dx. \tag{18}
$$

By applying Lemma 5 on the right hand side of inequality (18) and $v = d^{\alpha}u$, then we obtain

$$
C(\alpha, s, p, Q)\left(\int_{\mathbb{G}} |d^{\alpha}\nabla_H u|^p dx\right)^{\frac{Q-s}{Q-p}} \geq \int_{\mathbb{G}} \frac{|\nabla_H d|^s}{d^s}|d^{\alpha}u|^{p_*(s)} dx.
$$

This proves inequality (17) in the case $1 < p \leq 2$.

Case $2 < p < Q$. The outline of proof is to estimate in both the upper and lower bound of the following expression

$$
2^{p-2}\frac{p(p-1)}{2}\int_{\mathbb{G}} (|\alpha d^{-1} v \nabla_H d| + |\nabla_H v|)^{p-2}|\nabla_H v - \alpha d^{-1} v \nabla_H d|^2 dx. \tag{19}
$$

The upper bound estimate is obtained by applying Hölder inequality, Minkowski inequality and Hardy type inequality (14), respectively. On the other hand, lower bound estimate is acquired by making use of Proposition 2 with $v_1 = \alpha d^{-1} v \nabla_H d$ and $v_2 = \nabla_H v - \alpha d^{-1} v \nabla_H d$, integral by parts, the divergence theorem and Hardy type inequality (14), respectively.

Let us estimate the upper bound of expression (19), by denoting $C_p = 2^{p-2}\frac{p(p-1)}{2}$ we have

$$\begin{aligned}
&C_p\int_{\mathbb{G}}(|\alpha d^{-1}v\nabla_H d|+|\nabla_H v|)^{p-2}|\nabla_H v-\alpha d^{-1}v\nabla_H d|^2dx\\
\leq & C_p\left(\int_{\mathbb{G}}(|\alpha d^{-1}v\nabla_H d|+|\nabla_H v|)^{p}dx\right)^{\frac{p-2}{p}}\left(\int_{\mathbb{G}}|\nabla_H v-\alpha d^{-1}v\nabla_H d|^pdx\right)^{\frac{2}{p}}\\
\leq & C_p\left[\left(\int_{\mathbb{G}}|\alpha d^{-1}v\nabla_H d|^pdx\right)^{\frac{1}{p}}+\left(\int_{\mathbb{G}}|\nabla_H v|^pdx\right)^{\frac{1}{p}}\right]^{p-2}\left(\int_{\mathbb{G}}|\nabla_H v-\alpha d^{-1}v\nabla_H d|^pdx\right)^{\frac{2}{p}}\\
\leq & C_p\left(1+|\alpha\frac{p}{Q-p}|\right)^{p-2}\left(\int_{\mathbb{G}}|\nabla_H v|^pdx\right)^{\frac{p-2}{p}}\left(\int_{\mathbb{G}}|\nabla_H v-\alpha d^{-1}v\nabla_H d|^pdx\right)^{\frac{2}{p}}. \qquad (20)
\end{aligned}$$

Now we estimate the lower bound of expression (19), by using Proposition 2. We have

$$\begin{aligned}
&2^{p-2}\frac{p(p-1)}{2}(|\alpha d^{-1}v\nabla_H d|+|\nabla_H v|)^{p-2}|\nabla_H v-\alpha d^{-1}v\nabla_H d|^2\\
\geq & \frac{p(p-1)}{2}(2|\alpha d^{-1}v\nabla_H d|+|\nabla_H v|)^{p-2}|\nabla_H v-\alpha d^{-1}v\nabla_H d|^2\\
\geq & \frac{p(p-1)}{2}(|\alpha d^{-1}v\nabla_H d|+|\nabla_H v-\alpha d^{-1}v\nabla_H d|)^{p-2}|\nabla_H v-\alpha d^{-1}v\nabla_H d|^2\\
\geq & |\nabla_H v|^p-|\alpha d^{-1}v\nabla_H d|^p-p|\alpha d^{-1}v\nabla_H d|^{p-2}\langle\alpha d^{-1}v\nabla_H d,\nabla_H v-\alpha d^{-1}v\nabla_H d\rangle\\
= & |\nabla_H v|^p+(p-1)|\alpha|^p\frac{|v|^p}{d^p}|\nabla_H d|^p-p\alpha|\alpha|^{p-2}v|v|^{p-2}|\nabla_H d|^{p-2}\langle\nabla_H d,\nabla_H v\rangle.
\end{aligned}$$

By integrating both side of the above inequality, we arrive at

$$\begin{aligned}
&C_p\int_{\mathbb{G}}(|\alpha d^{-1}v\nabla_H d|+|\nabla_H v|)^{p-2}|\nabla_H v-\alpha d^{-1}v\nabla_H d|^2dx\\
\geq & \int_{\mathbb{G}}|\nabla_H v|^pdx+(p-1)|\alpha|^p\int_{\mathbb{G}}\frac{|v|^p}{d^p}|\nabla_H d|^pdx-\alpha|\alpha|^{p-2}\int_{\mathbb{G}}|\nabla_H d|^{p-2}\langle\nabla_H d,\nabla_H|v|^p\rangle dx\\
= & \int_{\mathbb{G}}|\nabla_H v|^pdx+(p-1)|\alpha|^p\int_{\mathbb{G}}\frac{|v|^p}{d^p}|\nabla_H d|^pdx+\alpha|\alpha|^{p-2}\int_{\mathbb{G}}|v|^p\mathrm{div}_H(|\nabla_H d|^{p-2}\nabla_H d)dx\\
\geq & \int_{\mathbb{G}}|\nabla_H v|^pdx+\alpha|\alpha|^{p-2}(Q-p+\alpha(p-1))\int_{\mathbb{G}}\frac{|\nabla_H d|^p}{d^p}|v|^pdx\\
\geq & C_1\int_{\mathbb{G}}|\nabla_H v|^pdx. \qquad (21)
\end{aligned}$$

By combining (20) and (21), we obtain

$$C(p)\int_{\mathbb{G}}|d^\alpha\nabla_H u|^pdx\geq C_1(\alpha,Q,p)\int_{\mathbb{G}}|\nabla_H v|^pdx. \qquad (22)$$

By applying Lemma 5 on the right hand side of inequality (22) and $v = d^{\alpha}u$, then we obtain

$$C(\alpha, s, p, Q)\left(\int_{\mathbb{G}} |d^{\alpha}\nabla_H u|^p dx\right)^{\frac{Q-s}{Q-p}} \geq \int_{\mathbb{G}} \frac{|\nabla_H d|^s}{d^s}|d^{\alpha}u|^{p_*(s)}dx.$$

This proves inequality (17) in the case $2 < p \leq Q$. □

Theorem 8 (Weighted Caffarelli-Kohn-Nirenberg inequality)**.** *Let* $\mathbb{G}$ *be a stratified group. Let*

$$\begin{aligned} &1 < p < Q,\ q \geq 1,\ r > 0,\ 0 \leq a \leq 1, \\ &\frac{1}{p} + \frac{\alpha}{Q} > 0,\ \frac{1}{q} + \frac{\beta}{Q} > 0,\ \frac{1}{r} + \frac{\gamma}{Q} > 0, \end{aligned} \tag{23}$$

where $\gamma = a\sigma + (1-a)\beta$. *Then there exists a positive constant* C *such that*

$$|||\nabla_H d|^{\alpha-\gamma} d^{\gamma} u||_{L^r(\mathbb{G})} \leq C||d^{\alpha}|\nabla_H u|||^a_{L^p(\mathbb{G})}|||\nabla_H d|^{\alpha-\beta} d^{\beta} u||^{1-a}_{L^q(\mathbb{G})}, \tag{24}$$

holds for all functions $u \in C_0^{\infty}(\mathbb{G})$, *and if and only if the following conditions hold:*

$$\frac{1}{r} + \frac{\gamma}{Q} = a\left(\frac{1}{p} + \frac{\alpha - 1}{Q}\right) + (1-a)\left(\frac{1}{q} - \frac{\beta}{Q}\right), \tag{25}$$

$$0 \leq \alpha - \sigma \leq 1,\ \textit{if } a > 0 \textit{ and } \frac{1}{p} + \frac{\alpha-1}{Q} = \frac{1}{r} + \frac{\gamma}{Q}. \tag{26}$$

Note that in Theorem 8 if we choose $a = 1$ then the condition (26) has the following form

$$0 \leq \alpha - \sigma = \alpha - \gamma \leq 1,\ \text{ and }\ r = \left(\frac{1}{p} + \frac{\alpha - \sigma - 1}{Q}\right)^{-1}, \tag{27}$$

since $\gamma = \sigma$. Furthermore, we have $p \leq r \leq p^* = \frac{Qp}{Q-p}$ that allows to write r as follows

$$r = tp + (1-t)p^* = \frac{p(Q-tp)}{Q-p}. \tag{28}$$

By combining (27), (28) and $(\alpha - \sigma)r = tp$ we obtain the following relations:

$$\frac{p(Q-tp)}{Q-p} = \left(\frac{1}{p} + \frac{\alpha - \sigma - 1}{Q}\right)^{-1}, \tag{29}$$

$$tp = (\alpha - \sigma)\left(\frac{1}{p} + \frac{\alpha - \sigma - 1}{Q}\right)^{-1}. \tag{30}$$

Note that tp equals s from Theorem 6. If we insert (29) and (30) into inequality (17) then we arrive at

$$\int_{\mathbb{G}}\left(\frac{|\nabla_H d|}{d}\right)^{(\alpha-\sigma)\left(\frac{1}{p}+\frac{\alpha-\sigma-1}{Q}\right)^{-1}}|d^{\alpha}u|^{\left(\frac{1}{p}+\frac{\alpha-\sigma-1}{Q}\right)^{-1}}dx\leq C\left(\int_{\mathbb{G}}|d^{\alpha}\nabla_H u|^p dx\right)^{\frac{1}{p}\left(\frac{1}{p}+\frac{\alpha-\sigma-1}{Q}\right)^{-1}}. \tag{31}$$

Here, we showed that Theorem 8 implies Theorem 6 and inequality (31) will be used in the proof of Theorem 8.

Proof of Theorem 8. First, we calculate the following term

$$\begin{aligned}\int_{\mathbb{G}}|\nabla_H d|^{(\alpha-\gamma)r}d^{\gamma r}|u|^r dx &= \int_{\mathbb{G}}\left[|\nabla_H d|^{\alpha-a\sigma-(1-a)\beta}d^{a\sigma+(1-a)\beta}|u|\right]^r dx \\ &= \int_{\mathbb{G}}\left[|\nabla_H d|^{a(\alpha-\sigma)+(1-a)(\alpha-\beta)}d^{a\sigma+(1-a)\beta}|u|\right]^r dx \\ &= \int_{\mathbb{G}}\left[|\nabla_H d|^{(\alpha-\sigma)}d^{\sigma}|u|\right]^{ar}\left[|\nabla_H d|^{(\alpha-\beta)}d^{\beta}|u|\right]^{r(1-a)} dx \\ &\geq \left(\int_{\mathbb{G}}\left[|\nabla_H d|^{(\alpha-\sigma)}d^{\sigma}|u|\right]^{\left(\frac{1}{p}+\frac{\alpha-\sigma-1}{Q}\right)^{-1}}dx\right)^{ra\left(\frac{1}{p}+\frac{\alpha-\sigma-1}{Q}\right)} \\ &\quad\times\left(\int_{\mathbb{G}}\left[|\nabla_H d|^{(\alpha-\beta)}d^{\beta}|u|\right]^q dx\right)^{\frac{r(1-a)}{q}}.\end{aligned} \tag{32}$$

Now we prove inequality (24) by applying the inequalities (32) and (31),

$$\begin{aligned}\int_{\mathbb{G}}|\nabla_H d|^{(\alpha-\gamma)r}d^{\gamma r}|u|^r dx &= \left(\int_{\mathbb{G}}|\nabla_H d|^{(\alpha-\gamma)r}d^{\gamma r}|u|^r dx\right)^{\frac{1}{r}}\left(\int_{\mathbb{G}}|\nabla_H d|^{(\alpha-\gamma)r}d^{\gamma r}|u|^r dx\right)^{\frac{r-1}{r}} \\ &\geq \left(\int_{\mathbb{G}}\left[|\nabla_H d|^{(\alpha-\sigma)}d^{\sigma-\alpha}|d^{\alpha}u|\right]^{\left(\frac{1}{p}+\frac{\alpha-\sigma-1}{Q}\right)^{-1}}dx\right)^{a\left(\frac{1}{p}+\frac{\alpha-\sigma-1}{Q}\right)} \\ &\quad\times\left(\int_{\mathbb{G}}|\nabla_H d|^{q(\alpha-\beta)}d^{\beta q}|u|^q dx\right)^{\frac{1-a}{q}}\left(\int_{\mathbb{G}}|\nabla_H d|^{(\alpha-\gamma)r}d^{\gamma r}|u|^r dx\right)^{\frac{r-1}{r}} \\ &\geq C\left(\int_{\mathbb{G}}|d^{\alpha}\nabla_H u|^p dx\right)^{\frac{a}{p}}\left(\int_{\mathbb{G}}|\nabla_H d|^{(\alpha-\beta)q}d^{\beta q}|u|^q dx\right)^{\frac{1-a}{q}} \\ &\quad\times\left(\int_{\mathbb{G}}|\nabla_H d|^{(\alpha-\gamma)r}d^{\gamma r}|u|^r dx\right)^{\frac{r-1}{r}}.\end{aligned}$$

We arrive at

$$\left(\int_{\mathbb{G}} |\nabla_H d|^{(\alpha-\gamma)r} d^{\gamma r} |u|^r dx\right)^{\frac{1}{r}} \geq C \left(\int_{\mathbb{G}} |d^{\alpha} \nabla_H u|^p dx\right)^{\frac{a}{p}} \left(\int_{\mathbb{G}} |\nabla_H d|^{(\alpha-\beta)q} d^{\beta q} |u|^q dx\right)^{\frac{1-a}{q}}.$$

We finish the proof. □

When $\alpha = \gamma = \beta$ in Theorem 8, we obtain the following Corollary.

Corollary 9. *Let $\mathbb{G}$ be a stratified group. Let*

$$\begin{aligned} &1 < p < Q,\ q \geq 1,\ r > 0,\ 0 \leq a \leq 1, \\ &\frac{1}{p} + \frac{\alpha}{Q} > 0,\ \frac{1}{q} + \frac{\alpha}{Q} > 0,\ \frac{1}{r} + \frac{\alpha}{Q} > 0. \end{aligned} \tag{33}$$

Then, there exists a positive constant C such that

$$||d^{\alpha} u||_{L^r(\mathbb{G})} \leq C ||d^{\alpha} |\nabla_H u|||^a_{L^p(\mathbb{G})} ||d^{\alpha} u||^{1-a}_{L^q(\mathbb{G})}, \tag{34}$$

holds for all functions $u \in C_0^{\infty}(\mathbb{G})$, and if and only if the following conditions hold:

$$\frac{1}{r} = a\left(\frac{1}{p} - \frac{1}{Q}\right) + \frac{1-a}{q}. \tag{35}$$

Note that Corollary 9 is a main ingredient to acquire the logarithmic Caffarelli-Kohn-Nirenberg type inequalities in the next section.

3 Logarithmic Caffarelli-Kohn-Nirenberg type inequalities

Lemma 10. *Let $\mathbb{G}$ be a stratified group. Let the parameters*

$$1 < p \leq r \leq q \leq \infty,\ \ \theta \in [0,1],$$

satisfy

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q},$$

then we have the following Hölder inequality

$$||d^{\alpha} u||_{L^r(\mathbb{G})} \leq ||d^{\alpha} u||^{\theta}_{L^p(\mathbb{G})} ||d^{\alpha} u||^{1-\theta}_{L^q(\mathbb{G})}, \tag{36}$$

for $d^{\alpha} u \in L^p(\mathbb{G}) \cap L^q(\mathbb{G})$.

Remark 11. *Note that the Hölder inequality* (36) *is equivalent to the convexity of the function*

$$\phi\left(\frac{1}{r}\right) = \frac{1}{r}\log\left(\int_{\mathbb{G}}(d^{\alpha}u)^{r}dx\right),$$

that is

$$\phi\left(\frac{1}{r}\right) \leq \theta\phi\left(\frac{1}{p}\right) + (1-\theta)\phi\left(\frac{1}{q}\right),$$

for $\frac{1}{r} = \frac{\theta}{r} + \frac{1-\theta}{q}$.

Proof of Lemma 10. A simple computation gives

$$\begin{aligned}\int_{\mathbb{G}}(d^{\alpha}u)^{r}dx &= \int_{\mathbb{G}}(d^{\alpha}u)^{\theta r}(d^{\alpha}u)^{(1-\theta)r}dx \\ &\leq \left(\int_{\mathbb{G}}(d^{\alpha}u)^{p}dx\right)^{\frac{\theta r}{p}}\left(\int_{\mathbb{G}}(d^{\alpha}u)^{q}dx\right)^{\frac{(1-\theta)r}{q}},\end{aligned}$$

since

$$1 = \frac{r\theta}{p} + \frac{(1-\theta)r}{q}.$$

□

Lemma 12 (A logarithmic Hölder inequality)**.** *Let* $\mathbb{G}$ *be a stratified group. Let* $1 < p < q \leq \infty$, *then we have*

$$\int_{\mathbb{G}}\frac{(d^{\alpha}u)^{p}}{||d^{\alpha}u||^{p}_{L^{p}(\mathbb{G})}}\log\left(\frac{(d^{\alpha}u)^{p}}{||d^{\alpha}u||^{p}_{L^{p}(\mathbb{G})}}\right)dx \leq \frac{p}{q-p}\log\left(\frac{\int_{\mathbb{G}}(d^{\alpha}|u|)^{q}dx}{\int_{\mathbb{G}}(d^{\alpha}|u|)^{p}dx}\right), \tag{37}$$

for $d^{\alpha}u \in L^{p}(\mathbb{G}) \cap L^{q}(\mathbb{G})$.

Proof of Lemma 12. Observe that the derivative of convexity of the function

$$\phi(h) = h\log\left(\int_{\mathbb{G}}(d^{\alpha}|u|)^{\frac{1}{h}}dx\right),$$

that is,

$$\frac{d\phi(h)}{dh} = \log\left(\int_{\mathbb{G}}(d^{\alpha}|u|)^{\frac{1}{h}}dx\right) - \frac{1}{h}\frac{\int_{\mathbb{G}}(d^{\alpha}|u|)^{\frac{1}{h}}\log(d^{\alpha}|u|)dx}{\int_{\mathbb{G}}(d^{\alpha}|u|)^{\frac{1}{h}}dx}.$$

Then convexity of function ϕ in $(0,\infty)$ is equivalent to

$$\frac{d\phi(h)}{dh} \geq \frac{\phi(h_1)-\phi(h)}{h_1-h},$$

for $h > h_1 \geq 0$. Now by taking $h := \frac{1}{p}$ and $h_1 := \frac{1}{q}$, we derive to

$$\frac{\int_{\mathbb{G}} (d^\alpha |u|)^p \log(d^\alpha |u|) dx}{\int_{\mathbb{G}} (d^\alpha |u|)^p dx} - \frac{1}{p} \log \left(\int_{\mathbb{G}} (d^\alpha |u|)^p dx \right) \leq \frac{p}{q-p} \log \left(\frac{\int_{\mathbb{G}} (d^\alpha |u|)^q dx}{\int_{\mathbb{G}} (d^\alpha |u|)^p dx} \right).$$

The left side of the above inequality can be rearranged in the following form

$$\begin{aligned} &\frac{\int_{\mathbb{G}} (d^\alpha |u|)^p \log(d^\alpha |u|) dx}{\int_{\mathbb{G}} (d^\alpha |u|)^p dx} - \frac{1}{p} \log \left(\int_{\mathbb{G}} (d^\alpha |u|)^p dx \right) \\ &= \frac{\int_{\mathbb{G}} (d^\alpha |u|)^p \log(d^\alpha |u|) dx}{\int_{\mathbb{G}} (d^\alpha |u|)^p dx} - \frac{\int_{\mathbb{G}} (d^\alpha |u|)^p \log(||d^\alpha u||_{L^p(\mathbb{G})}) dx}{\int_{\mathbb{G}} (d^\alpha |u|)^p dx} \\ &= \int_{\mathbb{G}} \frac{(d^\alpha |u|)^p}{||d^\alpha u||^p_{L^p(\mathbb{G})}} \log \left(\frac{(d^\alpha |u|)^p}{(||d^\alpha u||^p_{L^p(\mathbb{G})})} \right) dx. \end{aligned}$$

The proof of Lemma 12 is finished. □

Now we state the logarithmic Caffarelli-Kohn-Nirenberg inequality on $\mathbb{G}$.

Theorem 13 (Logarithmic Caffarelli-Kohn-Nirenberg inequality)**.** *Let $\mathbb{G}$ be a stratified group. For a positive constant C, the inequality*

$$\int_{\mathbb{G}} \frac{(d^\alpha u)^q}{||d^\alpha u||^q_{L^q(\mathbb{G})}} \log \left(\frac{(d^\alpha u)^q}{||d^\alpha u||^q_{L^q(\mathbb{G})}} \right) dx \leq \frac{1}{1 - \frac{q}{p^*}} \log \left(C^q \frac{||d^\alpha |\nabla_H u|||^q_{L^q(\mathbb{G})}}{||d^\alpha u||^q_{L^q(\mathbb{G})}} \right) \quad (38)$$

is valid for parameters

$$1 < q < p^*, \; 1 < p < Q, \; \alpha p + Q > 0, \; \alpha q + Q > 0,$$

for every function $d^\alpha |\nabla_H u| \in L^p(\mathbb{G})$ and $d^\alpha u \in L^q(\mathbb{G})$.

Proof of Theorem 13. We use Lemma 37 with $q = p$, $p = r$, $1 < q < r \leq \infty$ and inequality (34). This gives

$$\begin{aligned} \int_{\mathbb{G}} \frac{(d^\alpha u)^q}{||d^\alpha u||^q_{L^q(\mathbb{G})}} \log \left(\frac{(d^\alpha u)^q}{||d^\alpha u||^q_{L^q(\mathbb{G})}} \right) dx &\leq \frac{1}{1 - q/r} \left[\log \left(||d^\alpha u||^q_{L^r(\mathbb{G})} \right) - \log \left(||d^\alpha u||^q_{L^q(\mathbb{G})} \right) \right] \\ &\leq \frac{1}{1 - q/r} \left[\log \left(C^q ||d^\alpha |\nabla_H u|||^{aq}_{L^p(\mathbb{G})} ||d^\alpha u||^{(1-a)q}_{L^q(\mathbb{G})} \right) - \log \left(||d^\alpha f||^q_{L^q(\mathbb{G})} \right) \right] \\ &= \frac{a}{1 - q/r} \log \left(C^q \frac{||d^\alpha |\nabla_H u|||^q_{L^p(\mathbb{G})}}{||d^\alpha u||^q_{L^q(\mathbb{G})}} \right) \\ &= \frac{1}{1 - q/p^*} \log \left(C^q \frac{||d^\alpha |\nabla_H u|||^q_{L^p(\mathbb{G})}}{||d^\alpha u||^q_{L^q(\mathbb{G})}} \right). \end{aligned}$$

In the last line, we have used $1 - q/r = a(p^* - q)/p^*$ from $1/r = a/p^* + (1-a)/q$. That finishes the proof. $\square$

Theorem 14 (Parametric logarithmic Caffarelli-Kohn-Nirenberg inequality)**.** *Let $\mathbb{G}$ be a stratified group. Suppose*

$$1 < p < Q,\ p^* = \frac{pQ}{Q-p},\ 1 < p^2/q < p^*,$$
$$\alpha p + Q > 0,\ \alpha p^2/q + Q > 0,\ \mu > 0.$$

There exists a positive constant C such that

$$\int_{\mathbb{G}} \frac{|d^\alpha u|^p}{||d^\alpha u||^p_{L^p(\mathbb{G})}} \log\left(\frac{|d^\alpha u|^p}{||d^\alpha u||^p_{L^p(\mathbb{G})}}\right) dx - \mu \frac{||\nabla_H (d^{\alpha u})^{q/p}||^p_{L^p(\mathbb{G})}}{||d^\alpha u||^p_{L^p(\mathbb{G})}} \tag{39}$$
$$\leq \frac{p}{q - p^2/p^*} \log\left(\frac{pC^p}{e(q - p^2/p^*)\mu}\right) + \frac{p}{q - p^2/p^*} \log ||d^\alpha u||^{p-q}_{L^p(\mathbb{G})},$$

for all functions $u \in L^p(\mathbb{G})$ and $\nabla_H (d^\alpha u)^{q/p} \in L^p(\mathbb{G})$.

Proof of Theorem 14. When $\alpha = 0$ in the logarithmic Caffarelli-Kohn-Nirenberg inequality (38), we have

$$\int_{\mathbb{G}} \frac{u^q}{||u||^q_{L^q(\mathbb{G})}} \log\left(\frac{u^q}{||u||^q_{L^q(\mathbb{G})}}\right) dx \leq \frac{1}{1 - \frac{q}{p^*}} \log\left(C^q \frac{|||\nabla_H u|||^q_{L^q(\mathbb{G})}}{||u||^q_{L^q(\mathbb{G})}}\right).$$

By taking p^2/q instead of q and substituting u with $(d^\alpha u)^{q/p}$ in above inequality, we arrive at

$$\int_{\mathbb{G}} \frac{|d^\alpha u|^p}{||d^\alpha u||^p_{L^p(\mathbb{G})}} \log\left(\frac{|d^\alpha u|^p}{||d^\alpha u||^p_{L^p(\mathbb{G})}}\right) dx \leq \frac{p}{q - p^2/p^*} \log\left(C^p \frac{||\nabla_H (d^\alpha u)^{q/p}||^p_{L^p(\mathbb{G})}}{||d^\alpha u||^p_{L^p(\mathbb{G})}}\right)$$
$$+ \frac{p}{q - p^2/p^*} \log ||d^\alpha u||^{p-q}_{L^p(\mathbb{G})}.$$

In the last line, we drag p/q from inside the expression of *log* and arrange to have a form as (40). Now we add the following term

$$-\mu \frac{||\nabla_H (d^\alpha u)^{q/p}||^p_{L^p(\mathbb{G})}}{||d^\alpha u||^p_{L^p(\mathbb{G})}} \tag{40}$$

to both sides of the above inequality, then we get

$$\int_{\mathbb{G}} \frac{|d^\alpha u|^p}{||d^\alpha u||^p_{L^p(\mathbb{G})}} \log\left(\frac{|d^\alpha u|^p}{||d^\alpha u||^p_{L^p(\mathbb{G})}}\right) dx - \mu \frac{||\nabla_H (d^\alpha u)^{q/p}||^p_{L^p(\mathbb{G})}}{||d^\alpha u||^q_{L^p(\mathbb{G})}}$$
$$\leq \frac{p}{q - p^2/p^*} \log\left(C^p \frac{||\nabla_H (d^\alpha u)^{q/p}||^p_{L^p(\mathbb{G})}}{||d^\alpha u||^p_{L^p(\mathbb{G})}}\right) - \mu \frac{||\nabla_H (d^\alpha u)^{q/p}||^p_{L^p(\mathbb{G})}}{||d^\alpha u||^q_{L^p(\mathbb{G})}}$$
$$+ \frac{p}{q - p^2/p^*} \log ||d^\alpha u||^{p-q}_{L^p(\mathbb{G})}$$
$$= \frac{p}{q - p^2/p^*} \log (C^p z) - \mu z + \frac{p}{q - p^2/p^*} \log ||d^\alpha u||^{p-q}_{L^p(\mathbb{G})},$$

where

$$z = \frac{||\nabla_H (d^\alpha u)^{q/p}||^p_{L^p(\mathbb{G})}}{||d^\alpha u||^p_{L^p(\mathbb{G})}}.$$

If we maximize the right-hand side of the above inequality with respect to z, we get

$$\int_{\mathbb{G}} \frac{|d^\alpha u|^p}{||d^\alpha u||^p_{L^p(\mathbb{G})}} \log\left(\frac{|d^\alpha u|^p}{||d^\alpha u||^p_{L^p(\mathbb{G})}}\right) dx - \mu \frac{||\nabla_H (d^\alpha u)^{q/p}||^p_{L^p(\mathbb{G})}}{||d^\alpha u||^q_{L^p(\mathbb{G})}}$$
$$\leq \frac{p}{q - p^2/p^*} \log\left(C^p \frac{p}{e\mu(q - p^2/p^*)}\right) + \frac{p}{q - p^2/p^*} \log ||d^\alpha u||^{p-q}_{L^p(\mathbb{G})}.$$

That proves inequality (39). □

4 Application

Theorem 15. *Let $\mathbb{G}$ be a stratified group. Suppose*

$$1 < p < Q, \ \frac{1}{p-1} < m < a_0 < \infty, \ t > 0.$$

Let $u(t)$ be a positive strong solution to $d^\alpha \dot{u} = \mathcal{L}_p (d^\alpha u)^m$. Then for a function $d^\alpha u(0) \in L^{a_0}(\mathbb{G})$ and $d^\alpha u(t) \in L^\infty(\mathbb{G})$, we have

$$||d^\alpha u(t)||_{L^\infty(\mathbb{G})} \leq C(Q, p, m, a_0) ||d^\alpha u(0)||^{\frac{a_0 p}{a_0 p + Q(m(p-1)-1)}}_{L^{a_0}(\mathbb{G})} t^{-\frac{Q}{a_0 p + Q(m(p-1)-1)}}, \tag{41}$$

for such $C(Q, p, m, a_0)$ is a positive constant.

Proof of Theorem 15. Suppose that $a_0 \leq r(t) < \infty$ and $\dot{r}(t) > 0$ hold for $r(t)$. A straightforward calculation gives

$$\frac{d}{dt}\left(\int_{\mathbb{G}} d^{\alpha r(t)} u^{r(t)} dx\right)^{\frac{1}{r(t)}} = ||d^\alpha u||_{L^{r(t)}(\mathbb{G})}\left(\frac{-\dot{r}(t)}{r^2(t)}\right)\log\left(\int_{\mathbb{G}} d^{\alpha r(t)} u^{r(t)} dx\right)$$
$$+\frac{||d^\alpha u||^{1-r(t)}_{L^{r(t)}(\mathbb{G})}}{r(t)}\int_{\mathbb{G}}\Big((d^\alpha u(t))^{r(t)}\log(d^\alpha u(t))\dot{r} + r(t)(d^\alpha u(t))^{r(t)-1} d^\alpha \dot{u}(t)\Big)dx$$
$$= ||d^\alpha u||_{L^{r(t)}(\mathbb{G})}\frac{\dot{r}(t)}{r^2(t)}\left[\frac{r(t)}{||d^\alpha u||^{r(t)}_{L^{r(t)}(\mathbb{G})}}\int_{\mathbb{G}}[d^\alpha u(t)]^{r(t)}\log(d^\alpha u(t))dx\right.$$
$$\left.-\log||d^\alpha u(t)||^{r(t)}_{L^{r(t)(\mathbb{G})}} + \frac{r^2(t)}{\dot{r}(t)||d^\alpha u||^{r(t)}_{L^{r(t)}(\mathbb{G})}}\int_{\mathbb{G}} d^\alpha \dot{u}(t)(d^\alpha u(t))^{r(t)-1}dx\right]$$
$$= ||d^\alpha u||_{L^{r(t)}(\mathbb{G})}\frac{\dot{r}^2(t)}{r^2(t)}\left[\int_{\mathbb{G}}\frac{(d^\alpha u(t))^{r(t)}}{||d^\alpha u||^{r(t)}_{L^{r(t)}(\mathbb{G})}}\log\left(\frac{(d^\alpha u(t))^{r(t)}}{||d^\alpha u||^{r(t)}_{L^{r(t)}(\mathbb{G})}}\right)dx\right.$$
$$\left.+\frac{r^2(t)}{\dot{r}(t)||d^\alpha u||^{r(t)}_{L^{r(t)}(\mathbb{G})}}\int_{\mathbb{G}}(d^\alpha u(t))^{r(t)-1} d^\alpha \dot{u}(t)dx\right].$$

$$\int_{\mathbb{G}}(d^\alpha u(t))^{r(t)-1} d^\alpha \dot{u}(t)dx = \int_{\mathbb{G}}(d^\alpha u(t))^{r(t)-1}\mathcal{L}_p(d^\alpha u(t))^m dx$$
$$= -\int_{\mathbb{G}}(|\nabla_H (d^\alpha u(t))^m|^{p-2}\nabla_H (d^\alpha u(t))^m)\cdot\nabla_H((d^\alpha u(t))^{r(t)-1})dx$$
$$= -m^{p-1}(r(t)-1)\int_{\mathbb{G}}(d^\alpha u(t))^{(m-1)(p-1)+r(t)-2}|\nabla_H(d^\alpha u(t))|^p dx$$
$$= -\frac{p^p m^{p-1}(r(t)-1)}{(r(t)+m(p-1)-1)^p}\int_{\mathbb{G}}\left||\nabla_H(d^\alpha u)|^{\frac{r(t)+m(p-1)-1}{p}}\right|^p dx.$$

Let us have a function v such that $(d^\alpha v)^r = (d^\alpha u)^p$, then we get

$$||d^\alpha u||^r_{L^r(\mathbb{G})} = ||d^\alpha v||^p_{L^p(\mathbb{G})}.$$

We insert to

$$\begin{aligned}&\int_{\mathbb{G}} \frac{(d^\alpha u(t))^r}{||d^\alpha u||^r_{L^r(\mathbb{G})}} \log\left(\frac{(d^\alpha u(t))^r}{||d^\alpha u||^r_{L^r(\mathbb{G})}}\right) dx\\
&- \frac{p^p m^{p-1}(r(t)-1)}{(r(t)+m(p-1)-1)^p} \frac{r^2}{\dot{r}||d^\alpha u||^r_{L^r(\mathbb{G})}} \int_{\mathbb{G}} \left| |\nabla_H (d^\alpha u)|^{\frac{r(t)+m(p-1)-1}{p}} \right|^p dx\\
=&\int_{\mathbb{G}} \frac{(d^\alpha v)^p}{||d^\alpha v||^p_{L^p(\mathbb{G})}} \log\left(\frac{(d^\alpha v)^p}{||d^\alpha v||^p_{L^p(\mathbb{G})}}\right) dx\\
&- \frac{p^p m^{p-1}(r(t)-1)}{(r(t)+m(p-1)-1)^p} \frac{r^2}{\dot{r}||d^\alpha v||^p_{L^p(\mathbb{G})}} \int_{\mathbb{G}} |\nabla_H (d^\alpha v)^{\frac{q}{p}}|^p dx,\end{aligned}$$

where $q = p\frac{r+m(p-1)-1}{r}$. Here a parameter μ from the parametric logarithmic Caffarelli-Kohn-Nirenberg type inequalities is given by $\mu = \frac{r^2}{\dot{r}} \frac{p^p m^{p-1}(r-1)}{(r+m(p-1)-1)^p}$. By applying Theorem 39, we obtain

$$\begin{aligned}&\frac{d}{dt}||d^\alpha u||_{L^r(\mathbb{G})}\\
&= ||d^\alpha u||_{L^r(\mathbb{G})} \frac{\dot{r}}{r^2} \left[\int_{\mathbb{G}} \frac{(d^\alpha v)^p}{||d^\alpha v||^p_{L^p(\mathbb{G})}} \log\left(\frac{(d^\alpha v)^p}{||d^\alpha v||^p_{L^p(\mathbb{G})}}\right) dx - \frac{\mu}{||d^\alpha v||^p_{L^p(\mathbb{G})}} \int_{\mathbb{G}} |\nabla_H (d^\alpha v)^{\frac{q}{p}}|^p dx\right]\\
&\leq ||d^\alpha u||_{L^r(\mathbb{G})} \frac{\dot{r}}{r^2} \frac{p}{q-p^2/p^*} \left[\log\left(C^p \frac{p}{e\mu(q-p^2/p^*)}\right) + \frac{r(p-q)}{p} \log ||d^\alpha u||_{L^r(\mathbb{G})}\right].\end{aligned} \tag{42}$$

Then inequality (42) can be seen as

$$\dot{h} \leq F(t) h \log(h) + G(t) h, \tag{43}$$

where

$$\begin{aligned}h(t) &:= ||d^\alpha u(t)||_{L^r(\mathbb{G})},\\
F(t) &:= \frac{\dot{r}}{r} \frac{p-q}{q-p^2/p^*},\\
G(t) &:= \frac{\dot{r}}{r^2} \frac{p}{q-p^2/p^*} \log\left(C^p \frac{p}{e\mu(q-p^2/p^*)}\right).\end{aligned}$$

The solving (43) is a simple calculation but very long, so we refer to [19] where Q instead of n in our case. Then for $a_0 < b < \infty$ we have

$$||d^\alpha u(t)||_{L^b(\mathbb{G})} \leq C(Q,p,m,a_0,b) ||d^\alpha u(0)||^{\frac{a_0(bp+Q(m(p-1)-1))}{b(a_0 p+Q(m(p-1)-1))}}_{L^{a_0}(\mathbb{G})} t^{-\frac{(b-a_0)Q}{b(a_0 p+Q(m(p-1)-1))}}, \tag{44}$$

where $C(Q,p,m,a_0,b) = C\left(\frac{b-m_1}{bm_2}\right)^{\frac{b-m_3}{bm_4}}$ with $m_1 = m_3 = a_0$, and

$$m_2 = \left(1 - \frac{p}{p^*}\right)\left(a_0\left(1 - \frac{p}{p^*}\right) + m(p-1) - 1\right),\ m_4 = a_0\left(1 - \frac{p}{p^*}\right) + m(p-1) - 1.$$

Making use of L'Hospital, it can be shown that for $b \to \infty$ the constant is $C(Q,p,m,a_0,b) < \infty$. This proves

$$||d^\alpha u(t)||_{L^\infty(\mathbb{G})} \leq C(Q,p,m,a_0)||d^\alpha u(0)||_{L^{a_0}(\mathbb{G})}^{\frac{a_0 p}{a_0 p + Q(m(p-1)-1)}} t^{-\frac{Q}{a_0 p + Q(m(p-1)-1)}}.$$

□

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Сәбитбек Б.М. Салмақталған және логарифмдік Каффарелли-Кон-Ниренберг типті теңсіздіктер стратификацияланған топтар мен қолданбалар бойынша

Классикалық Каффарелли–Кона–Ниренберг теңсіздіктері алғаш рет 1980-жылдары Евклид кеңістігінде құрылып, Соболев және Харди теңсіздіктері арасындағы интерполяция үшін біртұтас негіз ұсынды. Бұл теңсіздіктерді стратификацияланған (немесе біртекті Карно) Ли топтарына жалпылау 2000-жылдардың басында субэллиптикалық талдау мен геометриялық өлшем теориясының ықпалымен басталды. Бұл кеңейту топтың құрылымы, масштабтау симметриясы және функционалдық теңсіздіктер арасындағы бай байланыстарды ашты. Бұл мақалада біз стратификацияланған Ли тобында салмақталған және логарифмдік Каффарелли-Кон-Ниренберг типті теңсіздіктерді орнатамыз. Нәтижесінде, біз оны келесі теңдеудің оң күшті шешімдерінің салмақталған ультра контрактивтілік (ultracontractivity) дәлелдеу үшін қолдана аламыз:

$$d^{\alpha}\frac{\partial u}{\partial t} = \mathcal{L}_p(d^{\alpha}u)^m,$$

мұндағы $\mathcal{L}_p f = \nabla_H(|\nabla_H f|^{p-2}\nabla_H f)$ - p-суб-Лапласиан, d суб-Лаплациан үшін іргелі шешімімен байланысты біртекті норма, және $\alpha \in \mathbb{R}$, $1 < p < Q$.

Түйін сөздер: Кафарелли-Кон-Ниренберг теңсіздігі; логарифмдік Каффарелли-Кон-Ниренберг теңсіздігі; стратификацияланған Ли тобы.

Сабитбек Б.М. Весовое и логарифмические неравенства типа Каффарелли-Кона-Ниренберга на стратифицированных группах и их приложения

Классические неравенства Каффарелли–Кона–Ниренберга, впервые установленные в евклидовом пространстве в 1980-х годах, предоставили единую основу для интерполяции между неравенствами Соболева и Харди. Их обобщение на стратифицированные (или однородные Карно) группы Ли началось в начале 2000-х годов под влиянием субэллиптического анализа и теории геометрической меры, выявив богатые взаимосвязи между структурой группы, симметрией растяжений и функциональными неравенствами. В этой статье мы устанавливаем весовые и логарифмические неравенства типа Каффарелли-Кона-Ниренберга на стратифицированной группе Ли. Как следствие, мы можем применить их для доказательства весовой ультрасжимаемости положительных сильных решений

$$d^{\alpha}\frac{\partial u}{\partial t} = \mathcal{L}_p(d^{\alpha}u)^m,$$

где $\mathcal{L}_p f = \nabla_H(|\nabla_H f|^{p-2}\nabla_H f)$ — p-суб-Лапласиан, d — однородная норма, связанная с фундаментальным решением для суб-Лапласиана и $\alpha \in \mathbb{R}$, $1 < p < Q$.

Ключевые слова: Неравенство Каффарелли-Кона-Ниренберга; логарифмическое неравенство Каффарелли-Кона-Ниренберга; стратифицированная группа Ли.

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