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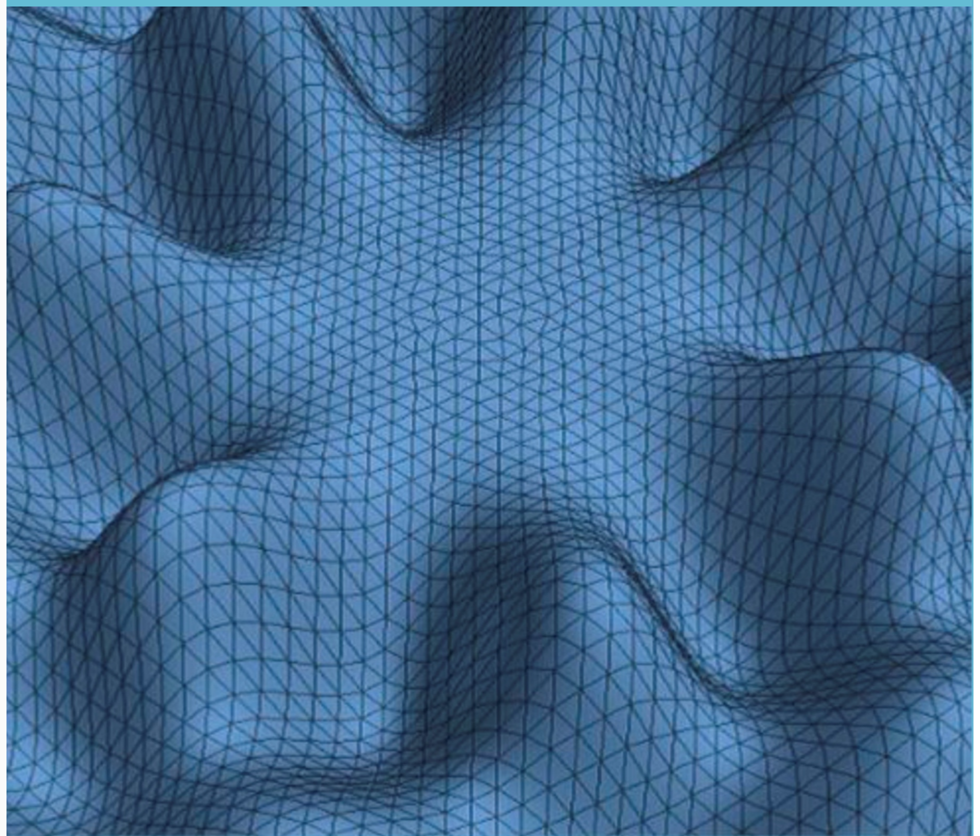

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The properties of amalgamation and joint embedding in the meaning of positive Jonsson theories

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Abstract. In this paper, we study special subclasses of theories based on the connection between the amalgamation property and the joint embedding property, as well as between the h-amalgamation property and the joint continuation property. Our results are presented in both the classical first-order logic and the positive logic, which exhibit a parallel structure. We establish sufficient conditions under which the amalgamation property implies the joint embedding property, and conversely; the h-amalgamation property implies the joint continuation property and vice versa. Furthermore, we investigate the preservation of these subclass links under extensions of the given theory.

Keywords. Existentially closed models, amalgamation property, joint embedding property, positive model theory, positively closed models, h-amalgamation property, joint continuation property, positively existentially prime Jonsson theories.

Introduction

This paper relates to both the so-called "East" direction of Model Theory that originated from Abraham Robinson's work [1] and the positive model theory, which was first studied by Itai Ben Yaacov and Bruno Poizat in [2].

Model theory, a fundamental branch of mathematical logic, has evolved through two distinct historical and methodological traditions. The first, often referred to as the "Western" tradition, originated from the pioneering works of Alfred Tarski and Robert Vaught. This approach primarily emphasizes the study of complete theories, and classification of models

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via stability and stability hierarchies, and techniques based on compactness and completeness theorems. A crucial aspect of this tradition is the use of elementary embeddings as the primary morphisms, ensuring that the logical structure is preserved precisely. It has been deeply connected with algebraic geometry, topological model theory, and, more recently, the development of o-minimality and geometric stability theory.

In contrast, the "Eastern" tradition of model theory, rooted in the works of Abraham Robinson and Anatoly Maltsev, focuses on methods derived from algebra and non-classical logic, particularly the model-theoretic study of algebraic structures through the lens of syntactical constructs. This tradition places a strong emphasis on non-elementary classes, Robinsonian concepts such as model-completeness, and the use of methods from universal algebra to investigate the nature of mathematical structures. Unlike the Western tradition, which predominantly operates within first-order logic with complete theories and elementary embeddings, the Eastern tradition explores weaker axiomatizations where the morphisms under consideration are more general, often allowing isomorphic embeddings. The research presented in this article is primarily aligned with the latter tradition.

A significant development within the Eastern tradition is the emergence of positive logic and positive model theory, introduced by Itai Ben Yaacov and Bruno Poizat [2]. Unlike classical first-order logic, where negation plays a central role, positive logic restricts itself to the study of theories that are preserved under positive embeddings—embeddings that respect existentially quantified formulae without involving negation. Instead of considering isomorphic embeddings, positive logic focuses on homomorphisms restricted to positive formulae. This perspective leads to a reformation of classical notions: the concept of inductive theories is generalized to h-inductive theories, existentially closed models are replaced by positively closed models, the amalgamation property transforms into the h-amalgamation property, and the joint embedding property is reformulated as the joint continuation property. These modifications provide a more flexible and structurally rich approach to model theory, particularly in contexts where classical elementary embeddings are too restrictive.

Further advancements in this area include the work of B. Poizat and A.R. Yeshkeyev [3] on positive Jonsson theories, whose attributes are h-inductiveness, h-amalgamation property, and joint continuation property. In this development, the authors extend the classical Jonsson theory framework into the realm of positive logic, redefining key Robinsonian notions to fit within this weaker logical setting. Their work provides new insights into the behavior of models under positive conditions, enriching the Eastern tradition's approach to structural analysis. Thus, the main object of this study is positive Jonsson theories.

More studies in positive model theory have been conducted in the framework of the Robinsonian tradition by Itai Ben Yaacov [4], Bruno Poizat and Aibat Yeshkeyev [5], Almaz Kungozhin [6] and Mohammed Belkasmi [7, 8, 9, 10, 11, 12].

Previously, Aibat Yeshkeyev defined subclasses of inductive theories regarding the amalgamation and joint embedding properties. It is well known that the amalgamation property and joint embedding property are independent of each other; however, there are cases where one implies the other, depending on the specificity of the class of models of the theory under consideration. In that work, we considered these classes of theories in both a classical first-order logic and a positive logic context. In this paper, we provide some sufficient conditions for inductive theories to belong to some of the distinguished classes, and show when the property of the connection between amalgamation and joint embedding can be preserved in extensions of the given theory. We also generalize these notions for positive model theory and similar results in which the h-amalgamation property implies the joint continuation property and vice versa.

Paper structure. This paper is structured as follows. The introduction is followed by two main sections and a reference list. In Section 1, we provide an overview of fundamental concepts in Robinsonian model theory, describe the specific subclasses of inductive theories concerning the amalgamation and joint embedding properties, and show some results in the framework of the presented notions. Section 2 describes the necessary concepts of positive model theory and introduces new subclasses of h-inductive theories based on the connection between the h-amalgamation property and the joint continuation property, along with key results characterizing these notions.

1 Amalgamation and joint embedding properties in classical logic

As mentioned in Introduction, model theory can be studied in various frameworks. In classical Robinsonian model theory, the central objects of interest include the specific axiomatization of theories under consideration, key properties of embeddings such as the amalgamation and joint embedding properties, the existence of specific models such as existentially closed models, algebraically prime models, and some others regarding the considered types of embeddings. In contrast, positive model theory provides a different perspective, modifying fundamental notions in restricted signature while preserving key structural aspects. In this section, we outline these concepts in the Robinsonian framework and give related results before comparing them to their positive counterparts in the next section.

Let us start with the notion of an inductive theory, which plays a crucial role in the classical "eastern" tradition of model theory.

Definition 1. [13, p. 62] A theory T is called inductive if it is closed under inductive unions, that is, whenever $(M_i)_{i\in I}$ is a chain of models of T, their union $\bigcup_{i\in I} M_i$ is also a model of T.

It is well known that a theory T is inductive iff it is $\forall\exists$-axiomatizable. Another well-known fact on inductive theories is related with the existentially closed models of such theories. Firstly, we recall the definition of an existentially closed model of a theory.

Definition 2. [13, p. 97] A model M of a theory T is said to be existentially closed if for every embedding $M \to N$ into another model N of T, and every existential formula $\varphi(x)$ with parameters from M that is satisfiable in N, is also satisfiable in M.

The following fact in existentially closed models is well-known from [14].

Theorem 3. *Let T be an L-theory, $M \in Mod(T)$, and let $T_\forall$ be the set of all universal L-consequences of T. Then the following conditions are equivalent:*

1. *M is existentially closed over T;*
2. *M is existentially closed over $T_\forall$.*

It is known that for any inductive theory T, for any model A of T, there is a model M existentially closed over A such that A is embedded in M, and this fact emphasizes the significance of the notion of existentially closed models in the study of inductive theories. However, another considerable class of models is the class of algebraically prime models. While existentially closed models ensure the maximal satisfaction of existential conditions within embeddings, algebraically prime models represent the minimal elements in the class of models of a theory. Recall the definition of an algebraically prime model.

Definition 4. [15] A model M of a theory T is called algebraically prime if for every model N of T, there is an embedding of M into N.

If a theory T has algebraically prime models, the class of all such models of T is denoted by $\mathcal{A}_T$. Similarly, $\mathcal{E}_T$ denotes the class of all existentially closed models of T.

Note that the notion of an algebraically prime model generalizes the concept of a prime model [13, p. 85], where elementary embeddings are considered.

Given a theory T, the existence of existentially closed models and algebraically prime models provides the special properties of T, and there are cases when a theory admits a model that is both existentially closed and algebraically prime. In this way, the concept of existentially prime theory was defined by A. Yeshkeyev in [16].

Definition 5. [16] An L-theory T is called an existentially prime theory if there is a model M of T such that $M \subseteq \mathcal{A}_T \cap \mathcal{E}_T$, that is, M is both algebraically prime and existentially closed.

Another key aspect of studying models of a theory is the structure of embeddings between them. Two fundamental properties in this regard are the amalgamation property (AP) and the joint embedding property (JEP), which play a significant role in this study.

Definition 6. [13, p. 80] A theory T has the amalgamation property if for any three models M_1, M_2, M_3 of T such that there exist elementary embeddings $M_1 \to M_2$ and $M_1 \to M_3$, there exists a model M_4 of T and embeddings $M_2 \to M_4$ and $M_3 \to M_4$ that make the corresponding diagram commute.

Definition 7. [13, p. 80] A theory T has the joint embedding property if for any two models M_1 and M_2 of T, there exists a model M_3 of T into which both M_1 and M_2 can be embedded.

One of the classical results on theories admitting JEP is the following theorem:

Theorem 8. *[17, p. 365] Suppose T is an L-theory that admits JEP. Let A and B be existentially closed models of T. Then each $\forall\exists$-sentence that is true in A is true in B as well.*

Generally, AP and JEP are independent of each other, which is supported by counterexamples of W. Forrest in [18]. However, there are partial cases where the specific construction of the class of models of a theory provides the implication of JEP from AP and vice versa. In this manner, the following definitions were introduced in [19] by A. Yeshkeyev:

Definition 9. Let K be a class of L-theories. We call this class (or a theory from K, for short, when the class can be recovered from the context)

1. an AP-class (an AP-theory), if each theory from K, which has the amalgamation property (AP), also admits the joint embedding property.
2. a JEP-class (a JEP-theory), if each theory from K, which admits the joint embedding property (JEP), also satisfies the amalgamation property (AP).

There are examples for each type of the theories. As mentioned in [19], the group theory, the theory of fields of a fixed characteristic, the theory of differential fields of characteristic 0, the theory of differentially perfect fields of characteristic p are strongly convex theories, which means that the class of strongly convex theories is an AP class. The class of complete inductive theories, which are also model complete, is an example of a JEP-class. This class contains theories such as the theory of dense linear orders without endpoints, the theory of algebraically closed fields of a fixed characteristic, and the theory of differentially closed fields of a fixed characteristic.

In this paper, we demonstrate some sufficient conditions of being an AP-theory or JEP-theory for an inductive L-theory. In this context, the following two propositions describe AP-theories and JEP-theories.

Proposition 10. *Let T be an L-theory such that $\mathcal{A}_T \neq \varnothing$ and T admits AP. Then T is an AP-theory, that is, the class of all L-theories, which have algebraically prime models, is an AP-class.*

Proof. We need to show that, under the given conditions, T has JEP. Let A and B be two arbitrary models of T. Since T has an algebraically prime model, there is a model M such that M is embedded both into A and B. Then due to the fact that T admits the amalgamation property, there is a model N such that both A and B are embedded into N. Therefore, T has the joint embedding property. $\square$

Proposition 11. *Let T be an L-theory such that T admits JEP and for any two models A and B of T, if there is an embedding $f : A \to B$ then f is unique. Then T is an AP-theory.*

Proof. Here we need to show that T admits AP. Let A, B and C be models of T such that there are embeddings $f_1 : A \to B$ and $f_2 : A \to C$. Since T has the joint embedding property, there exists a model $D \in Mod(T)$ and embeddings $g_1 : B \to D$ and $g_2 : C \to D$. In force of the fact that every embedding of models in $Mod(T)$ is unique, we obtain that $f_1 \cdot g_1 = f_2 \cdot g_2$. Thus, T has the amalgamation property. □

The given conditions demonstrate the semantic specificity of the connection of AP and JEP within the class of models of a single theory. However, the property of being an AP-theory (JEP-theory) can be obtained for the extensions of the given theories in L. In this context, we consider the case of two theories T and T' such that $T \subseteq T'$. To specify the link between the classes of models of T and T', we also restrict this case to mutually model consistent theories. Recall the definition of mutually model consistent theories.

Definition 12. [13, p. 157] Let T_1 and T_2 be L-theories. T_1 and T_2 are called mutually model consistent, if for any model A of T_1, there is a model B of T_2 such that there exists an embedding $A \to B$, and vice versa.

The following fact on mutually model consistent theories is well-known:

Proposition 13. [13, p. 158] *If T_1 and T_2 are mutually model consistent then $T_{1\forall} = T_{2\forall}$, where $T_{i\forall}$ is the set of all universal L-sentences that are deduced from T_i.*

We apply Propositions 10 and 11 to the case of two mutually model consistent theories and obtain the following results.

Theorem 14. *Let T be an inductive existentially prime L-theory, and let T' be an inductive L-theory such that $T \subseteq T'$ and T' is mutually model consistent with T. Then if T is an AP-theory, T' is also an AP-theory. In other words, let K be a class of inductive existentially prime L-theories, and let K' extend K in the following way: if an inductive theory T' contains some $T \in K$ and T' is mutually model consistent with T, then $T' \in K'$; then K' is an AP-class.*

Proof. Firstly, let us show that $\mathcal{E}_T = \mathcal{E}_{T'}$. Note that T and T' are mutually model consistent, which means that $T_\forall = T'_\forall$. Let $A \in \mathcal{E}_T$, then, according to Theorem 3, A is existentially closed over $T_\forall = T'_\forall$ and, consequently, over T'. Therefore, $A \in \mathcal{E}_{T'}$. Conversely, if $B \in \mathcal{E}_{T'}$, B is an existentially closed structure of $T'_\forall = T_\forall$ and T. Hence, $\mathcal{E}_T = \mathcal{E}_{T'}$.

Now, let M be an existentially closed algebraically prime model of T, that is $M \in \mathcal{E}_T \cap \mathcal{A}_T$. As we showed, $\mathcal{E}_T = \mathcal{E}_{T'}$, therefore $M \in \mathcal{E}_{T'}$. Since $T \subseteq T'$, $Mod(T') \subseteq Mod(T)$, then M is an existentially closed model of T' that is embedded into any model of T', which means that T' is an existentially prime L-theory. Hence, $\mathcal{A}_{T'} \neq \varnothing$.

Now, we show that T' admits AP. Let A, B, C be models of T' such that there exist embeddings $f_1 : A \to B$ and $f_2 : A \to C$. Note that A, B, $C \in Mod(T)$. Then there is a model $D \in Mod(T)$ and embeddings $g_1 : B \to D$ and $g_2 : C \to D$ and the diagram of these embeddings commutes, as T admits AP according to the condition of the theorem. If D is a model of T', then T' has AP. If D is not, there is an existentially closed model N of T such that $D \to N$. Since T is an AP-theory, T has JEP, and according to Theorem 8, M and N satisfy the same $\forall\exists$-sentences in L. Since T' is inductive and any inductive theory is $\forall\exists$-axiomatizable, $N \in Mod(T')$. Let $g : D \to N$. Then the embeddings $g_1 \cdot g : B \to N$ and $g_2 \cdot g : C \to N$ complete the diagram of the amalgamation of the models A, B, C, N in $Mod(T')$, and this diagram commutes. Therefore, T' admits AP.

Owing to Theorem 10, T' is an AP-theory. □

Theorem 15. *Let T be an inductive L-theory such that for any embedding $f : A \to B$, where A and $B \in Mod(T)$, f is unique. Let T' be an L-theory such that $T \subseteq T'$ and T is mutually model consistent with T'. Then if T is a JEP-theory, then T' is also a JEP-theory.*

Proof. Let $A, B \in Mod(T')$. Since $T \subseteq T'$, the inclusion $Mod(T') \subseteq Mod(T)$ holds; hence any embedding $g : A \to B$ is unique.

Now we show that T' admits JEP. Let $A, B \in Mod(T')$. It is clear that A and B are also models of T. Then there is a model C of T and embeddings $f : A \to C$ and $g : B \to C$. If C is a model of T', then T' is also has JEP. If C is not, we may consider an existentially closed model M of T such that C is embedded into M. Since T and T' are mutually model consistent, $E_T = E_{T'}$; therefore M is a model of T'. Thus, A and B are embedded in a model of T', and T' admits JEP.

Applying Theorem 11, we obtain that T' is a JEP-theory. □

2 The connection of APh and JCP for h-inductive

In this section, we present the results on model-theoretic link between h-amalgamation property and joint continuation property that are positive-logic analogues of the results of the previous section.

First, we give some fundamental definitions and facts on positive model theory.

During this article, we will use the denotation L^+ for a language in positive logic by the meaning of [3].

Let L^+ be a countable language involving individual constants, functions, and relations. L^+ also contains the binary relation of equality and a 0-ary symbol $\perp$ denoting antilogy.

Unlike classical Robinsonian model theory, where embeddings are typically isomorphic in nature, positive model theory focuses on homomorphisms as the primary type of model-theoretic inclusion. This shift reflects the broader semantic framework of positive logic, which emphasizes the preservation of positive formulae rather than isomorphisms and elementary equivalence.

Definition 16. [3] A map h from an L^+-structure M to an L^+-structure N is called a homomorphism between M and N, if for every individual constant c, every function symbol f and every relation symbol r of L^+, and every tuple $\bar{a} = (a_1, \ldots, a_n)$ of elements of M the following holds:

1. $h(c_M) = h(c_N)$;
2. $h(f_M(a_1, ...a_n)) = f_N(h(a_1), ...h(a_n))$;
3. if $M \models r_M(a_1, ...a_n)$ then $N \models r_N(h(a_1), ...h(a_n))$.

When there exists a homomorphism from M to N, we say that N is a continuation of M. Note that a continuation of M is nothing but a model of the positive diagram $\mathrm{Diag}^+(\mathrm{M})$ of M, which is the set of atomic sentences satified by M in the language L^+ obtained by adding to the language individual constants naming the elements of M.

According to [3], a positive formula is obtained from the atomic formulae by the use of $\vee, \wedge$ and $\exists$. Note that there are no universal quantifiers. A positive formula can be written in prenex form as $(\exists \bar{x})\varphi(\bar{x})$, where φ is positive quantifier-free; φ in turn can be written as a finite disjunction of finite conjunctions of atomic formulae.

Definition 17. [3] Let M and N be L^+-structures, and let h be a homomorphism between M and N. If every tuple $\bar{a}$ in M satisfies the same positive formulae as its image $h(\bar{a})$ in N, we say that h is a pure homomorphism, or an immersion.

The next definition presents a positive version of the concept of an existentially closed model.

Definition 18. [3] An L^+-structure M is positively closed inside a class Γ of L^+-structures if every homomorphism from M to any N in Γ is an immersion.

We denote the class of all positively closed models of a theory T by PC_T.

To define a positive analogue of an inductive theory, we need the following definition.

Definition 19. [3] A sentence is said to be an *h-inductive sentence* if it is equivalent to a finite conjunction of sentences each of them declaring that a certain positively defined set is included into another. Such a simple h-inductive sentence has the form $(\forall \bar{x})(\exists \bar{y})\varphi(\bar{x}, \bar{y})$.

In positive logic only h-inductive sentences are under consideration.

In [3], B. Poizat and A. Yeshkeyev defined an inductive limit of a chain of models, where the authors considered homomorphisms, possibly not injective. The following definition presents the analogue of the concept of an inductive theory and given via the notion of inductive limits in the sense of positive logic.

Definition 20. [3] An L^+-theory T is said an h-inductive theory, if it is equivalent to a set of inductive L^+-sentences.

It is known that the class of models of any h-inductive theory is h-inductive, that is closed under the union of chains in sense of inductive limits. Moreover, in an h-inductive class, every point can be continued into a positively closed element.

The following definitions generalizes the notion of mutually model consistent theories in the context of positive logic.

Definition 21. [3] Two h-inductive L^+-theories T and T' are called companion, if every model of one of them can be continued into a model of the other.

Similarly to the classical fact in first-order model theory, companion theories admit the following property concerning positively closed models:

Proposition 22. [3] *Let T and T' be L^+-theories that are companion. Then $PC_T = PC_{T'}$, that is, the class of all positively closed models of T is equal to the class of all positively closed models of T'.*

Just as algebraically prime models serve as distinguished representatives in classical model theory, positive model theory features an analogous notion of canonicity. These models capture the minimal structural essence of a theory within the framework of homomorphisms under consideration.

Definition 23. [3] Let T be a theory in L^+. A model $A \in Mod(T)$ is called a prime model of T, if for any model $B \in Mod(T)$, there is a homomorphism $f : A \to B$.

We denote the class of all prime models of T by P_T.

By analogy with the concept of existentially prime theories in first-order logic, the following concept was defined by A. Yeshkeyev:

Definition 24. An L^+-theory T is called a positively existentially prime theory, if there is a model M of T such that $M \subseteq P_T \cap PC_T$.

Note that if M is a positively closed prime model of T, for any $A \in Mod(T)$, a homomorphism $f : M \to A$ is an immersion.

We now turn to two fundamental properties in positive model theory: the h-amalgamation property and the joint continuation property. These properties, originally studied in classical model theory as the amalgamation property and the joint embedding property, respectively, take on a distinct character in the positive setting, where homomorphic rather than isomorphic embeddings govern the structure of models.

Definition 25. [2] An h-inductive theory T has the h-amalgamation property (APh) if, whenever there are two homomorphisms $f : A \to B$ and $g : A \to C$, where A, B, and $C \in Mod(T)$, there is a model $D \in Mod(T)$, and homomorphisms $f' : B \to D$ and $g' : C \to D$ such that $f \cdot f' = g \cdot g'$.

Definition 26. [2] An h-inductive theory T has the joint continuation property (JCP) if for any two models $A, B \in Mod(T)$ there is a model $C \in Mod(T)$ and homomorphisms $f : A \to C$ and $g : B \to C$.

In terms of positive model theory, A. Yeshkeyev introduced specific subclasses of h-inductive theories that are distinguished by the connection between the h-amalgamation property and the joint continuation property. Understanding when one of these properties implies the other within a given class of models provides valuable insight into the structural behavior of positive theories.

Definition 27. Let K^+ be a class of L^+-theories T. K^+ is called

1. an APh-class, if any $T \in K^+$, which admits h-amalgamation property, admits also joint continuation property.
2. a JCP-class, if each theory $T \in K^+$, which admits joint continuation property, also has h-amalgamation property.

We call a theory T an APh-theory (JCP-theory), if $T \in K^+$, where K^+ is an APh-class (JCP-class) in cases when then class K^+ can be recovered by the context.

The following fact was observed in [3]:

Theorem 28. [3] *Let T be an L^+-theory such that $P_T \neq \varnothing$ and T admits APh. Then T admits JCP.*

Thus, Theorem 28 states that an L^+-theory T is an APh-theory, if it admits APh and has a prime model.

We now present our main results. First, we provide a sufficient condition for a theory to be a JCP-theory.

Theorem 29. *Let T be an L^+-theory such that T admits JCP and for any two models A and B of T, if there is a homomorphism $h : A \to B$ then h is unique. Then T is an APh-theory.*

Proof. First, we prove that T admits APh. Let A, B and C be models of T such that there are homomorphisms $h_1 : A \to B$ and $h_2 : A \to C$. T has joint continuation property; therefore, there exists a model $D \in Mod(T)$ and homomorphisms $h_1' : B \to D$ and $h_2' : C \to D$. In force of the fact that every homomorphism of models in $Mod(T)$ is unique, we obtain that $h_1 \cdot h_1' = h_2 \cdot h_2'$. We obtain that T admits h-amalgamation property. □

Next, we examine the preservation of these properties under companion extensions. Specifically, we establish conditions under which a theory's status as an APh-theory or a JCP-theory is preserved by its companion extension.

Theorem 30. *Let T be an h-inductive positively existentially prime L^+-theory, and let T' be an h-inductive L^+-theory such that $T \subseteq T'$ and T' is a companion of T. Then if T is an APh-theory, T' is also an APh-theory.*

Proof. Suppose that M is a positively closed prime model of T. Due to the condition of the theorem, T and T' are h-inductive theories that are companion. According to Proposition 22, $PC_T = PC_{T'}$; then $M \in PC_{T'}$. Since $T \subseteq T'$, $Mod(T') \subseteq Mod(T)$, then M is a positively closed model of T' that is continued in any model of T'; hence, T' is a positively existentially prime L^+-theory. Therefore, $P_{T'} \neq \varnothing$.

Now we show that T' has APh. Let A, B, C be models of T' such that there exist homomorphisms $h_1 : A \to B$ and $h_2 : A \to C$. Note that A, B, $C \in Mod(T)$. The theory T admits APh; therefore there is a model $D \in Mod(T)$ and homomorphisms $h_1' : B \to D$ and $h_2' : C \to D$ such that the diagram of these continuations commutes. Suppose that D is a model of T', then T' has APh. In case when $\nvDash T'$, there is a positively closed model N of T such that D is continued in N. Since $PC_T = PC_{T'}$, $N \in Mod(T')$. Let $h : D \to N$. Then we may consider homomorphisms $h_1' \cdot h : B \to N$ and $h_2' \cdot h : C \to N$ that complete the diagram of the h-amalgamation of the models A, B, C, N in $Mod(T')$; moreover, this diagram commutes. Therefore, T' has APh. Thus, according to Theorem 28, T' is an APh-theory. □

Theorem 31. *Let T be an h-inductive L^+-theory such that for any homomorphism $h : A \to B$, where $A, B \in Mod(T)$, h is unique. Let T' be an L^+-theory such that $T \subseteq T'$ and T is a companion of T'. Then if T is a JCP-theory, then T' is also a JCP-theory.*

Proof. Let A and B be two arbitrary models of T'. It is clear that $A, B \in Mod(T)$, then any continuation $h : A \to B$ is unique.

Let us show that T' has JCP. Since T admits JCP, there exists a model C of T and homomorphisms $h_1 : A \to C$ and $h_2 : B \to C$. In case when C is a model of T', then T' is also has JCP. If $C \nvDash T'$, we consider a positively closed model M of T such that C is continued in M. Since T and T' are companion theories, $PC_T = PC_{T'}$ according to Proposition 22; hence, $M \models T'$. We obtain that A and B are continued in a model of T'; therefore T' has JCP. Finally, we apply Theorem 29 and obtain that T' is a JCP-theory. □

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Аманбеков С.М., Онерхаан А., Тунгушбаева И.О. ПОЗИТИВТІ ЙОНСОНДЫҚ ТЕОРИЯЛАРДЫҢ АЯСЫНДА АМАЛЬГАМА ЖӘНЕ БІРЛЕСКЕН ЕНГІЗУ ҚАСИЕТТЕРІ

Бұл мақалада амальгама қасиеті мен бірлескен енгізу қасиетінің, сондай-ақ h-амальгама мен бірлескен жалғастыру қасиетінің өзара байланысына негізделген теориялардың арнайы ішкі кластары зерттеледі. Қарастырылған нәтижелер бірінші ретті классикалық логикада да, позитивті логикада да тұжырымдалады, алынған нәтижелер бойынша олардың құрылымы ұқсас болып қалады. Амальгама қасиеті бірлескен енгізу қасиетін және керісінше, h-амальгама бірлескен жалғастыру қасиетін тудыратын жеткілікті шарттар орнатылады. Сонымен қатар, осы қасиеттердің зерттелетін теориялардың кеңейтулерінде сақталу мәселесі қарастырылады.

Түйін сөздер: экзистенциалды тұйық модель, амальгама қасиеті, бірлескен енгізу қасиеті, позитивті модельдер теориясы, позитивті йонсондық теориялар, позитивті тұйық модельдер, h-амальгама қасиеті, бірлескен жалғастыру қасиеті, позитивті тұйық жай йонсондық теориялар.

Аманбеков С.М., Онерхаан А., Тунгушбаева И.О. СВОЙСТВА МАЛЬГАМИРОВАНИЯ И СОВМЕСТНОГО ВЛОЖЕНИЯ В КОНТЕКСТЕ ПОЗИТИВНЫХ ЙОНСОНОВСКИХ ТЕОРИЙ

В данной статье исследуются специальные подклассы теорий, определяемые связью между свойством амальгамирования и свойством совместного вложения, а также между h-амальгамированием и свойством совместного продолжения. Рассмотренные результаты формулируются как в классической логике первого порядка, так и в позитивной логике, причем структура результатов представляется аналогичной. Нами показаны достаточные условия, при которых свойство амальгамирования влечет свойство совместного вложения, и наоборот, а также условия, при которых h-амальгамирование влечет свойство совместного продолжения, и наоборот. Кроме того, исследуется вопрос сохранения принадлежности теории к данным подклассам при расширении рассматриваемой теории.

Ключевые слова: экзистенциально замкнутая модель, свойство амальгамирования, свойство совместного вложения, позитивная теория моделей, позитивные йонсоновские теории, позитивно замкнутые модели, свойство h-амальгамирования, свойство совместного продолжения, позитивно экзистенциально простые йонсоновские теории.

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On T-pseudofinite models of universal theories T

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Abstract. The concept of pseudofinite structures emerged in the 1960s as part of efforts to understand infinite structures that behave, in certain respects, like finite ones. A structure is pseudofinite if it satisfies every first-order sentence that holds in all finite structures of the same language. This idea gained importance through works by Ax, who studied pseudofinite fields, and later by Hrushovski and others in the context of model-theoretic algebra. Pseudofiniteness has since played a key role in finite model theory and asymptotic classes. The article considers universal theories T, the number of isomorphism types of whose finite models is finite. It is proved that all cyclic submodels of a T-pseudofinite model of this theory are finite.

Keywords. universal axiomatizable class of L-structures, theory of a class of L-structures, pseudofinite structure, T-pseudofinite structure.

1 Introduction

The concept of pseudofinite structures emerged as part of a broader effort in model theory to understand the relationship between finite and infinite models, particularly within first-order logic. The development of this idea can be traced back to the 1960s, at the intersection of logic, algebra, and number theory, when logicians began to investigate infinite structures that could be characterized by the same first-order sentences as finite structures.

A structure is called pseudofinite if it is infinite, yet satisfies every first-order sentence that holds in all finite structures of the same language. More precisely, a model $\mathfrak{M}$ is pseudofinite if it is elementarily equivalent to an ultraproduct of finite structures. This notion allows logicians to treat certain infinite models as “limits” or idealizations of finite ones, enabling the application of model-theoretic methods to problems originally rooted in finite mathematics.

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One of the foundational developments in the area was James Ax's work in the late 1960s, particularly his characterization of pseudofinite fields. In his landmark paper "The Elementary Theory of Finite Fields" (1968), Ax proved that the theory of finite fields is complete and that every pseudofinite field is elementarily equivalent to an ultraproduct of finite fields. He further showed that pseudofinite fields are perfect, have exactly one extension of each finite degree, and are pseudo-algebraically closed. This result brought substantial attention to the utility of ultraproducts in connecting finite and infinite model-theoretic behavior.

Throughout the 1970s and 1980s, pseudofiniteness became an increasingly important concept in various branches of mathematical logic and algebra. Researchers began to apply it not only in the study of fields but also to groups, rings, and other algebraic structures. The broader idea of interpreting "finiteness-like" behavior in infinite models proved valuable in understanding the asymptotic properties of classes of finite structures, which became central in finite model theory and descriptive complexity theory.

In the 1990s and 2000s, Ehud Hrushovski significantly expanded the theoretical framework surrounding pseudofinite structures. His work on Zariski geometries and non-standard finite fields used pseudofinite techniques to derive deep results in number theory and algebraic geometry. Hrushovski's application of model-theoretic tools to diophantine geometry and the Mordell-Lang conjecture marked a new era in the interplay between logic and classical mathematics.

Pseudofiniteness also plays a crucial role in the study of asymptotic classes, random structures, and finite model theory, particularly in the context of computer science. The work of Macpherson, Pillay, and others on simple theories and the classification of pseudofinite groups has led to further connections with permutation group theory and stability theory.

Today, the study of pseudofinite structures continues to be a vibrant area within model theory. It lies at the crossroads of logic, algebra, and combinatorics, providing a unifying framework for analyzing infinite structures through the lens of finite approximations. This duality remains a powerful conceptual and technical tool in both pure and applied model theory.

Recently, the model theory of pseudofinite structures is an actively developing area of mathematics. In [1–3] and [4], the model-theoretic properties of theories of pseudofinite fields, groups, rings, and acts over monoids are studied. Clearly, given a pseudofinite model $\mathfrak{M}$ of some theory T in a language L and a sentence true in $\mathfrak{M}$, a finite model of this sentence may not be a model of T. For example, if ${}_SA$ is a pseudofinite act over a monoid S and ${}_SA \models \Phi$, then $\mathfrak{B} \models \Phi$ for some finite structure $\mathfrak{B}$ in the language of acts over a monoid S, but $\mathfrak{B}$ may not be an act over S. So, it is natural to consider the concept of T-pseudofiniteness for a theory T of a language L, which was introduced in [7]. A model $\mathfrak{M}$ of a theory T in a language L is called T-pseudofinite if every sentence in a language L true in $\mathfrak{M}$ has a finite model, which is a model of the theory T. It is clear that T-pseudofiniteness implies pseudofiniteness, and pseudofiniteness implies T-pseudofiniteness for every finite axiomatizable theory T. In [7, 8],

T-pseudofinite acts over a monoid S are considered, where T is a theory of all acts over S.

Also, we can note the articles [5] and [6]. In [5], S. Malyshev gives the description of types of pregeometries with an algebraic closure operator for acyclic theories. In [6], N. Markhabatov and Ye. Baisalov consider acyclic graphs approximated by finite acyclic graphs.

In this work, we consider universal theories T, the number of isomorphism types of whose finite models is finite. We prove that all cyclic submodels of a T-pseudofinite model of this theory are finite.

2 Preliminaries

A structure $\mathfrak{M}$ in a language L is called *pseudofinite* if every sentence true in $\mathfrak{M}$ has a finite model. Let T be a theory of a language L. A model $\mathfrak{M}$ of a theory T in the language L is called *T-pseudofinite* if every sentence in a language L true in $\mathfrak{M}$ has a finite model, which is a model of the theory T.

Theorem 1 (A.A. Stepanova, E.L. Efremov, S.G. Chekanov [7])**.** *Let T be a theory of a language L and $\mathfrak{M}$ be a model of T. Then $\mathfrak{M}$ is a T-pseudofinite structure if and only if $\mathfrak{M}$ is elementarily equivalent to the ultraproduct of finite models of the theory T.*

Theorem 2 (A.A. Stepanova, E.L. Efremov, S.G. Chekanov [3])**.** *Every coproduct of finite S-acts is a T-pseudofinite S-act, where T is the theory of all S-acts.*

A class K of L-structures is called *axiomatizable* if there exists a set Z of sentences of the language L such that for any structure $\mathfrak{A}$,

$$\mathfrak{A} \in K \iff (\text{the language of } \mathfrak{A} \text{ is } L \text{ and } \mathfrak{A} \models \Phi \text{ for all } \Phi \in Z). \tag{1}$$

An axiomatizable class K of L-structures is called *universal axiomatizable* if there exists a set Z of $\forall$-sentences of the language L for which (1) holds.

A substructure $\mathfrak{B}$ of an L-structure $\mathfrak{A}$ is called *one-generated* or *cyclic* if there exists $b \in B$ such that the intersection of all substructures of $\mathfrak{A}$ containing b coincides with $\mathfrak{B}$. In this case, we denote the substructure $\mathfrak{B}$ by $\langle b \rangle$.

3 Main result

Theorem 3. *Let $l \in \omega$, let K be a universal axiomatizable class of L-structures such that the cardinality of any finite cyclic structure in K is less than $l+1$, and let T be a theory of K. If $\mathfrak{A}$ is a T-pseudofinite structure, then the cardinality of any cyclic substructure of $\mathfrak{A}$ is less than $l+1$.*

Proof. Let the hypotheses of the theorem be satisfied, and let $\mathfrak{A} \in K$ be a T-pseudofinite structure. By Theorem 1, $\mathfrak{A} \equiv \mathfrak{B}$, where $\mathfrak{B} = \prod_{i \in I} \mathfrak{B}_i / D$, $\mathfrak{B}_i$ are finite models of T, D is an ultrafilter on I. Since T is a theory of an axiomatizable class K, then $\mathfrak{B} \in K$ and $\mathfrak{B}_i \in K$ for all $i \in I$. We prove that the cardinalities of all cyclic substructures of $\mathfrak{B}$ are less than $l+1$. Assume the opposite, that is, there exists a cyclic substructure $\langle b/D \rangle$ of $\mathfrak{B}$ such that $|\langle b/D \rangle| > l$. Then $\mathfrak{B} \models \Phi_{t_0,\ldots,t_l}(b/D)$, where

$$\Phi_{t_0,\ldots,t_l}(x) \leftrightharpoons \exists x_0 \ldots \exists x_l \left(\bigwedge_{0 \le i < j \le l} x_i \ne x_j \wedge \bigwedge_{0 \le i \le l} x_i = t_i(x) \right),$$

$t_i(x)$ are some terms of the language L. By Los's theorem,

$$J = \{ i \in I \mid \mathfrak{B}_i \models \Phi_{t_0,\ldots,t_l}(b(i)) \} \in D,$$

that is, $|\langle b(i) \rangle| > l$ for all $i \in J$. Since K is a universal axiomatizable class, then $\langle b(i) \rangle \in K$ for all $i \in J$. But the cardinality of any finite cyclic structure in K is less than $l+1$. A contradiction. Consequently, the cardinalities of all cyclic substructures of $\mathfrak{B}$ are less than $l+1$.

Now we prove that the cardinality of any cyclic substructure of $\mathfrak{A}$ is less than $l+1$. Assume the converse, that is, there exists $a \in A$ such that $|\langle a \rangle| > l$. Then $\mathfrak{A} \models \Phi_{g_0,\ldots,g_l}(a)$ for some terms $g_0, \ldots, g_l$ of the language L. Since the structures $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent, $\mathfrak{B} \models \exists x \Phi_{g_0,\ldots,g_l}(x)$. Therefore, there exists a cyclic substructure $\mathfrak{B}$ of cardinality greater than l. This contradiction proves the theorem. □

4 Corollaries from the main result

Corollary 4. *Let S be a monoid, $l \in \omega$, K be the class of all S-acts such that the cardinality of any finite cyclic S-act is less than $l+1$, and T be a theory of K. If ${}_SA$ is a T-pseudofinite S-act, then the cardinality of any cyclic subact of ${}_SA$ is less than $l+1$.*

Proof. Since the class K is universal axiomatizable, this Corollary follows from Theorem 3. □

By Corollary 4, we obtain the following corollary.

Corollary 5 (A.A. Stepanova, E.L. Efremov, S.G. Chekanov [7])**.** *Let S be a monoid, the number of isomorphism types of finite cyclic S-acts be finite, and T be a theory of S-acts. If ${}_SA$ is a T-pseudofinite S-act, then every cyclic subact of ${}_SA$ is finite.*

Corollary 6. *Let G be an abelian group, the number of finite index subgroups of G be finite, and T be a theory of all G-acts. Then ${}_GA$ is a T-pseudofinite G-act if and only if ${}_GA$ is a coproduct of finite G-acts.*

Proof. Let ${}_GA$ be a T-pseudofinite G-act. It is well known that every act over a group G is a coproduct of cyclic acts, and every cyclic act over G is isomorphic to a G-act ${}_GG/H$, where H is a subgroup of G and the unary operations on G/H are defined as follows: $g(aH) = (ga)H$ for any $g, a \in G$. Then by Corollary 4 ${}_GA$ is a coproduct of finite G-acts.

If ${}_GA$ is a coproduct of finite G acts, then by Theorem 2 ${}_GA$ is T-pseudofinite. □

Corollary 7. *Let K be the class of all abelian groups such that the number of isomorphism types of finite cyclic subgroups of groups in K is finite, T be the theory of class K, and let G be a pseudofinite (T-pseudofinite) group. Then G is a periodic group.*

Proof. Since the class K is universal axiomatizable, this corollary follows from Theorem 3. □

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Степанова А. А., Ефремов Е. Л., Чеканов С.Г. УНИВЕРСАЛДЫҚ T ТЕОРИЯЛАРЫНЫҢ T-ПСЕВДО-ШЕКТІ МҮЛДЕЛЕРІНДЕ

Жалған ақырлы құрылымдар ұғымы 1960-жылдары, шексіз құрылымдарды белгілі бір мағынада ақырлы құрылымдар сияқты сипаттау мақсатында пайда болды. Құрылым жалған ақырлы деп аталады, егер ол сол сигнатурадағы барлық ақырлы құрылымдарда орындалатын бірінші реттік тұжырымдарға бағынса. Бұл идея алғаш рет псевдоақырлы өрістерді зерттеген Акс еңбектерінде маңызға ие болды, кейінірек Хрушевский және басқа зерттеушілер модельдік теориялық алгебра саласында оны әрі қарай дамытты. Содан бері жалған ақырлылық ақырлы модельдер теориясында және асимптотикалық кластарда маңызды рөл атқарады. Мақалада T әмбебап теориялары қарастырылады, олардың шекті модельдерінің изоморфизм түрлерінің саны шекті. Бұл теорияның T-псевдофинитті моделінің барлық циклдік ішкі модельдері ақырлы болатыны дәлелденді.

Түйін сөздер: L-құрылымдардың әмбебап аксиоматизацияланатын класы, L-құрылымдар класының теориясы, псевдофинитті құрылым, T-псевдофинитті құрылым.

Степанова А. А., Ефремов Е. Л., Чеканов С.Г. О T-ПСЕВДОКОНЕЧНЫХ МОДЕЛЯХ УНИВЕРСАЛЬНЫХ ТЕОРИЙ T

Понятие псевдоконечных структур возникло в 1960-х годах в рамках попыток понять бесконечные структуры, которые в определённом смысле ведут себя как конечные. Структура называется псевдоконечной, если она удовлетворяет каждому предложению первого порядка, которое выполняется во всех конечных структурах того же языка. Эта идея приобрела значение благодаря работам Акса, изучавшего псевдоконечные поля, а позже Хрушевского и других — в контексте модельно-теоретической алгебры. Псевдоконечность с тех пор играет важную роль в теории конечных моделей и асимптотических классов. В статье рассматриваются универсальные теории T, число типов изоморфизмов конечных моделей которых конечно. Доказывается, что все циклические подмодели T-псевдоконечной модели этой теории конечны.

Ключевые слова: универсально аксиоматизирумый класс L-структур, теория класса L-структур, псевдоконечная структура, T-псевдоконечная структура.

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On the function depth in an o-stable ordered group of a finite convexity rank

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Abstract. We investigate the monotonicity properties of unary functions definable in ordered groups whose elementary theories are o-stable and have finite convexity rank. The notion of o-stability, combining o-minimality and stability, ensures tameness of types around cuts. Prior work established piecewise or local monotonicity of definable functions in weakly o-minimal structures, with key contributions by Pillay, Steinhorn, Wencel, and others. We build on these results by focusing on local monotonicity, n-tidiness, and the depth of definable functions. In particular, we show that any such function is piecewise n-tidy for some finite n, extending the theory of monotonicity beyond weakly o-minimal structures to a broader o-stable context.

Keywords. O-minimal theory, NIP theory, piecewise monotonicity, local monotonicity, o-stable theory, the convexity rank.

1 Preliminaries

The notion of o-stability combines both notions of o-minimality and stability. Roughly speaking, a linearly ordered structure is o-stable if, for any cut, there exist a few complete one-types that are consistent with this cut. B. Baizhanov and V. Verbovskiy showed in [1] that a weakly o-minimal theory is o-stable. A sharper result follows from the description of weakly o-minimal structures by B. Kulpeshov, that a linearly ordered structure is weakly o-minimal if and only if any cut has at most two extensions up to complete one-types over this structure and the sets of all realizations of these one-types are convex [5]. So, we can say that any weakly o-minimal theory has Morley o-rank 1 and Morley o-degree at most 2, so, these theories are o-ω-stable.

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A. Pillay and C. Steinhorn started the investigation of the piecewise monotonicity of definable unary functions in linearly ordered structures in [9]. R. Wencel extended their result to the class of non-valuational weakly o-minimal ordered groups [14]. D. Macpherson, D. Marker, and C. Steinhorn introduced the notion of a local monotonicity and a tidy function and proved that any unary function that is definable in a structure of a weakly o-minimal theory is tidy. V. Verbovskiy introduced the notion of the depth of a function and proved that any function definable in a structure of a weakly o-minimal theory has finite depth and is piecewise n-tidy for some finite natural n [10]. Also, the question of monotonicity of unary functions has been studied in many other articles for different classes of theories. V. Verbovskiy and A. Dauletiyarova proved piecewise monotonicity of a unary function definable in an ordered non-valuational group with an o-stable theory [13]. Here, we aim to consider local monotonicity and the notions of n-tidy and the depth of a unary function definable in an ordered group with an o-stable theory of a finite convexity rank.

In the following section, we provide some standard definitions and notations.

Let $\mathcal{M} = (M, <, \dots)$ be a totally ordered structure, a be an element of M, and let A and B be subsets of M. As usual, we write

$$\begin{aligned} a &< B, \text{ if } a < b \text{ for any } b \in B, \\ A &< B, \text{ if } a < b \text{ for any } a \in A \text{ and } b \in B. \end{aligned}$$

A partition $\langle C, D\rangle$ of M is called a *cut* if $C < D$. Given a cut $\langle C, D\rangle$, one can construct a partial type $\{c < x < d : c \in C,\ d \in D\}$, which we also call a cut and use the same notation $\langle C, D\rangle$. If the set C is definable, then the cut is called *quasirational*; if in addition $\sup C \in M$, then the cut $\langle C, D\rangle$ is called *rational*. A non-definable cut is called *irrational*. If $C = (-\infty, c)$ we denote this cut by c^-, and if $C = (-\infty, c]$ we denote it by c^+. If $C = M$, we denote this cut $+\infty$. The notation $\sup A$ stands for such a cut $\langle C, D\rangle$, that $C = \{c \in M : c < \sup A\}$. If the set C is definable we sometimes distinguish cuts defined by $\sup C$ and $\inf D$ as: $\sup C$ stands for $\langle C, D\rangle \cup \big\{C(x)\big\}$ and $\inf D$ stands for $\langle C, D\rangle \cup \big\{\neg C(x)\big\}$. A cut $\langle C, D\rangle$ in an ordered group is called *non-valuational* [7, 14] if $d - c$ converges to 0 whenever c and d converge to $\sup C$ and $\inf D$, respectively. A cut, which is not non-valuational, is called *valuational*. Observe that for a valuational cut $\langle C, D\rangle$, there is a convex non-trivial subgroup H such that $\sup C = \sup(a + H)$ for some a, and this cut is definable iff the subgroup H is definable. An ordered group G is said to be of *non-valuational type*, if any quasirational cut is non-valuational. Note that G is of non-valuational type if and only if there is no definable non-trivial convex subgroup in G.

The set of all cuts $\langle C, D\rangle$ that are definable in $\mathcal{M}$ and such that the set D has no smallest element will be denoted by $\overline{M}$. The set M can be regarded as a subset of $\overline{M}$ by identifying an element $a \in M$ with the cut $\big\langle(-\infty, a], (a, +\infty)\big\rangle$. After such identification, $\overline{M}$ is naturally equipped with a linear ordering extending $(M, <)$: $\langle C_1, D_1\rangle \le \langle C_2, D_2\rangle$ if and only if $C_1 \subseteq C_2$. Clearly, $(M, <)$ is a dense substructure of $(\overline{M}, <)$.

A subset A of a totally ordered set M is called *convex* if for any a and $b \in A$ the interval $[a, b]$ is a subset of A. The *length* of a convex set A is defined as $\sup\{a - b : a, b \in A\}$. A *convex component* of a set A is a maximal convex subset of A. The *convex hull* A^c of A is defined as

$$A^c = \big\{b \in M : \exists a_1, a_2 \in A\,(a_1 \le b \le a_2)\big\},$$

that is, it is the least convex set containing the set A.

2 Introduction

The aim of this paper is to investigate the properties of unary functions that are definable in an o-stable ordered group whose convexity rank is finite, say, n.

Let $\mathcal{M} = (M, <, \dots)$ be a totally ordered structure. Recall that a function can be defined from its graph and for each function, it is easy to construct its graph. So, we consider an arbitrary formula: $\Phi(x, \bar{y})$. Let B be the set of all such $\bar{b}$, that $\Phi(\mathcal{M}, \bar{b}) \neq \emptyset$. Given an element $\bar{b}$ we consider the definable set $\Phi(\mathcal{M}, \bar{b})$. Then we can consider $\sup \Phi(\mathcal{M}, \bar{b})$ as an element of $\overline{M}$. So, the set

$$\{(\bar{b}, \sup \Phi(\mathcal{M}, \bar{b})) : \bar{b} \in B\}$$

defines the graph of some function f from B to $\overline{M}$. The main property we consider here is the monotonicity of a function. So, below, we define $f(\bar{b}_1) \ge f(\bar{b}_2)$ in terms of Φ. We prefer to work with a formula $\Phi(x; \bar{y})$ rather than with a function $f(\bar{y})$.

So, let $\Phi(x, \bar{y})$ be an M-definable formula. We write

$$\Phi(\mathcal{M}, \bar{y}_1) \ge \Phi(\mathcal{M}, \bar{y}_2), \text{ if } \mathcal{M} \models \forall x_2 \exists x_1 [\bigwedge_{i=1}^{2} \Phi(x_i, \bar{y}_i) \to x_2 \le x_1]$$

it means that $\sup \Phi(\mathcal{M}, \bar{y}_1) \ge \sup \Phi(\mathcal{M}, \bar{y}_2)$. Then

$$\Phi(\mathcal{M}, \bar{y}_1) = \Phi(\mathcal{M}, \bar{y}_2) \Leftrightarrow \Phi(\mathcal{M}, \bar{y}_1) \le \Phi(\mathcal{M}, \bar{y}_2) \wedge \Phi(\mathcal{M}, \bar{y}_1) \ge \Phi(\mathcal{M}, \bar{y}_2)$$

$$\Phi(\mathcal{M}, \bar{y}_1) < \Phi(\mathcal{M}, \bar{y}_2) \Leftrightarrow \Phi(\mathcal{M}, \bar{y}_1) \le \Phi(\mathcal{M}, \bar{y}_2) \wedge \Phi(\mathcal{M}, \bar{y}_1) \neq \Phi(\mathcal{M}, \bar{y}_2)$$

Now we consider the case where the length of the tuple $\bar{y}$ is 1, that is, y is a variable.

We say that $\Phi(x, y)$ is strictly increasing on a set I, if

$$\forall y \forall z [y, z \in I \wedge y < z \to \Phi(\mathcal{M}, y) < \Phi(\mathcal{M}, z)].$$

If $E(y, z)$ is an equivalence relation with convex classes on a set I, then $\Phi(\mathcal{M}, y)$ is strictly increasing on the quotient set $I/_E$, if

$$\forall y \forall z [y, z \in I \wedge \neg E(y, z) \wedge y < z \to \Phi(\mathcal{M}, y) < \Phi(\mathcal{M}, z)].$$

We define strictly decreasing and constant behavior in a similar way to strictly increasing behavior.

We assume that $\operatorname{dom}\Phi(\mathcal{M}, y) = \{y \in M : \mathcal{M} \models (\exists x)\Phi(x, y)\}$.

Definition 1 (M. Dickmann; D. Macpherson, D. Marker, C. Steinhorn)**.**

- A weakly o-minimal structure is a totally ordered structure $\mathcal{M} = (M, <, \dots)$ such that any definable subset of M is a finite union of convex disjoint sets under the ordering $<$.
- A theory is weakly o-minimal if all of its models are.

Definition 2 (D. Macpherson, D. Marker, C. Steinhorn,[7])**.** If $\mathcal{M}$ is a totally ordered structure, $\Phi(x, y)$ is an M-definable formula, $I \subset \operatorname{dom}(\Phi(\mathcal{M}, y))$, then we say that Φ is tidy on I, if one of the following holds:

1. $\forall x \in I$ there is an infinite interval $J \subset I$ such that $x \in J$ and Φ is strictly increasing on J (we say that Φ is locally increasing on I).
2. $\forall x \in I$ there is an infinite interval $J \subset I$ such that $x \in J$ and Φ is strictly decreasing on J (we say that Φ is locally decreasing on I).
3. $\forall x \in I$ there is an infinite interval $J \subset I$ such that $x \in J$ and Φ is constant on J (we say that Φ is locally constant on I).

and, if for some $x \in I$ the set $\{y \in I \mid \Phi(\mathcal{M}, t)$ is strictly monotonic on $(x, y) \cup (y, x)\}$ and has a maximum or a minimum, then $\Phi(\mathcal{M}, t)$ is strictly monotonic on I.

Definition 3. If Φ and I are like in Definition 2, then we say that Φ is n-tidy on I if the following holds:

- $\forall z \forall y \forall t\ [\Phi(\mathcal{M}, z) = \Phi(\mathcal{M}, y) \wedge z < t < y \to \Phi(\mathcal{M}, z) = \Phi(\mathcal{M}, t)]$
- $\Phi^{(n)}$ is tidy on I/E_{n-1}, where $\Phi^{(n)}(x, y) := \exists z[E_{n-1}(y, z) \wedge \Phi(x, z)]$
- $(\forall y \in I)\ E_n(I, y)/E_{n-1}$ has no minimum and maximum.
- $|I/E_n| \geq \omega$.

Where E_n is an equivalence relation on I such that

$$\begin{array}{lll} E_n(z, y) & \Leftrightarrow & E_{n-1}(z, y) \vee \\ & \vee & [[z < y \wedge \neg E_{n-1}(z, y) \to \Phi^{(n)} \upharpoonright [z, y]/E_{n-1} \text{ is strictly monotonic}] \wedge \\ & \wedge & [y < z \wedge \neg E_{n-1}(z, y) \to \Phi^{(n)} \upharpoonright [y, z]/E_{n-1} \text{ is strictly monotonic}]] \end{array}$$

Here 0-tidy is tidy, $\Phi^{(0)} = \Phi,\ E_0(z, y) \Leftrightarrow z = y$.

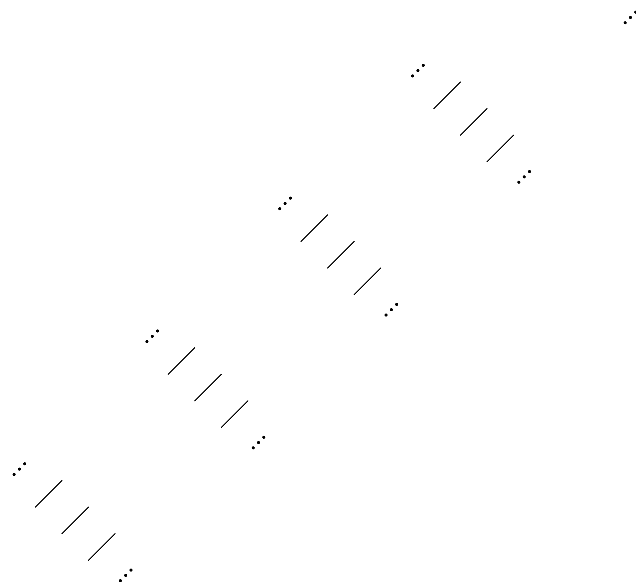

Figure 1: The example of the graph of a function of depth 3

Definition 4. If Φ and I like in Definition 2, then we say that Φ is strongly tidy on I if there exists $n \in N$ such that Φ is $(n-1)$-tidy on I and $\Phi^{(n)}$ is strictly monotonic on I/E_{n-1}. So we say that the depth of Φ on I equals n.

Definition 5 (D. Macpherson, D. Marker, C. Steinhorn, [7])**.** Let $\mathcal{M}$ be a weakly o-minimal structure. We say that $\mathcal{M}$ has monotonicity if the following holds: whenever $\Phi(x, y, \bar{a})$ is a formula with $\bar{a} \in M$, there is $m \in N$ and a partition of $\mathrm{dom}(\Phi(M, y, \bar{a}))$ into definable sets $X, I_1, \ldots, I_m$ such that X is finite, each I_i is convex and on each I_i the formula $\Phi(x, y, \bar{a})$ is tidy.

Definition 6 (V. Verbovskiy, [10])**.** Let $\mathcal{M}$ be a weakly o-minimal structure. We say that $\mathcal{M}$ has strong monotonicity (monotonicity [7]), if the following holds: whenever $\Phi(x, y, \bar{a})$ is a formula with $\bar{a} \in M$, there is $m \in N$ and a partition of $\mathrm{dom}(\Phi(M, y, \bar{a}))$ into definable sets X, I_1, $\ldots$, I_m such that X is finite, each I_i is convex and on each I_i the formula $\Phi(x, y, \bar{a})$ is strongly tidy.

Definition 7. Let $\mathcal{M}$ be a weakly o-minimal structure. Then we say that $\mathcal{M}$ has finite depth, if the following holds: whenever $\Phi(x, y, \bar{z})$ is a formula, there exists $n \in N$ such that for any $\bar{a} \in M$ and for any convex set $I \subset \mathrm{dom}(\Phi(\mathcal{M}, y, \bar{a}))$, on which $\Phi(\mathcal{M}, y, \bar{a})$ is strongly tidy, the depth of $\Phi(\mathcal{M}, y, \bar{a})$ on I is less than n.

Theorem 8 (D. Macpherson, D. Marker, C. Steinhorn, [7])**.** *If all models of $Th(\mathcal{M})$ are weakly o-minimal, then $\mathcal{M}$ has monotonicity.*

Theorem 9 (V. Verbovskiy, [10])**.** *If all models of $Th(M)$ are weakly o-minimal, then M has strong monotonicity and finite depth.*

Definition 10 (B. Baizhanov, V. Verbovskiy, [1], [11])**.**

1. An ordered structure $\mathcal{M}$ is *o-stable in* λ if for any $A \subseteq M$ with $|A| \leq \lambda$ and for any cut $\langle C, D\rangle$ in $\mathcal{M}$ there are at most λ 1-types over A which are consistent with the cut $\langle C, D\rangle$, i.e.
$$\left|S^1_{\langle C,D\rangle}(A)\right| \leq \lambda.$$
2. A theory T is *o-stable in* λ if every model of T is. Sometimes, we write T is o-λ-stable.
3. A theory T is *o-stable* if there exists an infinite cardinal λ in which T is o-stable.

In [11], V. Verbovskiy proved that any ordered group whose elementary theory is o-stable is Abelian.

Lemma 11 (V. Verbovskiy, [11])**.** *Let G be an ordered group of non-valuational type whose elementary theory is o-stable. Then any equivalence relation in G has at most finitely many infinite convex classes.*

Let $\Gamma = \{(x, f(x)) : x \in \mathrm{dom}(f)\}$ be the graph of a function f.

We denote by $\lim_{x\to a+0} f$ the set of all elements $b \in \overline{G}$ such that (a, b) is a limit point of the set $\{(x, f(x)) : x \in \mathrm{dom}(f), x > a\}$. In other words, $\lim_{x\to a+0} f$ is the set of all the right-hand limit points of the function f at the point a.

Similarly, we define $\lim_{x\to a-0} f$ as the set of all such b that (a, b) is a limit point of the set $\{(x, f(x)) : x \in \mathrm{dom}(f), x < a\}$.

Furthermore, we define
$$\lim_{x\to a} f \triangleq \lim_{x\to a-0} f \cup \lim_{x\to a+0} f.$$

Fact 12 (J. Goodrick, [4])**.** *For any densely ordered structure A and any function $f : A \to \overline{A}$, for any $a \in A$, the set $\lim_{x\to a} f(x)$ is nonempty.*

Lemma 13 (V. Verbovskiy, A. Dauletiyarova, [13])**.** *Let an ordered group $(G, <, +, f, 0, \dots)$, whose order is dense, have an o-stable theory. Then there exists a natural number k such that for any element $a \in \mathrm{dom}\, f$, the set $\lim_{x\to a} f(x)$ has at most k elements.*

Due to Lemma 13, we can define k functions $f_1, \dots, f_k$, where k is taken from Lemma 13, as follows: $f_i(x)$ is the i-th element of $\lim_{x\to a} f(x)$. So, we can define finitely many functions which have at most one limit $\lim_{x\to a+0} f$ and at most one limit $\lim_{x\to a-0} f$. So, without loss of generality, we may assume that a function under consideration has at most one left-hand limit and at most one right-hand limit.

The definition of the convexity rank of a formula with one free variable was introduced in [5] and extended on an arbitrary set in [6] by B. Kulpeshov:

Definition 14 (B. Kulpeshov, [5, 6])**.** Let T be a weakly o-minimal theory, $\mathcal{M} \models T$, $A \subseteq M$. *The rank of convexity of the set* A ($RC(A)$) is defined as follows:

1. $RC(A) = -1$ if $A = \emptyset$.
2. $RC(A) = 0$ if A is finite and non-empty.
3. $RC(A) \geq 1$ if A is infinite.
4. $RC(A) \geq \alpha + 1$ if there exists a parametrically definable equivalence relation $E(x, y)$ and an infinite sequence of elements $b_i \in A, i \in \omega$, such that:

- For every $i, j \in \omega$ whenever $i \neq j$ we have $M \models \neg E(b_i, b_j)$;
- For every $i \in \omega$, $RC(E(x, b_i)) \geq \alpha$ and $E(M, b_i)$ is a convex subset of A.

5. $RC(A) \geq \delta$ if $RC(A) \geq \alpha$ for all $\alpha < \delta$, where δ is a limit ordinal.

If $RC(A) = \alpha$ for some ordinal α, we say that $RC(A)$ is defined. Otherwise (that is, if $RC(A)) \geq \alpha$ for all α), we put $RC(A) = \infty$.

The rank of convexity of a formula $\phi(x, \bar{a})$, where $\bar{a} \in M$, is defined as the rank of convexity of the set $\phi(M, \bar{a})$, that is, $RC(\phi(x, \bar{a})) \triangleq RC(\phi(M, \bar{a}))$.

The convexity rank of a 1-type p is defined as the rank of convexity of the set $p(M)$, that is, $RC(p) \triangleq RC(p(M))$.

Obviously, a theory that extends the theory of a linear order has the convexity rank 1 if there are no parametrically definable equivalence relations with infinitely many infinite convex classes.

3 Main result

Let A be a definable convex set. We define the following convex subgroups:

$$H_A^+ = \{g \in G : a + |g| \in A \text{ for any } a \in A\},$$

$$H_A^- = \{g \in G : a - |g| \in A \text{ for any } a \in A\}.$$

Following [8], we say that the right shore of A is long if H_A^+ is trivial; similarly, the left shore of A is long if H_A^- is trivial.

Lemma 15. *Let $\mathcal{G}$ be an ordered o-stable group and E a definable equivalence relation with convex classes. Then the number of infinite E-classes with a long shore is finite.*

Proof. Let E be a definable equivalence relation with infinitely many convex classes. Assume to the contrary that there exist infinitely many infinite E-classes with a long shore. By Dirichlet's principle, without loss of generality, we may assume that there are infinitely many infinite E-classes with the right long shore. Moreover, without loss of generality, we may assume that there exists an infinite increasing sequence $\langle a_i : i < \omega \rangle$ of representatives of infinite E-classes with a long right shore, such that $\neg E(a_i, a_j)$ for each $i < j < \omega$. Since we can consider a sufficiently saturated model, we may suppose that there exists a positive element $b \in G$ such that $E(a_i, a_i + b)$ holds for each $i < \omega$.

Let $\varphi(x; b)$ say that x belongs to an E-class whose length is at least b and whose right shore is long and the distance between x and the right shore of $[x]_E$ is less than b. Since the right shore is long, this set is not empty.

Let $C = \{g \in G : g < a_i \text{ for some } i\}$ and $D = G \setminus C$. Since for each $c \in C$ and each b_1 and b_2 with $0 < b_1 < b_2 < b$ it holds that

$$\varphi(\mathcal{G}, b_1) \cap (c, \sup C) \subset \varphi(\mathcal{G}, b_2) \cap (c, \sup C)$$

we obtain that $\mathcal{G}$ has the strict order property inside the cut $\sup C$. Since an expansion of a model of an o-stable theory by a convex unary predicate preserves o-stability [11], we may add a convex predicate P which names C. So, we obtain the strict order property inside the cut

$$\{c < x < d : c \in C,\ d \in D\} \cup \{P(x)\},$$

that contradicts o-stability, [1]. □

Theorem 16. *Let $\mathcal{G} = (G, <, +, \dots)$ be an ordered group with an o-stable theory. Let $\mathcal{G}$ have finitely many, say k, non-trivial proper definable convex subgroups. Then $RC(G) = k$. And vice versa, if $RC(G)$ is finite, say k, then the number of non-trivial proper definable convex subgroups is equal to k.*

Proof. Let $\{0\} < H_1 < \dots < H_k < G$ be a chain of all definable convex subgroups of $\mathcal{G}$. Then $E_k(x, y) \triangleq x - y \in H_k$ is an equivalence relation with convex classes, and the chain $E_1, \dots, E_k$ of equivalence relations demonstrates that $RC(G)$ is at least k.

Assume that $RC(G) = k$ and a chain of equivalence relations $E_1, \dots, E_k$ witnesses it, where E_i refines E_{i+1} for each positive $i < k$. By Lemma 15, without loss of generality, after removing finitely many equivalence classes, we may assume that each shore of each class of each equivalence relation is short, that is, not long, so, it defines a non-trivial subgroup. By the first paragraph of the proof of this theorem, the number of definable convex non-trivial proper subgroups is at most k, say, n. Let

$$\{0\} < H_1 < \dots < H_n < G$$

be the sequence of all definable convex non-trivial proper subgroups of $\mathcal{G}$.

First, we consider E_1. Since H_1 is the least non-trivial definable convex subgroup, and each shore of each infinite E_1-class defines a subgroup, each infinite E_1-class consists of cosets of H_1.

Assume that there are infinitely many cosets of H_2 such that each contains an E_1-class as a proper subset and this E_1-class consists of infinitely many cosets of H_1. Then we consider $\mathcal{G}/H_1$ with the full induced structure. Its elementary theory is o-stable [11]. We obtain an equivalence relation in $\mathcal{G}/H_1$ with infinitely many infinite convex classes, but these classes are proper subsets of cosets of H_2/H_1. So, there exists a definable convex subgroup $H'/H_1 < H_2/H_1$. Let H' be the pre-image of H'/H_1 in G. It is definable and convex. We obtain a contradiction with the fact that there are exactly n definable convex non-trivial proper subgroups. So, either an E_1-class consists of finitely many cosets of H_1 or consists of cosets of H_2. Note that if an E_1-class consists of finitely many cosets of H_1, then the order in G/H_1 is discrete. Moreover, there is a positive integer m_1 such that if a E_1-class consists of finitely many cosets of H_1, then the number of these cosets is at most m_1. Indeed, otherwise by compactness there are infinite E_1-classes and we obtain a contradiction as above.

Considering G/H_i we can conclude by the similar reasons that either an E_{i+1}-class consists of finitely many cosets of H_{i+1} or consists of cosets of H_{i+2}.

These imply that $k \leq n$. So, we obtain the equality $k = n$. □

Theorem 17. *Let G be an order-stable ordered group of non-valuational type with a dense order, let A be a sort in $\overline{G}$, and let $f : D \subset G \to A$ be a definable and continuous function. Then the function f is piecewise monotonic; that is, there exists some $m \in \mathbb{N}$ and a finite partition of the domain of the function $D = \operatorname{dom} f$ into definable sets $X, I_1, \ldots, I_m$ such that X is finite, each I_i is convex for $i < m$, and $f \restriction I_i$ is monotonic.*

Theorem 18. *Let $\mathcal{G}$ be an ordered group with an o-stable theory and let f be a definable continuous unary function. Let $RC(G) = n$. Let $\{0\} < H_1 < \cdots < H_n < G$ be the chain of all its definable convex subgroups and let G/H_i be dense for each i.*

Then f is strongly tidy and its depth is at most n. In other words, an ordered group with an o-stable theory and a finite convexity rank has strong monotonicity and finite depth bounded by its convexity rank.

Proof. As we mentioned it below Lemma 13, we may assume that the graph f has at most one limit point from the left and at most one limit point from the right for each element. By Theorem 17 the restriction of f to any coset of H_1 is piecewise monotone. Let E_1 be an equivalence relation on $\operatorname{dom} f$ with convex classes, on which f is monotone. Then, as we know, the E_1-classes are cosets of H_1. Now we consider G/H_1 with the complete induced structure and we find that f/H_1 is monotone on the cosets of H_2/H_1. Here,

$$f/H_1(a) \triangleq \sup\{f(x) : x \in a + H_1\}.$$

Proceeding by induction, we conclude that the statements of the theorem hold. The bound for the depth of f follows from Theorem 16. $\square$

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Вербовский В.В. ШЕКТЕУЛІ ДӨҢЕСТІК РАНГІ БАР РЕТТЕЛГЕН-ТҰРАҚТЫ ТОПТАҒЫ ФУНКЦИЯНЫҢ ТЕРЕҢДІГІ ТУРАЛЫ

Біз элементар теориялары o-тұрақты және шектеулі дөңестіктік рангі бар реттелген топтарда анықталатын унар функциялардың монотондық қасиеттерін зерттейміз. o-тұрақтылық ұғымы o-минималдылық пен тұрақтылықты біріктіріп, үзілістер маңындағы типтердің тәртіптілігін қамтамасыз етеді. Бұған дейінгі еңбектерде Пиллэй, Стайнхорн, Венцел және басқалардың маңызды үлесімен әлсіз o-минимал құрылымдарда анықталатын функциялардың кесінділік немесе локальді монотондығы көрсетілген. Біз бұл нәтижелерді жалғастырып, локальді монотондыққа, n-тәртіптілікке және анықталатын функциялардың тереңдігіне назар аударамыз. Атап айтқанда, мұндай функцияның кейбір шекті натурал n үшін кесінділік n-тәртіпті болатынын көрсетеміз. Бұл монотондық теориясын әлсіз o-минимал құрылымдардан кеңірек o-тұрақты контекске дейін жалғастырады.

Түйін сөздер: реттелген минимал теория, NIP теориясы, кесінділік монотондық, локальді монотондық, реттелген тұрақты теория, дөңестік ранг.

Вербовский В.В. О ГЛУБИНЕ ФУНКЦИИ В О-СТАБИЛЬНОЙ УПОРЯДОЧЕННОЙ ГРУППЕ С КОНЕЧНЫМ РАНГОМ ВЫПУКЛОСТИ

Мы исследуем свойства монотонности унарных функций, определимых в упорядоченных группах, элементарные теории которых являются o-стабильными и имеют конечный ранг выпуклости. Понятие o-стабильности, объединяющее o-минимальность и стабильность, обеспечивает упорядоченность типов в окрестности сечений. В предыдущих работах, в частности благодаря вкладу Пиллея, Стайнхорна, Венцеля и других, была установлена кусочная или локальная монотонность определимых функций в слабо o-минимальных структурах. Мы развиваем эти результаты, сосредотачиваясь на локальной монотонности, n-упорядоченности и глубине определимых функций. В частности, мы показываем, что любая такая функция является кусочно n-упорядоченной для некоторого конечного n, расширяя теорию монотонности за пределы слабо o-минимальных структур к более общему o-стабильному контексту.

Ключевые слова: o-минимальная теория, теория с NIP, кусочная монотонность, локальная монотонность, упорядоченно-стабильная теория, ранг выпуклости.

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Inverse initial problem for fractional wave equation with the Hadamard fractional derivative

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Abstract. This paper investigates an inverse initial problem for a time-fractional wave equation involving the Hadamard fractional derivative. Unlike the more widely studied Caputo and Riemann–Liouville derivatives, the Hadamard derivative is defined via a logarithmic kernel and exhibits distinct analytical features, making it suitable for modeling processes with slow memory decay and multiplicative structures. Building on prior work concerning the extremum principle and solvability of boundary value problems with Hadamard-type operators, we establish sufficient conditions for the unique solvability of the inverse problem. The analysis is carried out in terms of eigenfunction expansions and leverages properties of the two-parameter Mittag–Leffler function. The findings contribute to the theory of inverse problems for fractional wave equations and highlight the role of Hadamard derivatives in capturing complex temporal dynamics in mathematical models.

Keywords. Fractional wave equation, inverse initial problem, Hadamard fractional derivative, Mittag-Leffler function.

1 Introduction

Inverse problems for fractional partial differential equations have gained considerable attention in recent years due to their broad applicability to modeling complex phenomena in physics, engineering, and other scientific fields. In particular, fractional-wave equations, which incorporate memory effects and nonlocal behavior, provide a more accurate representation of various real-world processes than their classical counterparts [1].

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This paper addresses an inverse initial problem for a time-fractional wave equation where the fractional derivative is understood in the sense of Hadamard. The Hadamard fractional derivative, characterized by its logarithmic kernel, introduces unique analytical challenges and properties distinct from those of the more commonly studied Riemann–Liouville and Caputo derivatives. More properties of this derivative can be found in [2], [3].

In [4], an extremum principle for Hadamard fractional derivatives was considered. The authors established new estimates for the Hadamard fractional derivatives at extreme points of functions. This extremum principle was instrumental in proving the uniqueness and continuous dependence of solutions for initial boundary value problems related to linear and nonlinear time-fractional diffusion equations.

Sequential differential equations with the Hadamard fractional derivative were the subject of [5]. The Ulam–Hyers stability of Caputo-Hadamard fractional stochastic differential equations was studied in [6]. Variable-order Caputo-Hadamard fractional derivative was considered in [7].

We note works [8] and [9], where sub-diffusion and fractional diffusion-wave equations involving the Hadamard fractional derivative were analyzed. In [10], a problem with the terminal integral condition for a nonlinear fractional-differential equation with the bi-ordinal Hilfer-Hadamard derivative was targeted for the unique solvability.

We investigate the well-posedness of this problem under specific assumptions on the given data. By carefully analyzing the structure of the equation and utilizing appropriate functional analytic tools, we establish conditions that guarantee the existence and uniqueness of a solution. The results presented contribute to a broader understanding of inverse problems associated with fractional wave equations and highlight the potential of the Hadamard derivative in modeling and analysis.

2 Direct problem

Consider an initial boundary value problem for the time-fractional wave equation with the Hadamard fractional derivative in a rectangular domain. Let us consider an equation

$$ {}_H D_{1t}^{\alpha} u(t,x) - u_{xx}(t,x) = f(t,x) \tag{1} $$

in a rectangular domain $\Omega = \{(t,x) : 0 < x < 1, 1 < t < T\}$. Here $f(t,x)$ is a given function, $T > 1$ is a positive real number, and

$$ {}_H D_{1t}^{\alpha} g(t) = \left(t\frac{d}{dt}\right)^n \frac{1}{\Gamma(n-\alpha)} \int\limits_1^t \left(\log\frac{t}{s}\right)^{n-\alpha+1} \frac{g(s)}{s} ds \ \ (t > 1) $$

represents the Hadamard fractional derivative of order $\alpha(1 < \alpha \leq 2$, $\log(..) = \ln(..)$ [1].

Let us formulate a direct problem for equation (1).

Direct problem. To find a function $u(t,x)$ satisfying

- the equation (1) in Ω;
- regularity conditions $u(\cdot,x)\in C^{\alpha}_{\gamma,\log}[1,T]$, $u(t,\cdot)\in C^1[0,1]\cap C^2(0,1)$;
- boundary conditions

$$u(t,0)=u(t,1)=0,\ 1\le t\le T; \tag{2}$$

- initial conditions

$${}_HI^{2-\alpha}_{1t}u(t,x)\big|_{t=1+}=\varphi(x),\ 0\le x\le 1,\ {}_HD^{\alpha-1}_{1t}u(t,x)\big|_{t=1+}=\psi(x),\ 0<x<1. \tag{3}$$

Here $\varphi(x)$ and $\psi(x)$ are given functions, ${}_HI^{\beta}_{1t}$ represents the Hadamard fractional integral of order $\beta>0$ given

$${}_HI^{\beta}_{1t}g(t)=\frac{1}{\Gamma(\beta)}\int\limits_1^t\left(\log\frac{t}{s}\right)^{\beta-1}\frac{g(s)}{s}ds,\ \ t>1,$$

the class of functions $C^{\alpha}_{\delta,\gamma}(..)$ with $0<\gamma\le 1$ is given by (see [1])

$$\begin{aligned}C^n_{\delta,\gamma}[a,b] &= \left\{g:\ \|g\|_{C^n_{\delta,\gamma}}=\sum_{k=0}^{n-1}\|\delta^k g\|_C+\|\delta^n g\|_{C_{\gamma,\log}}\right\},\\ C^0_{\delta,\gamma}[a,b] &= C_{\gamma,\log}[a,b],\ \delta=t\frac{d}{dt}.\end{aligned}$$

We search a solution to the direct problem as follows:

$$u(t,x)=\sum_{k=1}^{\infty}U_k(t)\sin k\pi x. \tag{4}$$

Substituting (4) into (1) at $f(t,x)\equiv 0$ we obtain

$${}_HD^{\alpha}_{1t}U_k(t)+(k\pi)^2U_k(t)=f_k(t), \tag{5}$$

where $f_k(t)=2\int\limits_0^1 f(t,x)\sin k\pi x dx$ are Fourier coefficients of the function $f(t,x)$ represented by Fourier-Sine series, i.e.

$$f(t,x)=\sum_{k=1}^{\infty}f_k(t)\sin k\pi x.$$

Initial conditions (2) give us

$${}_HI^{2-\alpha}_{1t}U_k(t)\big|_{t=1+}=\varphi_k,\ 0\le x\le 1,\ {}_HD^{\alpha-1}_{1t}U_k(t)\big|_{t=1+}=\psi_k,\ 0<x<1. \tag{6}$$

Here, φ_k and ψ_k are Fourier coefficients of functions $\varphi(x)$ and $\psi(x)$, respectively.

The solution of the Cauchy-type problem (5)-(6) can be represented as [1]

$$U_k(t) = \varphi_k (t-1)^{\alpha-1} E_{\alpha,\alpha}\left[-(k\pi)^2 (\log t)^\alpha\right] + \psi_k (t-1)^{\alpha-2} E_{\alpha,\alpha-1}\left[-(k\pi)^2 (\log t)^\alpha\right] + \\ + \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left[-(k\pi)^2\left(\log\frac{t}{s}\right)^\alpha\right] f_k(s)\frac{ds}{s}, \quad (7)$$

where $E_{a,b}(z) = \sum\limits_{n=0}^{\infty} \frac{z^n}{\Gamma(an+b)}$, with $a > 0$ and $b \in \mathbb{R}$, represents two-parameter Mittag-Leffler function [1].

It is easy to prove the following statement.

Lemma 1. *If $g(x) \in C^2[0,1]$ is such that $g(0) = g(1) = 0$, $g''(0) = g''(1) = 0$, and $g'''(x) \in L_2(0,1)$, then*

$$\sum_{k=1}^{\infty} |g_k| (k\pi)^2 \le \sum_{k=1}^{\infty} \frac{1}{(k\pi)^2} + \left\|g'''(x)\right\|_2^2.$$

The proof of this lemma can be done using integration by parts, considering Bessel's inequality and Parseval's identity.

The convergence of the infinite series corresponding to the functions $u(t,x)$ and $u_{xx}(t,x)$ can be proved using Lemma 1 and the well-known estimate of the two-parameter Mittag-Leffler function $E_{a,b}(-z) \le \frac{C}{1+|z|}$ for $z > 0$ [1].

Regarding the solvability of the direct problem, we can state the following.

Theorem 2. *If the functions $\varphi(x)$, $\psi(x)$, and $f(t,x)$ (with respect to the variable x) satisfy the condition of Lemma 1 and $f(\cdot,x) \in C_{\gamma,\log}[1,T]$, then a solution of the direct problem does exist, moreover, it is unique and is represented by Formula (4), where $U_k(t)$ will be found using (7).*

3 Inverse initial problem

In this section, we consider an inverse problem of finding an initial condition using the additional data at a fixed time.

Inverse initial problem. To find a pair of functions $\{u(t,x); \psi(x)\}$ satisfying

- the equation (1) in Ω;
- regularity conditions $u(\cdot,x) \in C^\alpha_{\gamma,\log}[1,T]$, $u(t,\cdot) \in C^1[0,1] \cap C^2(0,1)$, $\psi(x) \in C[0,1]$;
- boundary conditions (2);

- the first initial condition of (3);
- over-determination condition $u(\xi, x) = \zeta(x)$, $0 \le x \le 1$ for fixed $\xi \in (1, T]$.

Here, $\varphi(x)$ and $\zeta(x)$ are given functions.

Inverse initial problems for differential equations were considered in many works. For example, see [11]–[14]. Namely, in [11], authors investigated the unique solvability of the inverse initial problem for the heat equation with the Bessel operator. Then in [12], the result was generalized for the time-fractional heat equation with the same operator in the space variable. In [13], a similar inverse problem was targeted at the sub-diffusion equation with a variable coefficient involving a more general fractional derivative. The work [14] is devoted to the unique solvability of the inverse initial problem for the fractional wave equation.

The following statement holds:

Theorem 3. *Let $1 < \alpha \le 4/3$. Then if the functions $\varphi(x)$, $\zeta(x)$, and $f(t, x)$ (concerning the variable x) satisfy the condition of Lemma 1 and $f(\cdot, x) \in C_{\gamma,\log}[1, T]$, then a solution of the inverse initial problem does exist, moreover, it is unique and represented by the following formula:*

$$u(t,x) = \sum_{k=1}^{\infty} \Bigg[\varphi_k (t-1)^{\alpha-1} E_{\alpha,\alpha}[-(k\pi)^2 (\log t)^\alpha] + \psi_k (t-1)^{\alpha-2} E_{\alpha,\alpha-1}\left[-(k\pi)^2 (\log t)^\alpha\right] + \\ + \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left[-(k\pi)^2 \left(\log\frac{t}{s}\right)^\alpha\right] f_k(s)\frac{ds}{s} \Bigg] \sin k\pi x, \quad (8)$$

$$\psi(x) = \sum_{k=1}^{\infty} \frac{1}{(\xi-1)^{\alpha-2} E_{\alpha,\alpha-1}\left[-(k\pi)^2 (\log \xi)^\alpha\right]} \Bigg\{ \zeta_k - \varphi_k (\xi-1)^{\alpha-1} E_{\alpha,\alpha}[-(k\pi)^2 (\log \xi)^\alpha] - \\ - \int_1^\xi \left(\log\frac{\xi}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left[-(k\pi)^2 \left(\log\frac{\xi}{s}\right)^\alpha\right] f_k(s)\frac{ds}{s} \Bigg\} \sin k\pi x; \quad (9)$$

Proof. We assume that function $\psi(x)$ is given and then use the solution of the direct problem, given by

$$u(t,x) = \sum_{k=1}^{\infty} \sin k\pi x \Bigg\{ \varphi_k (t-1)^{\alpha-1} E_{\alpha,\alpha}\left[-(k\pi)^2 (\log t)^\alpha\right] + \psi_k (t-1)^{\alpha-2} \times \\ \times E_{\alpha,\alpha-1}\left[-(k\pi)^2 (\log t)^\alpha\right] + \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left[-(k\pi)^2 \left(\log\frac{t}{s}\right)^\alpha\right] f_k(s)\frac{ds}{s} \Bigg\}. \quad (10)$$

Substituting (10) into the over-determination condition, one will get

$$\zeta(x)=\sum_{k=1}^{\infty}\sin k\pi x\Bigg\{\varphi_k(\xi-1)^{\alpha-1}E_{\alpha,\alpha}\left[-(k\pi)^2(\log\xi)^{\alpha}\right]+\psi_k(\xi-1)^{\alpha-2}\times$$

$$\times E_{\alpha,\alpha-1}\left[-(k\pi)^2(\log\xi)^{\alpha}\right]+\int\limits_1^{\xi}\left(\log\frac{\xi}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left[-(k\pi)^2\left(\log\frac{\xi}{s}\right)^{\alpha}\right]f_k(s)\frac{ds}{s}\Bigg\}. \quad (11)$$

In [15], it was shown that the Mittag-Leffler function $E_{a,b}(z)$ does not have zeros for $1<a\leq 4/3,\ z,\ b\in\mathbb{R}$. You can also see [14]. Therefore, dividing the coefficient of the function $\psi(x)$ in (11), one can easily get (9).

The convergence of infinite series representing the solution can be proved using Lemma 1. Namely, using (8) and considering the estimate $|zE_{a,b}(-z)|\leq C$ for $z>0$, we get

$$|u(t,x)|\leq\sum_{k=1}^{\infty}\left[C_1|\varphi_k|+C_2|\psi_k|+C_3\int\limits_1^t|f_k(s)|\frac{ds}{s}\right].$$

Here $C_i\ (i=\overline{1,3})$ are positive real numbers. Further, since they do not have principal importance, we denote them as C. Integration by parts and the well-known inequality $2ab\leq a^2+b^2$ yield

$$|u(t,x)|\leq C\sum_{k=1}^{\infty}\frac{1}{(k\pi)^2}\left[|\varphi_k^{(1)}|^2+|\psi_k^{(1)}|^2+\int\limits_1^t|f_k^{(1)}(s)|^2\frac{ds}{s}\right],$$

where $\varphi_k^{(1)}=\int\limits_0^1\varphi'(x)\cos k\pi x dx,\ \psi_k^{(1)}=\int\limits_0^1\psi'(x)\cos k\pi x dx,\ f_k^{(1)}(t)=\int\limits_0^1 f_x(t,x)\cos k\pi x dx.$ Using Parseval's identity, one can easily get

$$|u(t,x)|\leq C\left(\sum_{k=1}^{\infty}\frac{2}{(k\pi)^2}+\left\|\varphi'(x)\right\|_2^2+\left\|\psi'(x)\right\|_2^2+\int\limits_1^t\|f_x(s,\cdot)\|_2^2\frac{ds}{s}\right).$$

Here $\|\cdot\|_2$ presents the $L_2(0,1)$-norm. Similarly, we will get

$$|\psi(x)|\leq C\left(\sum_{k=1}^{\infty}\frac{1}{(k\pi)^2}+\left\|\zeta'(x)\right\|_2^2+\left\|\varphi'(x)\right\|_2^2+\int\limits_1^{\xi}\|f_x(s,\cdot)\|_2^2\frac{ds}{s}\right).$$

Note that to get this, we have imposed the following conditions on the given functions:

$$\varphi(x),\ \zeta(x),\ f(t,x)\in C[0,1],\ \varphi(0)=\varphi(1)=0,\ \zeta(0)=\zeta(1)=0,\ f(t,0)=f(t,1)=0,$$

$$\varphi'(x),\ \zeta'(x),\ f_x(t,\cdot)\in L_2(0,1).$$

To prove the uniform convergence of infinite series corresponding to $u_{xx}(t,x)$, we will impose more conditions on the given functions as it was present in Lemma 1.

The uniqueness of the solution to the inverse initial problem follows from the completeness of the system $\{\sin k\pi x\}_{k=1}^{\infty}$. Namely, assuming that the problem has two different set of solutions $\{u_1(t,x),\,\psi_1(x)\}$, $\{u_2(t,x),\,\psi_2(x)\}$, and denoting

$$u(t,x)=u_1(t,x)-u_2(t,x),\quad \psi(x)=\psi_1(x)-\psi_2(x),$$

we will get the corresponding homogeneous problem. Then we multiply both sides of (4) by $\sin m\pi x$, and integrate along $[0,1]$:

$$\int_0^1 u(t,x)\sin m\pi x dx=\int_0^1\sum_{k=1}^{\infty}U_k(t)\sin(k\pi x)\sin(m\pi x)dx.$$

Based on orthogonality of the system $\{\sin k\pi x\}_{k=1}^{\infty}$, one can easily get

$$U_k(t)=2\int_0^1 u(t,x)\sin k\pi x dx. \tag{12}$$

The solution of the homogeneous case of Problem (5)–(6), from (7) easy to deduce that $U_k(t)\equiv 0$. Hence, due to (12), considering that the system $\{\sin k\pi x\}_{k=1}^{\infty}$ is complete, we obtain $u(t,x)\equiv 0$, which proves the uniqueness of the solution to the considered inverse initial problem.

Theorem 3 has been proved. □

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Алимов Зухридин, Кербал Себти, АДАМАР БӨЛШЕК ТУЫНДЫСЫ ҚАТЫСҚАН БӨЛШЕК РЕТТІ ТОЛҚЫН ТЕҢДЕУІ ҮШІН КЕРІ БАСТАПҚЫ ЕСЕП

Бұл мақалада уақыт айнымалысы бойынша Адамар бөлшек туындысы қатысқан бөлшек ретті толқын теңдеуі үшін кері бастапқы есеп қарастырылады. Көп зерттелетін Риман–Лиувилль мен Капуто туындыларынан айырмашылығы, Адамар туындысы логарифмдік ядро арқылы анықталып, баяу жад әсерлері мен мультипликативті құрылымдарды сипаттауға мүмкіндік береді. Авторлар Адамар типті операторлармен байланысты шекаралық есептердің шешілуі және экстремум принципі жөніндегі алдыңғы жұмыстарға сүйене отырып, кері есептің жалғыз шешімі үшін жеткілікті шарттарды дәлелдейді. Зерттеу Фурье қатарлары мен екі параметрлі Миттаг–Леффлер функциясының қасиеттеріне негізделген. Алынған нәтижелер бөлшек толқындық теңдеулерге арналған кері есептер теориясын толықтырады және Адамар туындыларының күрделі уақытша динамикаларды сипаттаудағы маңызын көрсетеді.

Түйін сөздер: Бөлшек ретті толқын теңдеуі, кері бастапқы есеп, Адамар бөлшек туындысы, Миттаг–Леффлер функциясы.

Алимов Зухридин, Кербал Себти, ОБРАТНАЯ НАЧАЛЬНАЯ ЗАДАЧА ДЛЯ ДРОБНОГО ВОЛНОВОГО УРАВНЕНИЯ С ДРОБНОЙ ПРОИЗВОДНОЙ АДАМАРА

В данной статье рассматривается обратная начальная задача для дробного волнового уравнения с дробной производной Адамара по времени. В отличие от более известных производных Римана–Лиувилля и Капуто, производная Адамара определяется с помощью логарифмического ядра и обладает особыми аналитическими свойствами, что делает её подходящей для моделирования процессов с медленным затуханием памяти и мультипликативной структурой. Основываясь на ранее полученных результатах по принципу экстремума и разрешимости краевых задач с производными Хадамара, авторы устанавливают достаточные условия единственности решения. Метод основан на разложении по собственным функциям и использовании свойств двухпараметрической функции Миттага–Леффлера. Полученные результаты вносят вклад в развитие теории обратных задач для дробных волновых уравнений и подчёркивают роль производных Адамара в моделировании сложной временной динамики.

Ключевые слова: Дробное волновое уравнение, обратная начальная задача, дробная производная Адамара, функция Миттаг–Леффлера.

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On the orthogonality of a system of solenoidal functions in a three-dimensional cube

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Abstract. Previously, we constructed a system of orthonormal functions (SOF) as a solution to a spectral problem for a fourth-order operator in a three-dimensional cube. Using a three-dimensional curl operator applied to SOF, we derived a system of solenoidal functions (SSF), which are crucial in the study of incompressible fluid dynamics and the theory of Navier-Stokes equations. However, the SSF obtained in this way did not possess the orthogonality property, which is often desirable in theoretical analysis and numerical applications. The main result of this work is the construction of a new system of solenoidal functions, based on the original SOF, which is shown to be almost orthogonal. This property makes the system suitable for use in spectral methods and other analytical approaches where near-orthogonality ensures better convergence and stability. The methodology developed in this study can be generalized to other types of boundary value problems involving higher-order differential operators and may contribute to the development of more efficient computational schemes in fluid mechanics.

Keywords. Spectral problem, fourth-order differential operator, system of solenoidal functions, orthogonality property

1 Introduction

In a number of works by Academician O.A. Ladyzhenskaya, the importance of constructing “a certain fundamental system” [1, p. 94] in the space of solenoidal functions (in particular, for the simplest domains such as a cube, a sphere, etc.) was pointed out. The existence of such a system is well known and requires no proof. This fact is actively employed by researchers in proving existence theorems for two- and three-dimensional Navier–Stokes systems, in both

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linear and nonlinear cases, as well as in further analysis of the qualitative properties of the solutions thus obtained. However, for the numerical solution of boundary value problems that arise in the Stokes as well as Navier-Stokes systems, it is necessary to construct the aforementioned fundamental system explicitly. R.S. Saks wrote in [2, p. 724], [3, p. 56]: "In particular, O.A. Ladyzhenskaya was interested in the possibility of calculating the eigenfunctions of the Stokes operator in domains of the simplest types (cube, sphere, etc.)".

The paper is organized as follows. In Section 2, we present the basic definitions, several auxiliary statements, the key ideas, and the preliminary results of the work. In Section 3, we formulate the main result of the paper, Theorem 11, and provide its proof. In the end, a brief conclusion is provided.

2 Definitions, Problem Statement and Preliminary results

2.1. On the concept of a "fundamental system". Let $d \geq 2$, $\Omega_1 \subset \mathbb{R}^d$ be an open bounded domain with Lipschitz boundary $\partial\Omega_1$. Ladyzhenskaya O.A. writes [1, 105]: 'We do not require the property of linear independence from the so-called "fundamental system" $\{\vec{\varphi}_k\}_{k=1}^{\infty}$ (for example, in space $(\overset{\circ}{W}{}_2^1(\Omega_1))^d$, $d \geq 2$). Instead, we require only the following: for any $\varepsilon > 0$ and any function $\vec{\varphi} \in (\overset{\circ}{W}{}_2^1(\Omega_1))^d$ there exists a sum $\vec{\varphi}^{\varepsilon} = \sum\limits_{k=1}^{N_\varepsilon} a_k \vec{\varphi}_k$, $N_\varepsilon < \infty$, such that the inequality $\|\nabla(\vec{\varphi} - \vec{\varphi}^{\varepsilon})\|_{L^2(\Omega_1)} \leq \varepsilon$, holds.'

In this paper, we adhere to the above definition of a fundamental system of functions. Moreover, throughout the paper, we denote by $\Omega_1 \subset \mathbb{R}^d$, $d \geq 2$ a domain with a Lipschitz boundary $\partial\Omega_1$, and by $\Omega = (0, l)^d \subset \mathbb{R}^d$ a square for $d = 2$ or a cube for $d = 3$.

We introduce the fundamental function spaces relevant to the analysis of the Navier-Stokes equations [4, 5, 6]

$$\mathbf{H}(\Omega_1) = \{\vec{w} |\, \vec{w} \in \mathbf{L}^2(\Omega_1),\ \operatorname{div} \vec{w} = 0,\ \vec{w} \cdot \vec{n}|_{\partial\Omega_1} = 0\}, \tag{1}$$

$$\mathbf{L}^2(\Omega_1) \equiv (L^2(\Omega_1))^d \equiv \mathbf{H}(\Omega_1) \oplus \mathbf{H}^{\perp}(\Omega_1),$$

where $\vec{w} \cdot \vec{n}$ is the normal component of the vector $\vec{w}$,

$$\mathbf{H}^{\perp}(\Omega_1) = \{\vec{w} |\, \vec{w} \in \mathbf{L}^2(\Omega_1),\ \vec{w} = \operatorname{grad} q,\ q \in W_2^1(\Omega_1)\},$$

$$\mathbf{V}(\Omega_1) = \{\vec{w} |\, \vec{w} \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega_1),\ \operatorname{div} \vec{w} = 0\},\ \ \overset{\circ}{\mathbf{W}}{}_2^1(\Omega_1) \equiv (\overset{\circ}{W}{}_2^1(\Omega_1))^d. \tag{2}$$

Let $\Omega = (0, l)^d$ denote a "d-dimensional cube". We now introduce the Hilbert spaces that will be used throughout this work.

Definition 1. Denote by $V_1(\Omega)$ and $V_2(\Omega)$ the Hilbert spaces equipped with the following inner products, respectively:

$$(\nabla u, \nabla v)_{L^2(\Omega)}\ \forall\, u, v \in \overset{\circ}{W}{}_2^1(\Omega), \tag{3}$$

$$((u,v)) = \sum_{k=1}^{d} \left(\partial_{x_k}^2 u, \partial_{x_k}^2 v\right)_{L^2(\Omega)} \ \forall\, u, v \in \overset{\circ}{W}{}_2^2(\Omega). \tag{4}$$

Definition 2. Let $V_{1k}(0,l)$ and $V_{2k}(0,l)$, $k = 1,...,d$, be Hilbert spaces equipped with the corresponding inner products

$$(\alpha'(x_k), \beta'(x_k))_{L^2(0,l)} \ \forall\, \alpha(x_k), \beta(x_k) \in \overset{\circ}{W}{}_2^1(0,l), \tag{5}$$

$$((\alpha(x_k), \beta(x_k))) = \left(\alpha''(x_k), \beta''(x_k)\right)_{L^2(0,l)} \ \forall\, \alpha(x_k), \beta(x_k) \in \overset{\circ}{W}{}_2^2(0,l). \tag{6}$$

We define the spaces $V_1(\Omega)$ (3) and $V_2(\Omega)$ (4) as the following direct products of the spaces $V_{1k}(0,l)$ (5) and $V_{2k}(0,l)$ (6):

$$V_1(\Omega) = \bigotimes_{k=1}^{d} V_{1k}(0,l), \tag{7}$$

$$V_2(\Omega) = \bigotimes_{k=1}^{d} V_{2k}(0,l). \tag{8}$$

Problem A. It is necessary to construct a system of functions belonging to the space $\mathbf{V}(\Omega)$ (2) that is fundamental (in the sense of O.A. Ladyzhenskaya) in the space $\mathbf{H}(\Omega)$ (1).

2.4. Let $\Omega = (0,l)^3$ be a three-dimensional cube. Previously, we studied the following spectral problem:

$$(\partial_{x_1}^4 + \partial_{x_2}^4 + \partial_{x_3}^4)U(x) = \lambda^2(-\Delta)U(x), \ x \in \Omega, \tag{9}$$

$$U(x) = 0, \quad \partial_{\vec{n}} U(x) = 0, \quad x \in \partial\Omega. \tag{10}$$

Note that the left-hand side of equation (9) defines a positive definite (elliptic) operator in the space $\overset{\circ}{W}{}_2^2(\Omega)$. Therefore, the following statement holds.

Theorem 3. *The set of generalized eigenfunctions $\{u_n(x),\ n \in \mathbb{N}\}$ of the spectral problem* (9)–(10) *belongs to the space $V_2(\Omega)$* (8) *and forms an orthogonal basis in the space $V_1(\Omega)$* (7)*. Moreover, all eigenvalues $\{\lambda_n^2\}_{n\in\mathbb{N}}$ lie on the positive real semi-axis, and the smallest eigenvalue λ_1^2 is strictly positive.*

Remark 4. *Note that when the biharmonic operator appears on the left-hand side of the equation* (9)*, the problem* (9)–(10) *admits an explicit solution only in the case of a circle [7, 8]. A more detailed discussion of this problem can be found in works [10, 11, 12].*

Theorem 5. *The spectral problem* (9)–(10) *has the following solution*

$$u_n(x) = \prod_{k=1}^{3} X_{k,n}(x_k), \quad \lambda_n^2, \quad n \in \mathbb{N}, \tag{11}$$

where $X_{k,n}(x_k) = \Phi_n(y)_{|y=x_k}$, $k = 1, 2, 3$*:*

$$\Phi_{2n-1}(y) = \sin^2 \frac{\lambda_{2n-1} y}{2}, \ \lambda_{2n-1}^2 = \left(\frac{2(2n-1)\pi}{l}\right)^2, \ n \in \mathbb{N}, \tag{12}$$

$$\Phi_{2n}(y) = [\lambda_{2n} l - \sin \lambda_{2n} l] \sin^2 \frac{\lambda_{2n} y}{2} - \sin^2 \frac{\lambda_{2n} l}{2} [\lambda_{2n} y - \sin \lambda_{2n} y], \quad \lambda_{2n}^2 = \left(\frac{2\nu_n}{l}\right)^2, \ n \in \mathbb{N}, \tag{13}$$

and $\{\nu_n, \ n \in \mathbb{N}\}$ *are the positive roots of the equation* $\tan \nu = \nu$.

From Theorems 3 and 5, we deduce the following:

Theorem 6. *The system of eigenfunctions* $\{u_n(x), \ n \in \mathbb{N}\}$ (11)–(13) *belongs to the space* $V_2(\Omega)$ (8) *and forms an orthogonal basis in the space* $V_1(\Omega)$ (7).

The system of eigenfunctions $\{u_n(x), \ n \in \mathbb{N}\}$ from (11)–(13) can be normalized to obtain an orthonormal basis. Thus, the following statement is true.

Corollary 7. *After normalization, the system of eigenfunctions* $\{\bar{u}_n(x_1, x_2, x_3), \ n \in \mathbb{N}\}$ *of the spectral problem* (9)–(10) *can be represented as*

$$\bar{u}_{2n-1}(x_1, x_2, x_3) = \frac{8\sqrt{2}}{3\sqrt{3}(2n-1)\pi\sqrt{l}} \prod_{k=1}^{3} \sin^2 \frac{\lambda_{2n-1} x_k}{2}, \ n \in \mathbb{N}, \tag{14}$$

$$\bar{u}_{2n}(x_1, x_2, x_3) = \frac{\sqrt{6}(1+\nu_n^2)^2}{5\sqrt{l}\,\nu_n^7} \prod_{k=1}^{3} \left[\frac{1}{\nu_n} \sin \frac{2\nu_n x_k}{l} - \frac{2}{l} x_k + 2\sin^2 \frac{\nu_n x_k}{l}\right], \ n \in \mathbb{N}, \tag{15}$$

where $\{\nu_n, \ n \in \mathbb{N}\}$ *are the positive roots of the equation* $\tan \nu = \nu$.

Moreover, the system of eigenfunctions (14)–(15) *belongs to the space* $V_2(\Omega)$ (8) *and forms an orthonormal basis in the space* $V_1(\Omega)$ (7).

As is known, the curl operator is determined by the formula

$$\operatorname{curl}\{U_1(x), U_2(x), U_3(x)\} = \{\partial_{x_2} U_3 - \partial_{x_3} U_2, \, \partial_{x_3} U_1 - \partial_{x_1} U_3, \, \partial_{x_1} U_2 - \partial_{x_2} U_1\}. \tag{16}$$

The following statement holds.

Proposition 8. *If* $U_1 = U_2 = U_3 = U(x) \in \overset{\circ}{W}{}_2^2(\Omega)$, *then we obtain*

$$\mathrm{curl}\{U(x), U(x), U(x)\} \subset \mathbf{V}(\Omega),$$

that is, there does not exist a scalar function $U(x) \in \overset{\circ}{W}{}_2^2(\Omega)$ *for every vector function* $\vec{w}(x) \in \mathbf{V}(\Omega)$ (2) *such that the following relations hold:*

$$\vec{w} = \mathrm{curl}\{U(x), U(x), U(x)\}.$$

Next, using the system of functions (14)–(15), we introduce an extended system of functions $\left\{u_m^1(x, y, z)\right\}_{m=0}^{\infty}$, where

$$u_0^1(x) \equiv 0, \;\; u_m^1(x) = \bar{u}_m(x), \;\; m \in \mathbb{N}, \tag{17}$$

and construct vector functions

$$\left\{\vec{u}_{mjk}^1(x) = \left(u_m^1(x), u_j^1(x), u_k^1(x)\right), \quad m, j, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\right\} \subset \mathbf{V}_2(\Omega), \tag{18}$$

which will form an orthogonal basis in the space $\mathbf{V}_1(\Omega) \equiv (V_1(\Omega))^3$ (7).

From Theorems 3–6, it follows that

Theorem 9. *Let* $d = 3,\ \Omega = (0, l)^3$. *Then the system of eigenfunctions* $\{\vec{u}_{mjk}^1(x),\ m, j, k \in \mathbb{N}_0\}$ (17)–(18) *belongs to the space* $(V_2(\Omega))^3$ (8) *and forms an orthonormal basis in the space* $\mathbf{V}_1(\Omega) \equiv (V_1(\Omega))^3$ (7).

Theorem 10. *Let* $d = 3,\ \Omega = (0, l)^3$. *Then, by applying the operator* curl (16) *to the extended system of vector functions* (17)–(18), *we obtain the desired fundamental system (in the sense of O.A. Ladyzhenskaya):*

$$\{\vec{w}_{mjk}(x) = (w_{1,jk}(x), w_{2,km}(x), w_{3,mj}(x)), \;\; m, j, k \in \mathbb{N}_0\} \subset \mathbf{V}(\Omega), \tag{19}$$

in space of solenoidal functions $\mathbf{H}(\Omega)$ (1), *where*

$$w_{1,jk}(x) = \left(\partial_{x_2} \bar{u}_k^1 - \partial_{x_3} \bar{u}_j^1\right)(x), \quad x \in \Omega, \quad j, k \in \mathbb{N}_0, \tag{20}$$

$$w_{2,km}(x) = \left(\partial_{x_3} \bar{u}_m^1 - \partial_{x_1} \bar{u}_k^1\right)(x), \quad x \in \Omega, \quad m, k \in \mathbb{N}_0, \tag{21}$$

$$w_{3,mj}(x) = \left(\partial_{x_1} \bar{u}_j^1 - \partial_{x_2} \bar{u}_m^1\right)(x), \quad x \in \Omega, \quad m, j \in \mathbb{N}_0, \tag{22}$$

$$\mathrm{div}\, \vec{w}_{mjk}(x) = 0, \;\; x \in \Omega, \;\; m, j, k \in \mathbb{N}_0,$$

$$\vec{w}_{mjk}(x) = 0, \;\; x \in \partial\Omega, \;\; m, j, k \in \mathbb{N}_0.$$

Thus, Theorem 10 provides a solution to Problem A for a cubic domain of independent variables. However, the system of functions (20)–(22) does not have the property of orthogonality.

We rewrite the orthogonal basis (17)–(18) (by Theorem 3) as three groups of orthogonal elements in the direct product of spaces $(V_1(\Omega))^3 \equiv V_1(\Omega) \otimes V_1(\Omega) \otimes V_1(\Omega)$:

$$\{\vec{A}_{1n} \equiv (\bar{u}_n^1, 0, 0), \quad \vec{A}_{2n} \equiv (0, \bar{u}_n^1, 0), \quad \vec{A}_{3n} \equiv (0, 0, \bar{u}_n^1), \quad n \in \mathbb{N}_0\} \subset (V_2(\Omega))^3. \tag{23}$$

Let $\vec{U}(x) = (U_1(x), U_2(x), U_3(x))$ be an arbitrary vector function from $(V_2(\Omega))^3$. It is obvious that the vector function $\vec{U}(x)$ can be represented as the sum of three vector functions:

$$\vec{U}(x) = (U_1(x), 0, 0) + (0, U_2(x), 0) + (0, 0, U_3(x)) \in (V_2(\Omega))^3, \tag{24}$$

In Section 3 we will construct a nearly orthogonal system of solenoidal functions.

3 Main result

From (23)–(24), the Theorems 9 and 10 we obtain a system of vector functions

$$\vec{A}(x) = \left\{\vec{A}_{1n}(x),\, \vec{A}_{2n}(x),\, \vec{A}_{3n}(x), \quad n \in \mathbb{N}\right\}, \quad \vec{A}_{kn}(x) = \left(A_{kn}^1(x), A_{kn}^2(x), A_{kn}^3(x)\right). \tag{25}$$

We formulate the main results of the paper.

Theorem 11 (Main result)**.** *Let $d = 3$, $\Omega = (0, l)^3$. Then, using* (17)–(18)*,* (23)–(24) *and* (25)*, as well as the curl operator* (16)*, we obtain the following six groups* (26)–(32) *of vector functions*

$$\{\vec{w}_{kn}(x) \equiv (w_{1,kn}(x), w_{2,kn}(x), w_{3,kn}(x)),\ k = 1, 2, 3,\ n \in \mathbb{N}\}: \tag{26}$$

$$\begin{cases} w_{1,1,2n-1}(x) = \partial_{x_2} A_{1,2n-1}^3(x) - \partial_{x_3} A_{1,2n-1}^2(x) = 0, \\ w_{2,1,2n-1}(x) = \partial_{x_3} A_{1,2n-1}^1(x) - \partial_{x_1} A_{1,2n-1}^3(x) = \partial_{x_3} \bar{u}_{2n-1}^1(x), \qquad k = 1, \quad n \in \mathbb{N}, \\ w_{3,1,2n-1}(x) = \partial_{x_1} A_{1,2n-1}^2(x) - \partial_{x_2} A_{1,2n-1}^1(x) = -\partial_{x_3} \bar{u}_{2n-1}^1(x), \end{cases} \tag{27}$$

$$\begin{cases} w_{1,2,2n-1}(x) = \partial_{x_2} A_{2,2n-1}^3(x) - \partial_{x_3} A_{2,2n-1}^2(x) = -\partial_{x_3} \bar{u}_{2n-1}^1(x), \\ w_{2,2,2n-1}(x) = \partial_{x_3} A_{2,2n-1}^1(x) - \partial_{x_1} A_{2,2n-1}^2(x) = 0, \qquad k = 2, \quad n \in \mathbb{N}, \\ w_{3,2,2n-1}(x) = \partial_{x_1} A_{2,2n-1}^2(x) - \partial_{x_2} A_{2,2n-1}^1(x) = \partial_{x_1} \bar{u}_{2n-1}^1(x), \end{cases} \tag{28}$$

$$\begin{cases} w_{1,3,2n-1}(x) = \partial_{x_2} A_{3,2n-1}^3(x) - \partial_{x_3} A_{3,2n-1}^2(x) = \partial_{x_2} \bar{u}_{2n-1}^1(x), \\ w_{2,3,2n-1}(x) = \partial_{x_3} A_{3,2n-1}^1(x) - \partial_{x_1} A_{3,2n-1}^3(x) = -\partial_{x_2} \bar{u}_{2n-1}^1(x), \qquad k = 3, \quad n \in \mathbb{N}, \\ w_{3,3,2n-1}(x) = \partial_{x_1} A_{3,2n-1}^2(x) - \partial_{x_2} A_{3,2n-1}^1(x) = 0, \end{cases} \tag{29}$$

$$\begin{cases} w_{1,1,2n}(x) = \partial_{x_2} A^3_{1,2n}(x) - \partial_{x_3} A^2_{1,2n}(x) = 0, \\ w_{2,1,2n}(x) = \partial_{x_3} A^1_{1,2n}(x) - \partial_{x_1} A^3_{1,2n}(x) = \partial_{x_3} \bar{u}^1_{2n}(x), \\ w_{3,1,2n}(x) = \partial_{x_1} A^2_{1,2n}(x) - \partial_{x_2} A^1_{1,2n}(x) = -\partial_{x_3} \bar{u}^1_{2n}(x), \end{cases} \quad k = 1, \quad n \in \mathbb{N}, \tag{30}$$

$$\begin{cases} w_{1,2,2n}(x) = \partial_{x_2} A^3_{2,2n}(x) - \partial_{x_3} A^2_{2,2n}(x) = -\partial_{x_3} \bar{u}^1_{2n}(x), \\ w_{2,2,2n}(x) = \partial_{x_3} A^1_{2,2n}(x) - \partial_{x_1} A^2_{2,2n}(x) = 0, \\ w_{3,2,2n}(x) = \partial_{x_1} A^2_{2,2n}(x) - \partial_{x_2} A^1_{2,2n}(x) = \partial_{x_1} \bar{u}^1_{2n}(x), \end{cases} \quad k = 2, \quad n \in \mathbb{N}, \tag{31}$$

$$\begin{cases} w_{1,3,2n}(x) = \partial_{x_2} A^3_{3,2n}(x) - \partial_{x_3} A^2_{3,2n}(x) = \partial_{x_2} \bar{u}^1_{2n}(x), \\ w_{2,3,2n}(x) = \partial_{x_3} A^1_{3,2n}(x) - \partial_{x_1} A^3_{3,2n}(x) = -\partial_{x_2} \bar{u}^1_{2n}(x), \\ w_{3,3,2n}(x) = \partial_{x_1} A^2_{3,2n}(x) - \partial_{x_2} A^1_{3,2n}(x) = 0, \end{cases} \quad k = 3, \quad n \in \mathbb{N}, \tag{32}$$

for which the following statements are true:

1^0. The set of vector functions (27)–(29) *are orthogonal to each other.*

2^0. The set of vector functions (30)–(32) *are orthogonal to each other.*

3^0. The set of elements from the sets of vector functions (27) *and* (30) *is orthogonal. So are the sets* (28) *and* (31)*, as well as* (29) *and* (32)*.*

4^0. However, the set of elements from the sets of vector functions (27) *and* (31)*,* (27) *and* (32)*,* (28) *and* (30)*,* (28) *and* (32)*,* (29) *and* (30)*,* (29) *and* (31) *do not have the property of orthogonality.*

Here we understand orthogonality in the sense of the space $\mathbf{H}(\Omega)$ (1)*.*

Proof of Theorem 11. Let us prove the validity of point 1^0. Let $k = 1$. Consider the scalar product in space $L^2(\Omega)$ of vector functions of set (27). Let $n \neq m$. We have

$$(\vec{w}_{1,2n-1}(x), \vec{w}_{1,2m-1}(x)) = \int\limits_{\Omega} \left[\partial_{x_3} \bar{u}^1_{2n-1}(x) \partial_{x_3} \bar{u}^1_{2m-1}(x) + \partial_{x_2} \bar{u}^1_{2n-1}(x) \partial_{x_2} \bar{u}^1_{2m-1}(x) \right] dx = 0. \tag{33}$$

Indeed, according to relations (11)–(13) from Theorem 5, the equality to zero of the first and second integrals from (33) is equivalent to the following relations

$$\int\limits_0^l X'_{3,2n-1}(x_3) X'_{3,2m-1}(x_3)\, dx_3 = 0, \quad \int\limits_0^l X'_{2,2n-1}(x_2) X'_{2,2m-1}(x_2)\, dx_2 = 0. \tag{34}$$

The validity of equalities (34) is verified by direct calculation.

We have

$$X'_{j,2n-1}(x_j) = \frac{\lambda_{2n-1}}{2} \tilde{X}'_{j,2n-1}(x_j), \tag{35}$$

where

$$\tilde{X}'_{j,2n-1}(x_j) = \sin\lambda_{2n-1}x_j, \tag{36}$$

$$X'_{j,2n}(x_j) = \frac{2}{l}(\nu_n)^3\cos^2\nu_n\,\tilde{X}'_{j,2n}(x_j), \tag{37}$$

$$\tilde{X}'_{j,2n}(x_j) = \nu_n\cdot\sin\frac{2\nu_n}{l}x_j - 2\sin^2\frac{\nu_n x_j}{l}. \tag{38}$$

Note that without paying attention to the coefficients of the functions (35) and (37), it is sufficient for us to study the orthogonality of the system of functions (36) and (38).

Lemma 12. *The equality*

$$\int\limits_0^l \tilde{X}'_{j,2n-1}(x_j)\tilde{X}'_{j,2m-1}(x_j)dx_j = 0, \quad j=1,2,3, \quad m\neq n, \quad m,n\in\mathbb{N},$$

is true.

Proof of Lemma 12. Indeed, we have

$$\int\limits_0^l \tilde{X}'_{j,2n-1}(x_j)\tilde{X}'_{j,2m-1}(x_j)\,dx_j = \int\limits_0^l \sin\lambda_{2n-1}x_j\cdot\sin\lambda_{2m-1}x_j\,dx_j =$$

$$= \frac{1}{2}\left[\frac{1}{\lambda_{2n-1}-\lambda_{2m-1}}\sin\left(\lambda_{2n-1}-\lambda_{2m-1}\right)x_j - \right.$$

$$\left. - \frac{1}{\lambda_{2n-1}+\lambda_{2m-1}}\sin\left(\lambda_{2n-1}+\lambda_{2m-1}\right)x_j\right]\Bigg|_{x_j=0}^{x_j=l} = 0.$$

The validity of Lemma 12 is established. □

From here we obtain the required orthogonality condition for 1^0 at $k=1,2,3$.

Next, it remains for us to show the orthogonality of the vector functions $\vec{w}_{k,2n-1}(x)$ and $\vec{w}_{j,2m-1}(x)$ for $k\neq j,\ \ k,j=1,2,3$. For $k=1,\ j=2$ we have

$$(\vec{w}_{1,2n-1}(x),\vec{w}_{2,2m-1}(x)) = -\int\limits_{\Omega}\partial_{x_2}\bar{u}^1_{2n-1}(x)\partial_{x_1}\bar{u}^1_{2m-1}(x)\,dx = 0 \quad \forall\, n,m\in\mathbb{N}. \tag{39}$$

Indeed, according to relations (11)–(13) from Theorem 5, the equality to zero of the integral from (39) will be satisfied if following equalities hold

$$\int\limits_0^l X_{2,2m-1}(x_2)X'_{2,2n-1}(x_2)\,dx_2 = 0 \quad \text{and} \quad \int\limits_0^l X_{1,2n-1}(x_1)X'_{1,2m-1}(x_1)\,dx_1 = 0. \tag{40}$$

The validity of equalities (40) is verified by direct calculation. This is true for other $k \neq j$, $k, j = 1, 2, 3$. Let us just note that it is necessary to consider two cases: $m = n$ and $m \neq n$.

Lemma 13. *The equality*

$$\int_0^l X_{j,2m-1}(x_j) X'_{j,2n-1}(x_j)\, dx_j = 0, \quad j = 1, 2, 3, \quad m \neq n, \quad n \in \mathbb{N},$$

is true.

Proof of Lemma 13. We have

$$\int_0^l X_{j,2m-1}(x_j) X'_{j,2n-1}(x_j)\, dx_j = \frac{\lambda_{2n-1}}{2} \int_0^l \sin^2 \frac{\lambda_{2m-1} x_j}{2} \sin \lambda_{2n-1} x_j \, dx_j = 0.$$

□

Remark 14. *Let $m = n$. The equality*

$$\int_0^l X_{j,2n-1}(x_j) X'_{j,2n-1}(x_j)\, dx_j = \frac{1}{2} \left[X_{j,2n-1}(x_j)\right]^2 \Big|_0^l = 0, \quad j = 1, 2, 3, \quad n \in \mathbb{N},$$

is true.

Let us prove the validity of point 2^0. Let $k = 1$. Consider the scalar product in space $L^2(\Omega)$ of vector functions of set (30). Let $n \neq m$. We have

$$(\vec{w}_{1,2n}(x), \vec{w}_{1,2m}(x)) = \int_\Omega \left[\partial_{x_3} \bar{u}^1_{2n}(x) \partial_{x_3} \bar{u}^1_{2m}(x) + \partial_{x_2} \bar{u}^1_{2n}(x) \partial_{x_2} \bar{u}^1_{2m}(x)\right] dx = 0. \tag{41}$$

Indeed, according to relations (11)–(13) from Theorem 5, the equality to zero of the integral from (41) will be satisfied if following equalities hold

$$\int_0^l X'_{3,2n}(x_3) X'_{3,2m}(x_3)\, dx_3 = 0 \quad \text{and} \quad \int_0^l X'_{2,2n}(x_2) X'_{2,2m}(x_2)\, dx_2 = 0. \tag{42}$$

The validity of equalities (42) is verified by direct calculation.

Lemma 15. *The equality*

$$\int_0^l \tilde{X}'_{j,2n}(x_j)\tilde{X}'_{j,2m}(x_j)\,dx_j = 0, \quad j=1,2,3, \quad m\neq n, \quad m,n\in\mathbb{N}, \tag{43}$$

is true.

Proof of Lemma 15. First of all, we write the left part of the relation (43) as:

$$\int_0^l \left[\nu_n \sin\frac{2\nu_n}{l}x_j - 2\sin^2\frac{\nu_n x_j}{l}\right]\cdot\left[\nu_m \sin\frac{2\nu_m}{l}x_j - 2\sin^2\frac{\nu_m}{l}x_j\right]dx_j = \sum_{k=1}^{4} I_k. \tag{44}$$

Next, we sequentially calculate the integrals from (44) $I_k,\ k=1,...,4$. We have

$$I_1 = \int_0^l \left[\nu_n \sin\frac{2\nu_n}{l}x_j \cdot \nu_m \sin\frac{2\nu_m}{l}x_j\right]dx_j = \frac{l\nu_n^2\nu_m^2}{1+\nu_n^2+\nu_m^2+\nu_n^2\nu_m^2}.$$

Here and in what follows we take into account the equation where ν_n are the positive roots of the equation

$$\tan\nu_n = \nu_n.$$

We calculate the integral I_2. We have

$$I_2 = -2\int_0^l \left[\nu_n \sin\frac{2\nu_n}{l}x_j \cdot \sin^2\frac{\nu_m}{l}x_j\right]dx_j = -\frac{l\nu_n^2\nu_m^2}{1+\nu_n^2+\nu_m^2+\nu_n^2\nu_m^2}.$$

Similarly to I_2, the expression for I_3 is calculated:

$$I_3 = -\frac{l\nu_n^2\nu_m^2}{1+\nu_n^2+\nu_m^2+\nu_n^2\nu_m^2}.$$

It remains to calculate the integral I_4, for which we obtain

$$I_4 = \int_0^l \left(1-\cos\frac{2\nu_n}{l}x_j\right)\left(1-\cos\frac{2\nu_m}{l}x_j\right)dx_j = \frac{l\nu_n^2\nu_m^2}{1+\nu_n^2+\nu_m^2+\nu_n^2\nu_m^2}.$$

So, finally, taking into account (44)–(3) we have

$$\sum_{k=1}^{4} I_k = \frac{l\nu_n^2\nu_m^2}{1+\nu_n^2+\nu_m^2+\nu_n^2\nu_m^2} - \frac{l\nu_n^2\nu_m^2}{1+\nu_n^2+\nu_m^2+\nu_n^2\nu_m^2} -$$

$$-\frac{l\nu_n^2\nu_m^2}{1+\nu_n^2+\nu_m^2+\nu_n^2\nu_m^2}+\frac{l\nu_n^2\nu_m^2}{1+\nu_n^2+\nu_m^2+\nu_n^2\nu_m^2}=0.$$

We have established the validity of the orthogonality condition (43) of Lemma 15. □

From here we obtain the required orthogonality condition for 2^0 at $k=1,2,3$.

Next, it remains for us to show the orthogonality of the vector functions $\vec{w}_{k,2n}(x)$ and $\vec{w}_{j,2m}(x)$ for $k\neq j,\ \ k,j=1,2,3$. For $k=1,\ j=2$ we have

$$(\vec{w}_{1,2n}(x),\vec{w}_{2,2m}(x))=-\int\limits_{\Omega}\partial_{x_2}\bar{u}^1_{2n}(x)\partial_{x_1}\bar{u}^1_{2m}(x)\,dx=0\quad\forall\,n,m\in\mathbb{N}.\tag{45}$$

Indeed, according to relations (11)–(13) from Theorem 5, the equality to zero of the integral from (45) will be satisfied if following equalities hold

$$\int\limits_0^l X_{2,2m}(x_2)X'_{2,2n}(x_2)\,dx_2=0\quad\text{and}\quad\int\limits_0^l X_{1,2n}(x_1)X'_{1,2m}(x_1)\,dx_1=0,\tag{46}$$

where

$$X_{2,2m}(x_2)=C(m)\tilde{X}_{2,2m}(x_2),\quad \tilde{X}_{2,2m}(x_2)=\frac{1}{\nu_m}\sin\frac{2\nu_m x_2}{l}-\frac{2}{l}x_2+2\sin^2\frac{\nu_m x_2}{l},$$

$$X_{2,2n}(x_2)=C(n)\tilde{X}_{2,2n}(x_2),\quad \tilde{X}'_{2,2n}(x_2)=\nu_n\sin\frac{2\nu_n x_2}{l}-2\sin^2\frac{\nu_n x_2}{l},$$

$C(m),C(n)$ are constants.

The validity of equalities (46) is verified by direct calculation. Let us just note that it is necessary to consider two cases: $m=n$ and $m\neq n$. Let $m\neq n$. We obtain

Lemma 16. *The equality*

$$I_{m,n}=\int\limits_0^l\tilde{X}_{2,2m}(x_2)\tilde{X}'_{2,2n}(x_2)\,dx_2=\sum_{s=1}^{6}I_{m,n,s}=0,\quad m\neq n,\quad m,n\in\mathbb{N},\tag{47}$$

is true.

Proof of Lemma 16. Let us calculate the integrals $I_{m,n,s},\ s=1,...,6$. We have

$$\begin{aligned}I_{m,n,1}&=\frac{\nu_n}{\nu_m}\int\limits_0^l\sin\frac{2\nu_m x_2}{l}\sin\frac{2\nu_n x_2}{l}\,dx_2\\&=-\frac{l\nu_n}{4\nu_m}\left[\frac{\sin 2(\nu_m+\nu_n)}{\nu_m+\nu_n}-\frac{\sin 2(\nu_m-\nu_n)}{\nu_m-\nu_n}\right]=l\frac{\nu_n^2}{(1+\nu_m^2)(1+\nu_n^2)},\end{aligned}\tag{48}$$

$$I_{m,n,2} = -\frac{2}{\nu_m}\int\limits_0^l \sin\frac{2\nu_m x_2}{l}\sin^2\frac{\nu_n x_2}{l}\,dx_2$$

$$= -\frac{1}{\nu_m}\int\limits_0^l \sin\frac{2\nu_m x_2}{l}\left[1-\cos\frac{2\nu_m x_2}{l}\right]dx_2 = -l\frac{\nu_n^2}{(1+\nu_m^2)(1+\nu_n^2)}, \tag{49}$$

$$I_{m,n,3} = -\frac{2\nu_n}{l}\int\limits_0^l x_2\sin\frac{2\nu_n x_2}{l}\,dx_2 = -l\frac{\nu_n^2}{1+\nu_n^2}, \tag{50}$$

$$I_{m,n,4} = \frac{4}{l}\int\limits_0^l x_2\sin^2\frac{\nu_n x_2}{l}\,dx_2 = l\frac{\nu_n^2}{1+\nu_n^2}, \tag{51}$$

$$I_{m,n,5} = 2\nu_n\int\limits_0^l \sin\frac{2\nu_n x_2}{l}\sin^2\frac{\nu_n x_2}{l}\,dx_2$$

$$= l\left[\frac{\nu_n^2}{1+\nu_n^2} - \frac{\nu_n(\nu_n+\nu_m)}{2(1+\nu_m^2)(1+\nu_n^2)} - \frac{\nu_n(\nu_n-\nu_m)}{2(1+\nu_m^2)(1+\nu_n^2)}\right] = l\frac{\nu_n^2\nu_m^2}{(1+\nu_m^2)(1+\nu_n^2)}, \tag{52}$$

$$I_{m,n,6} = -4\int\limits_0^l \sin^2\frac{\nu_m x_2}{l}\sin^2\frac{\nu_n x_2}{l}\,dx_2$$

$$= l\left[-1+\frac{1}{1+\nu_m^2}+\frac{1}{1+\nu_n^2}-\frac{1}{(1+\nu_m^2)(1+\nu_n^2)}\right] = -l\frac{\nu_n^2\nu_m^2}{(1+\nu_m^2)(1+\nu_n^2)}. \tag{53}$$

According to (48)–(53) we obtain relation (47).

□

The equalities (45) or (47) is true for other $k \neq j$, $k,j = 1,2,3$.

Remark 17. *Let $m = n$. The equality*

$$I_{m,m} = \int\limits_0^l \tilde{X}_{2,2n}(x_2)\tilde{X}'_{2,2n}(x_2)\,dx_2 = \frac{1}{2}\left[\tilde{X}_{2,2n}(x_2)\right]^2\Big|_0^l = 0, \quad n\in\mathbb{N},$$

is true.

Let us prove the validity of point 3^0. Consider the scalar product in space $L^2(\Omega)$ of vector functions of sets (27) and (30). We have

Lemma 18. *For any m, $n \in \mathbb{N}$ the equality*

$$\int_0^l \tilde{X}'_{j,2n-1}(x_j)\tilde{X}'_{j,2m}(x_j)\,dx_j = 0, \quad j = 1,2,3, \tag{54}$$

holds.

Proof of Lemma 18. Indeed, first of all, from (54) we obtain

$$\int_0^l \sin\lambda_{2n-1}x_j\left[\nu_m \sin\frac{2\nu_m}{l}x_j - 2\sin^2\frac{\nu_m}{l}x_j\right]dx_j = \sum_{k=1}^{2}(-1)^{k-1}J_k.$$

We calculate the integral J_1. We have

$$J_1 = \frac{\nu_m}{2}\int_0^l \left[\cos\left(\lambda_{2n-1} - \frac{2\nu_m}{l}\right)x_j - \cos\left(\lambda_{2n-1} + \frac{2\nu_m}{l}\right)x_j\right]dx_j =$$

$$= \frac{l^2\lambda_{2n-1}\nu_m^2}{2(1+\nu_m^2)\left[\nu_m^2 - \left(\frac{\lambda_{2n-1}l}{2}\right)^2\right]}. \tag{55}$$

We calculate the integral J_2. We have

$$J_2 = \int_0^l \sin\lambda_{2n-1}x_j\left[1 - \cos\frac{2\nu_m}{l}x_j\right]dx_j = -\frac{l^2\lambda_{2n-1}\nu_m^2}{2(1+\nu_m^2)\left[\left(\frac{\lambda_{2n-1}l}{2}\right)^2 - \nu_m^2\right]}. \tag{56}$$

So, finally, taking into account (55)–(56) we have

$$\sum_{k=1}^{2}(-1)^{k-1}J_k = \frac{l^2\lambda_{2n-1}\nu_m^2}{2(1+\nu_m^2)\left[\nu_m^2 - \left(\frac{\lambda_{2n-1}l}{2}\right)^2\right]} - \frac{l^2\lambda_{2n-1}\nu_m^2}{2(1+\nu_m^2)\left[\nu_m^2 - \left(\frac{\lambda_{2n-1}l}{2}\right)^2\right]} = 0.$$

The statement of Lemma 18 is proved. □

Similarly, as in case (27) and (30), one can show the orthogonality of the systems of vector functions (28) and (31), (29) and (32). This completes the proof of point 3^0 of Theorem 11.

Let us move on to the proof of point 4^0 Theorem 11.

Lemma 19. *Vector functions* $\vec{w}_{k,2n-1}(x)$ *and* $\vec{w}_{j,2m}(x)$ *for* $k \neq j,\ \ k,j = 1,2,3$ *are non-orthogonal.*

Proof of Lemma 19. For $k=1$, $j=2$ we show that the following inequality holds

$$(\vec{w}_{1,2n-1}(x), \vec{w}_{2,2m}(x)) = -\left(\partial_{x_2}\bar{u}^1_{2n-1}(x), \partial_{x_1}\bar{u}^1_{2m}(x)\right) = C(n,m)\prod_{k=1}^{3} T_k$$

$$= C(n,m)\int_0^l \tilde{X}_{1,2n-1}(x_1)\tilde{X}'_{1,2m}(x_1)\,dx_1 \int_0^l \tilde{X}'_{2,2n-1}(x_2)\tilde{X}_{2,2m}(x_2)\,dx_2$$

$$\times \int_0^l \tilde{X}_{3,2n-1}(x_3)\tilde{X}_{3,2m}(x_3)\,dx_3 \neq 0, \quad C(m,n) = \text{constant}. \tag{57}$$

To do this, it is sufficient to calculate the first and third factors represented by the integrals in (57). We have

$$2T_1 = 2\int_0^l \tilde{X}_{1,2n-1}(x_1)\tilde{X}'_{1,2m}(x_1)\,dx_1 = 2\int_0^l \sin^2\frac{\lambda_{2n-1}x_1}{2}\left[\nu_m \sin\frac{2\nu_m x_1}{l} - 2\sin^2\frac{\nu_m x_1}{l}\right]dx_1$$

$$= \int_0^l [1-\cos\lambda_{2n-1}x_1]\left[\nu_m \sin\frac{2\nu_m x_1}{l} - 2\sin^2\frac{\nu_m x_1}{l}\right]dx_1 = \sum_{s=1}^{6} I_s. \tag{58}$$

$$2T_1 = \sum_{s=1}^{6} I_s = \nu_m \int_0^l \sin\frac{2\nu_m x_1}{l}\,dx_1 = l\frac{1}{1+\nu_m^2}. \tag{59}$$

$$I_2 = -\int_0^1 dx_1 = -l. \tag{60}$$

$$I_3 = \int_0^l \cos\frac{2\nu_m x_1}{l}\,dx_1 = l\frac{1}{1+\nu_m^2}. \tag{61}$$

$$I_4 = -\nu_m \int_0^l \sin\frac{2\nu_m x_1}{l}\cos\lambda_{2n-1}x_1\,dx_1 = -l\frac{4\nu_m^4}{(1+\nu_m^2)\left[(2\nu_m)^2 - (\lambda_{2n-1}l)^2\right]}. \tag{62}$$

$$I_5 = \int\limits_0^l \cos \lambda_{2n-1} x_1 \, dx_1 = 0. \tag{63}$$

$$I_6 = -\nu_m \int\limits_0^l \cos \frac{2\nu_m x_1}{l} \cos \lambda_{2n-1} x_1 \, dx_1 = -l \frac{4\nu_m^4}{(1+\nu_m^2)\left[(2\nu_m)^2 - (\lambda_{2n-1} l)^2\right]}. \tag{64}$$

Thus, from (58)–(64) we obtain

$$2T_1 = -\nu_m \int\limits_0^l \cos \frac{2\nu_m x_1}{l} \cos \lambda_{2n-1} x_1 \, dx_1 = -l \frac{4\nu_m^2}{(2\nu_m)^2 - (\lambda_{2n-1} l)^2} \neq 0.$$

$$T_3 = \int\limits_0^l \tilde{X}_{3,2n-1}(x_3) \tilde{X}_{3,2m}(x_3) \, dx_3$$

$$= \int\limits_0^l \sin^2 \frac{\lambda_{2n-1} x_3}{2} \left[\frac{1}{\nu_m} \sin \frac{2\nu_m x_3}{l} - \frac{2}{l} x_3 + 2 \sin^2 \frac{\nu_m x_3}{l} \right] dx_3 = \sum_{s=1}^{3} J_s.$$

$$J_1 = \frac{1}{\nu_m} \int\limits_0^l \sin^2 \frac{\lambda_{2n-1} x_3}{2} \sin \frac{2\nu_m x_3}{l} \, dx_3 = -\frac{l}{2(1+\nu_m^2)} \left[1 + \frac{2\nu_m}{(2\nu_m)^2 - (\lambda_{2n-1} l)^2} \right].$$

$$J_2 = -\frac{2}{l} \int\limits_0^l x_3 \sin^2 \frac{\lambda_{2n-1} x_3}{2} \, dx_3 = -\frac{l}{2}. \tag{65}$$

$$J_3 = 2 \int\limits_0^l \sin^2 \frac{\lambda_{2n-1} x_3}{2} \sin^2 \frac{\nu_m x_3}{l} \, dx_3 = \frac{l}{2} - \frac{l}{2(1+\nu_m^2)} \left[1 - \frac{2\nu_m}{(2\nu_m)^2 - (\lambda_{2n-1} l)^2} \right]. \tag{66}$$

Thus, from (65)–(66) we obtain

$$T_3 = \sum_{s=1}^{3} J_s = -\frac{l}{1+\nu_m^2} \neq 0.$$

It remains to consider the factor T_2. We have

$$T_2 = \int\limits_0^l \tilde{X}'_{2,2n-1}(x_2) \tilde{X}_{2,2m}(x_2) \, dx_2 = -\int\limits_0^l \tilde{X}_{2,2n-1} \tilde{X}'_{2,2m}(x_2) \, dx_2. \tag{67}$$

From (67) the following relations follow:

$$T_2 = -T_1 \neq 0.$$

□

Similarly, as in the case of the pair of systems (27) and (31), it can be shown that pairwise systems of vector functions (27) and (32), (28) and (30), (28) and (32), (29) and (30), (29) and (32) do not possess the orthogonality property. This completes the proof of point 4^0 of Theorem 11.

Theorem 11 is completely proved.

□

Theorem 11 is a significant refinement of the statement made in Theorem 10.

Thus, Theorem 10 and Theorem 11 give us a solution to Problem A for a cubic domain of independent variables.

Conclusion

Fundamental systems in the space of solenoidal vector fields for a cube are constructed, which can easily be reformulated for an arbitrary rectangular parallelepiped.

These fundamental systems of functions can be used to approximate the solution of both direct and inverse boundary value problems for stationary and evolutionary systems of the Stokes and Navier-Stokes equations in a cubic domain, as well as in cylinders with cross-sections shaped like a cube.

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Жиеналиев М.Т., Ерғалиев М.Г. ҮШ ӨЛШЕМДІ КУБТАҒЫ СОЛЕНОИДАЛДЫҚ ФУНКЦИЯЛАР ЖҮЙЕСІНІҢ ОРТОГОНАЛДЫҒЫ ТУРАЛЫ

Бұрын біз үш өлшемді текшеде төртінші ретті дифференциалдық оператор үшін спектрлік есептің шешімі ретінде іргелі функциялар жүйесін (ІФЖ) құрған едік. Үш өлшемді ротор операторын ІФЖ-ге қолдану арқылы біз сұйықтықтың сығымдалмайтын қозғалысын сипаттайтын Навье–Стокс теңдеулерінің теориясында маңызды болып табылатын соленоидалдық функциялар жүйесін (СФЖ) алдық. Алайда, бұл жолмен алынған СФЖ ортогональдылық қасиетіне ие емес, бұл оны теориялық және сандық әдістерде пайдалануды шектейді. Бұл жұмыста ІФЖ негізінде құрылған және дерлік ортогоналдылық қасиеті бар жаңа СФЖ құрылымы ұсынылады. Мұндай жүйе спектралдық және вариациялық әдістерде тиімді қолдануға мүмкіндік береді, өйткені дерлік ортогоналдылық шешімдердің жинақтылығы мен орнықтылығын арттырады. Ұсынылған әдіс жоғары ретті дифференциалдық операторлармен байланысты басқа шеттік есептерге жалпылауға болады және гидродинамика есептерінде тиімді сандық алгоритмдерді әзірлеуге ықпал етуі мүмкін.

Түйін сөздер: спектралдық есеп, төртінші ретті дифференциалдық оператор, соленоидалдық функциялар жүйесі, ортогоналдық қасиет.

Дженалиев М. Т., Ергалиев М. Г. ОБ ОРТОГОНАЛЬНОСТИ СИСТЕМЫ СОЛЕНОИДАЛЬНЫХ ФУНКЦИЙ В ТРЕХМЕРНОМ КУБЕ

Ранее мы построили систему фундаментальных функций (СФФ) как решение спектральной задачи для дифференциального оператора четвёртого порядка в трёхмерном кубе. Применяя к СФФ трёхмерный оператор ротор, мы получили систему соленоидальных функций (ССФ), которая имеет важное значение в теории уравнений Навье–Стокса и моделировании движения несжимаемой жидкости. Однако построенная таким образом ССФ не обладает свойством ортогональности, что ограничивает её применение в теоретическом анализе и численных методах. В данной работе предложено новое построение ССФ на основе СФФ, обладающее свойством почти ортогональности. Это позволяет использовать такую систему в спектральных и вариационных методах, где почти ортогональность способствует улучшению сходимости и стабильности решений. Представленный подход может быть обобщён на другие краевые задачи с участием дифференциальных операторов высокого порядка и может быть полезен при разработке более эффективных численных алгоритмов в задачах гидродинамики.

Ключевые слова: спектральная задача, дифференциальный оператор четвертого порядка, система соленоидальных функций, свойство ортогональности.

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